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On the Motivic spectrum BO and Hermitian K-theory

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A mathematician said "Who
Can quote me a theorem that's true?
For the ones that I know
Are simply not so
When the characteristic is two!"

– An anonymous Irish poet

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Contents

Introduction

Conventions and notations

Throughout this thesis, by a scheme we will always mean a *quasi-compact quasi-separated scheme*. We will use the term *vector bundle over X* for both a locally free $\mathcal{O}_X$-module and the associated scheme over X, $V \to X$. Whenever we have an essentially small category $\mathcal{C}$, we will replace it with an equivalent small category if it is necessary for a construction to make sense. By an S-scheme or a scheme over S we will mean a scheme X, with a morphism $f : X \to S$.

Overview

Motivic homotopy theory was introduced by Morel and Voevodsky [MV99] as a way to extend the machinery of homotopy theory from topological spaces to schemes and algebraic varieties. Several cohomology theories in algebraic geometry are representable as objects in the stable motivic homotopy category $SH(S)$ [Voe98]. In topology, the Bott periodicity isomorphisms

$$\Omega^2 BU \simeq \mathbb{Z} \times BU$$

$$\Omega^2 BO \simeq \mathbb{Z} \times BSp \quad \Omega^2 BSp \simeq \mathbb{Z} \times BO$$

give spectra $\mathbf{BU} = (\mathbb{Z} \times BU, \Omega BU, \mathbb{Z} \times BU, \ldots)$ and $\mathbf{BO} = (\mathbb{Z} \times BO, \Omega BO, \mathbb{Z} \times BSp, \Omega BSp, \mathbb{Z} \times BO, \ldots)$ which represent complex and real K-theory respectively. Voevodsky in [Voe98] was able to show that for any regular Noetherian scheme S of finite dimension, algebraic K-theory is analogously represented by $\mathbf{BGL} = (\mathbb{Z} \times BGL, \mathbb{Z} \times BGL, \ldots)$ in the stable motivic homotopy category $SH(S)$. $\mathbf{BGL}$ has the property that its complex analytification $\mathbf{BGL}(\mathbb{C})^{an}$ ([Ser56]) is isomorphic to $\mathbf{BU}$ in the stable homotopy category $\mathbf{SH}$. Hermitian K-theory is the general term for the different approaches to extend this to real K-theory. One way to do this is to use that fact that real K-theory can be understood as complex K-theory of spaces equipped with a C_2-action. This leads to the algebraic K-theory of categories with duality (Sec. 2.4). Hornbostel in [Hor05] was able to show that applying this theory to the category of finitely generated projective R-modules equipped with the duality $P \mapsto Hom(P, R)$ gives us a hermitian K-theory spectrum $\mathbf{KO}$ over any scheme where 2 is invertible. Another approach is to use the algebraic groups $O(n)$ and $Sp(2n)$ and define the spectra explicitly using a motivic analog of Bott periodicity. Panin and Walter in [PW18] were able to construct such a spectrum $\mathbf{BO}$ over any scheme S containing $\frac{1}{2}$ and show that is it isomorphic to Hornbostel's $\mathbf{KO}$ in

$SH(S)$ whenever S is regular Noetherian of finite dimension. Using this new model, Röndigs and Østvær were able to compute the slice spectral sequence of hermitian K-theory and provide an alternate proof of Milnor's conjecture on quadratic forms [RØ16].

There are several people working on extending the hermitian K-theory spectra to the case where 2 is not necessarily invertible. One approach is by Heine, Spitzweck, and Verdugo [HSV19] using the machinery of ∞-categories to define a family of hermitian K-theory spaces associated to any exact ∞-category with a C_2-action. In a recent preprint, Calmés et al.[CDH+20] where able to extend several results about hermitians K-groups to arbitrary rings. Another approach is by Schlichting, using the theory of quadratic functors [Sch19]. No single approach given us all the desired properties.

The main goal of this thesis is to study Panin and Walter's construction of the motivic spectrum **BO** and extend it to arbitrary schemes. Let $HGr(2r, 2n)$ denote the subscheme of the Grassmannians $Gr(2r, 2n)$ which classifies symplectic subbundles of $\mathcal{O}^{\oplus 2n}$ equipped with the standard hyperbolic symplectic form. The infinite quaternionic Grassmannian is the ind-scheme $HGr = \operatorname{colim}_n HGr(2n, 4n)$. Panin and Walter showed that over schemes with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ ([PW18, Thm.8.2]), there are isomorphisms $\mathbb{Z} \times HGr \cong KSp$ in the unstable homotopy category $H(S)$, where KSp is the symplectic K-theory space. In chapter 3 we show that the isomorphism still holds without the invertibility of 2, provided we define KSp as a group completion of the groupoid of symplectic bundles. The fact that the proof extends is referenced in many places (cf.[AHW18]) but here we explicitly write it out. In chapter 4 we show that the structure maps

$$HP^1 \wedge HP^1 \wedge \mathbb{Z} \times HGr \to \mathbb{Z} \times HGr$$

used to define **BO** extend to arbitrary schemes. As this spectrum is constructed out of $HGr(2r, 2n)$, it is in fact cellular. This was shown in [RSØ18] when $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, the proof extends to the general case. The collected result of this thesis is then the following.

Theorem. *For any scheme S, there exists a motivic cellular HP^1-spectrum*

$$\mathbf{BO}_S = (\mathbf{BO}_0, \mathbf{BO}_1, \dots) \in SH(S)_{HP^1} \cong SH(S)$$

such that,

1. $\mathbf{BO}_{2n+1} \cong \mathbb{Z} \times HGr \cong KSp$ *in* $H_\bullet(S)$*;*
2. *for any morphism of schemes $f : S_1 \to S_2$, there exists a canonical isomorphism $Lf^*\mathbf{BO}_{S_2} \xrightarrow{\sim} \mathbf{BO}_{S_1}$ in $SH(S_1)$;*
3. *if $S \to Spec(\mathbb{Z})$ is any scheme with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, $Lf^*\mathbf{BO}$ is equal to the spectrum in [PW18].*

In particular when S is regular Noetherian of finite dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, **BO** *represents hermitian K-theory.*

We briefly describe the contents of the thesis below.

In the first chapter we recall the basics of motivic homotopy theory. We define $H(S)$ using the category of simplicial presheaves $sPSh(Sm_S)$ equipped with the injective local model structure w.r.t the Nisnevich topology and inverting all maps $\mathbb{A}^1 \times \mathcal{X} \to \mathcal{X}$. We also discuss homotopy theory of classifying spaces of groups and monoids, with an eye toward application to the symplectic groups Sp_{2n}.

In the second chapter we discuss K-theory. All the notations and results about algebraic K-theory are standard. When it comes to hermitian K-theory there is some ambiguity when 2 is not invertible. Hermitian K-theory is the general term for the study of K-theory of categories equipped with some kind of duality. We use the name orthogonal K-theory when the the duality comes from the standard duality on vector bundles $V \mapsto Hom(V, \mathcal{O})$. Our definition of symplectic K-theory will be using the group completion of $Bi\mathbf{Symp}(X)$, where $\mathbf{Symp}(X)$ is the category of symplectic bundles over a scheme X. Here we also discuss the relationship of symplectic K-theory to the classifying spaces BSp_{2n} and the hermitian K-theory of chain complexes.

In the third chapter we discuss the geometric properties of the Grassmannian schemes $Gr(r, n)$ and their open subschemes $RGr(r, n)$ and $HGr(2r, 2n)$. The final section discusses the unstable isomorphism result

$$\mathbb{Z} \times HGr \cong \mathbb{Z} \times BSp_\infty \cong KSp;$$

to prove this we tweak Morel and Voevodsky's proof of the analogous result for algebraic K-theory.

In the final chapter we prove the main result stated above using Panin and Walter's model of $SH(S)$ as the category of HP^1-spectra.

Open questions: We do not know what cohomology theory **BO** represents over $Spec(\mathbb{Z})$. The hope is that it represents some version of hermitian K-theory. In a recent paper Schlichting [Sch19] introduced the notion of K-theory of forms which generalises the K-theory of spaces with duality. In this formalism $\mathbf{Symp}(X)$ becomes the category of quadratic spaces for a suitable choice of category with forms structure on vector bundles $Vect(X)$. If this theory satisfies Nisnevich excision and $\mathbb{A}^1$-invariance then KSp will represent it in the unstable homotopy category. It is still unknown if this is true. **BO** is known to have an E_∞-ring structure over schemes where 2 is invertible [LÁ18]. This is still not known over $Spec(\mathbb{Z})$. However, a recent preprint by Bachmann and Wickelgren ([BW20]) suggests that there is a version of the hermitian K-theory ring spectrum which can be defined over arbitrary schemes (although it might not represent hermitian K-theory any more). It is unknown if this spectrum is stably equivalent to ours when 2 is not invertible.

Chapter 1

Motivic Homotopy theory

1.1 Model categories

Model categories give us a convenient framework to do homotopy theory. They were first introduced by Quillen in [Qui67]. Our main reference for this section will be [GJ09]. The beginning of [MV99] also has a good summary of the required results.

Definition 1.1.1. A model category is a category $\mathcal{M}$, with all small limits and colimits, equipped with three classes of morphisms W, F, C satisfying the following axioms.

M1 : Given composable morphisms f and g, if two out of the three morphisms f, g and $g \circ f$ are in W, then the third is as well.

M2 : W, F, C are all closed under retracts

M3 : Given any commutative square of the form given by the solid arrows below,

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle c}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

with $c \in C$ and $f \in F$, there exists a lift given by the dotted arrow which makes the diagram commute whenever either c or f is in W.

M4 : Any morphism $f : X \to Y$ in $\mathcal{M}$ admits two factorisations, $X \xrightarrow{c} A \xrightarrow{p} Y$ and $X \xrightarrow{i} B \xrightarrow{f} Y$ with $c \in C$, $p \in F \cap W$, $f \in F$ and $i \in C \cap W$ and these factorisations can be chosen to be functorial.

We call the morphisms in W, C and F, *weak equivalences, fibrations* and *cofibrations* respectively. We also call morphisms in $W \cap C$ and $W \cap F$, *acyclic cofibrations* and *acyclic fibrations* respectively.

From these axioms it follows that W, F, C are subcategories of $\mathcal{M}$ containing every object. When it is understood we will refer to the model category by just the underlying category. The definition of model categories in [Qui67] does not use closure under limits and colimits or the factorisation being functorial. However, Hovey in [Hov99] adds these properties as most model categories we care about satisfy them and they make certain constructions easier.

Definition 1.1.2. An object $M \in Ob(\mathcal{M})$ is called *fibrant* if the unique morphism to the terminal object $M \to *$ is a fibration. An object $M \in Ob(\mathcal{M})$ is called *cofibrant* if the unique morphism from the initial object $\emptyset \to M$ is a cofibration. We denote the full subcategories of fibrant and cofibrant objects by $\mathcal{M}_f$ and $\mathcal{M}_c$ respectively. We denote by $\mathcal{M}_{cf}$, the intersection $\mathcal{M}_f \cap \mathcal{M}_c$.

Applying axiom **M4** to the maps $M \to *$ and $\emptyset \to M$ gives us factorisations $M \to RM \to *$ and $\emptyset \to QM \to M$, where RM is fibrant, QM is cofibrant, and $M \to RM$, $QM \to M$ are acyclic cofibration and acyclic fibration respectively. Functoriality gives us fibrant and cofibrant replacement functors, $R : \mathcal{M} \to \mathcal{M}_f$ and $Q : \mathcal{M} \to \mathcal{M}_c$.

Definition 1.1.3. Let $\mathcal{M}$ be any category and W be a class morphisms in $\mathcal{M}$. The localization category $\mathcal{M}[W^{-1}]$, if it exists, is a category equipped with a functor $L : \mathcal{M} \to \mathcal{M}[W^{-1}]$ such that,

1. For every morphism $w \in W$, $L(w)$ is an isomorphism.
2. Given any functor $F : \mathcal{M} \to \mathcal{C}$ such that $F(w)$ is an isomorphism for all $w \in W$, there is a functor $F' : \mathcal{M}[W^{-1}] \to \mathcal{C}$ and a natural isomorphism $F' \circ L \xrightarrow{\sim} F$.
3. For any category $\mathcal{N}$, the functor $\circ L : Fun(\mathcal{M}[W^{-1}], \mathcal{N}) \to Fun(\mathcal{M}, \mathcal{N})$ is fully faithful.

In any model category $\mathcal{M}$ we call an object M *contractible* if the unique map to the terminal object $M \to *$ is a weak equivalence. From this definition it is clear that if $\mathcal{M}[W^{-1}]$ exists, then it is unique up to equivalence. When $\mathcal{M}$ is a small category, we can construct $\mathcal{M}[W^{-1}]$ by explicitly adding inverses to morphisms in W. From [Hov99, Th.1.2.10] we obtain the following theorem.

Theorem 1.1.1. *For any model category $(\mathcal{M}, W, F, C)$, there exists a construction of the localization category $\mathcal{M}[W^{-1}]$ such that,*

1. *$\mathcal{M} \to \mathcal{M}[W^{-1}]$ factors through the cofibrant-fibrant replacement functor $QR : \mathcal{M} \to \mathcal{M}_{cf}$;*
2. *the induced functor $\mathcal{M}_{cf} \to \mathcal{M}[W^{-1}]$ is full and identity on objects.*

We denote $\mathcal{M}[W^{-1}]$ by $\mathrm{Ho}\mathcal{M}$ and call it the homotopy category of $\mathcal{M}$.

Remark 1.1.1. In fact, $Hom_{\mathcal{M}}(A, B) \to Hom_{\mathrm{Ho}M}(A, B)$ is a surjection whenever A is cofibrant and B is fibrant.

We will sometimes use $\mathrm{Ho}\mathcal{M}$ for the localization category $\mathcal{M}[W^{-1}]$ even when $\mathcal{M}$ is not a model category, especially for subcategories of model categories.

Definition 1.1.4 (*Quillen adjunction*)**.** Let $(\mathcal{M}, W, F, C)$ and $(\mathcal{M}', W', F', C')$ be model categories. A Quillen adjunction from $\mathcal{M}$ to $\mathcal{M}'$ is an adjunction (P, U, ϕ) such that,

1. $P : \mathcal{M} \to \mathcal{M}'$ preserves cofibrations and acyclic cofibrations;

2. $U : \mathcal{M}' \to \mathcal{M}$ preserves fibrations and acyclic fibrations.

P and U are called a *left* and *right Quillen functors* respectively.

Quillen adjunctions induce morphisms between the corresponding homotopy categories.

Definition 1.1.5 (Derived functor)**.** Given a Quillen adjunction (P, U, ϕ) between model categories $(\mathcal{M}, W, F, C)$ and $(\mathcal{M}', W', F', C')$,

1. *the total left derived functor* $LP : \mathrm{Ho}\mathcal{M} \to \mathrm{Ho}\mathcal{M}'$ is the composite

$$\mathrm{Ho}\mathcal{M} \xrightarrow{\mathrm{Ho}Q} \mathrm{Ho}\mathcal{M}_c \xrightarrow{\mathrm{Ho}P} \mathrm{Ho}\mathcal{M}'$$

2. *the total right derived functor* $RU : \mathrm{Ho}\mathcal{M}' \to \mathrm{Ho}\mathcal{M}$ is the composite

$$\mathrm{Ho}\mathcal{M}' \xrightarrow{\mathrm{Ho}R} \mathrm{Ho}\mathcal{M}'_f \xrightarrow{\mathrm{Ho}U} \mathrm{Ho}\mathcal{M}$$

3. When LP and RU are isomorphisms, we call the Quillen pair a *Quillen equivalence*

These maps are well defined by Ken Brown's lemma [Hov99, Lem.1.1.12].

Example 1.1.1. We have the standard model structure on **sSet**. The cofibrations are monomorphisms, the weak equivalences are maps $f : \mathcal{X} \to \mathcal{Y}$ which induce isomorphisms of path connected components $\pi_0(\mathcal{X}) \to \pi_0(\mathcal{Y})$ and all higher homotopy groups $\pi_n(\mathcal{X}, x) \xrightarrow{\sim} \pi_n(\mathcal{Y}, f(x))$ and the fibrations are Kan fibrations. We will denote the corresponding homotopy category by Ho**sSet**. The category of pointed simplicial sets with the standard model structure with homotopy category Ho$\mathbf{sSet}_*$. We have a Quillen adjunction $(-)_+ : \mathbf{sSet} \leftrightarrows \mathbf{sSet}_* : F$ given by the disjoint base point and forgetful functor respectively.

Lemma 1.1.2 ([Hov99, Lem.1.3.10])**.** *A Quillen adjunction (P, U, ϕ) between model categories $(\mathcal{M}, W, F, C)$ and $(\mathcal{M}', W', F', C')$ induces an adjunction on the homotopy categories*

$$LP : \mathrm{Ho}\mathcal{M} \leftrightarrows \mathrm{Ho}\mathcal{M}' : RU.$$

The categories **sSet** and $\mathbf{sSet}_*$ have additional structure which makes constructions such as homotopy limits and colimits much easier. Several of these properties extend to categories of simplicial presheaves and sheaves.

1.2 Simplicial homotopy theory

Recall that a site is a category with a Grothendieck topology (T, τ), which allows us to define a category of sheaves on T, $Sh_\tau(T)$. We will mostly define Grothendieck topologies using covering families (cf.[Jar15, Chap.3]).

Definition 1.2.1. 1. A *point in a site* (T, τ) is an adjoint pair of functors $x : \mathbf{Set} \underset{x_*}{\overset{x^*}{\leftrightarrows}} Sh_\tau(T)$ such that x^* preserves finite limits. We call $x^*\mathcal{X}$ the *stalk of* $\mathcal{X}$ *at* x.

2. A *conservative set of points* in a site (T, τ) is a set $\{x_i\}_{i \in I}$ of points in (T, τ) such that

$$\prod_{i \in I} x_i^* : Sh_\tau(T) \to \prod_{i \in I} \mathbf{Set}$$

is a faithful functor.

3. We say a site (T, τ) has *enough points* if it has a conservative set of points.

For simplicity, we will assume all sites have enough points. Note that from this definition it follows that given a conservative set of points $\{x_i\}_{i \in I}$, a morphism of sheaves $\mathcal{X} \to \mathcal{Y}$ is an isomorphism if and only if it's image under $\prod_{i \in I} x_i^*$ is an isomorphism. We are interested in the local model structures on categories of simplicial presheaves and sheaves, $sPSh(T)$ and $sSh_\tau(T)$ respectively.

Definition 1.2.2. Let T be a site.

1. A morphism of simplicial sheaves $f : \mathcal{X} \to \mathcal{Y}$ is called a *local weak equivalence (resp.local fibration, local cofibration)*, if for every point $x : \mathbf{Set} \underset{x_*}{\overset{x^*}{\leftrightarrows}} Sh(T)$, the induced morphism of stalks $x^*\mathcal{X} \to x^*\mathcal{Y}$ is a weak equivalence (resp. fibration, cofibration) of simplicial sets;

2. a morphism of simplicial presheaves is called a local weak equivalence if its sheafification is one.

Example 1.2.1. We will mostly be interested in Grothendieck topologies on categories of schemes. For any scheme S, let Sch_S denote the (essentially small) category of schemes of finite type over S and Sm_S be its full subcategory of smooth schemes of finite type over S.

1. The collection of open covers of X, $\{U_i \to X\}$, for each $X \in Sch_S$, define a topology on Sch_S. We call this the *Zariski topology* and denote the corresponding Zariski site by $(Sch_S, \tau_{Zar}) = (Sch_S)_{Zar}$. Every (set theoretic) point $x \in X$ in an S-scheme defines a point in the Zariski site by

$$\mathcal{X} \mapsto \operatorname*{colim}_{x \in U} \mathcal{X}(U)$$

where the colimit is over open neighbourhoods of x. The stalks then coincide with stalks in the sense of sheaves on topological spaces. The set

$$\{x \in X | \forall X \in Sch_S\}$$

then defines a conservative set of points.

2. We call a collection of maps $\{Z_i \to X\}$ an *étale covering* of X if each $Z_i \to X$ is étale [Sta18, 02GI] and

$$\coprod_{i \in I} Z_i \to X$$

is a surjection of underlying sets. The collection of étale coverings define a topology on Sch_S called the *étale topology* and the corresponding étale site is denoted by $(Sch_S, \tau_{ét}) = (Sch_S)_{ét}$. Every open embedding is étale [Sta18, 02GP] and so $\tau_{Zar} \subset \tau_{ét}$. The set of geometric points of all schemes in Sch_S defines a conservative set of points in the étale topology (cf.[Sta18, 03PO]).

3. Every étale map is smooth and hence the topologies τ_{Zar} and $\tau_{ét}$ restrict to site structures on smooth S-schemes, $(Sm_S)_{Zar}$ and $(Sm_S)_{ét}$ respectively.

Both $sSh(T)$ and $sPSh(T)$ have model structures with weak equivalences given by these local weak equivalences.

Theorem 1.2.1. *There exist proper model structures on $sSh(T)$ and $sPSh(T)$ with weak equivalences the local weak equivalences of sheaves and presheaves respectively, and cofibrations the categorical monomorphisms. Furthermore, the inclusion and sheafification adjunction,*

$$a : sPSh(T) \leftrightarrows sSh(T) : i$$

is a Quillen equivalence.

These model structure are discussed in [Jar15]. The model structures above are called the local injective model structures. We denote the associated homotopy category by $\mathcal{H}(T)$. Properness implies that weak equivalences are preserved by pullbacks along fibrations and pushouts along cofibrations. Due to the Quillen equivalence above we will be a little lenient when going between these two model categories. There are different equivalent definitions of local weak equivalences. Given a simplicial presheaf $\mathcal{X}$, we define the n^{th} *homotopy sheaf* $\Pi_n(\mathcal{X})$ to be the sheaf of pointed sets over $\mathcal{X}_0$ associated to the pointed presheaf $(x_0 \in \mathcal{X}_0(U)) \mapsto \pi_n(\mathcal{X}(U), x_0)$.

Theorem 1.2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial presheaves. The morphism f is a local weak equivalence if and only if the induced map $a\pi_0\mathcal{X} \to a\pi_0\mathcal{Y}$ is an isomorphism and for all $n \geq 0$ the square,*

$$\begin{array}{ccc} \Pi_n(\mathcal{X}) & \longrightarrow & \Pi_n(\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{X}_0 & \longrightarrow & \mathcal{Y}_0 \end{array}$$

is cartesian.

Proof. Recall that the topological realization of a simplicial set $|X|$ is the coequalizer of $\coprod_n \Delta^n \times X_n \rightrightarrows \coprod_{n'} \Delta^{n'} \times X_{n'}$ where the maps are $(x, y) \mapsto (x, \theta^* y)$ and $(x, y) \mapsto (\theta(x), y)$ respectively. For any point t in the site T, t^* preserves colimits and hence $t^*|\mathcal{X}| = |t^*\mathcal{X}|$. Similarly the homotopy groups are also defined as quotients and hence $t^*\Pi_n(\mathcal{X}) = \Pi_n(t^*\mathcal{X})$ as sets. As t^* commutes with finite products it takes sheaves of (abelian) groups to (abelian) groups. Hence any $f : \mathcal{X} \to \mathcal{Y}$ is a local weak equivalence if and only if it induces an isomorphism of homotopy sheaves. □

Remark 1.2.1. The above description of weak equivalences implies that $f : \mathcal{X} \to \mathcal{Y}$ is a local weak equivalence if and only if for every object X in a site (T, τ) there exists a τ-cover $\{U_i \to X\}$ of X such that $\mathcal{X}(U_i) \to \mathcal{Y}(U_i)$ is a weak equivalence for each i.

By definition, every simplicial presheaf is cofibrant. 1.1.1 implies that for any pair of cofibrant-fibrant objects C, D in a model category $\mathcal{M}$, the map $Hom_{\mathcal{M}}(C, D) \to Hom_{\mathrm{Ho}\mathcal{M}}(C, D)$ is surjective. In $sPSh(T)$, we can go further and describe $Hom_{\mathrm{Ho}\mathcal{M}}(C, D)$ functorially as a quotient of $Hom_{\mathcal{M}}(C, D)$. Recall that for any Kan complex K and any simplicial set L,

$$Hom_{\mathrm{Ho}\mathbf{sSet}}(L, K) \cong \pi(L, K)$$

where $\pi(L, K)$ is the set of simplicial homotopy classes of maps $L \to K$. In particular for any $* \in K_0$,

$$\pi_n(K, *) = \pi((S^n, s_0), (K, *))$$

where $S^n = \Delta^n/\partial\Delta^n$, s_0 is the image of $\partial\Delta^n$ and $\pi((S^n, s_0), (K, *))$ is the set of maps $(S^n, s_0) \to (K, *)$ upto pointed simplicial homotopy.

Definition 1.2.3. Let $\mathcal{X}, \mathcal{Y} \in sPSh(T)$ and $f, g : \mathcal{X} \to \mathcal{Y}$. A simplicial homotopy from f to g is a map $H : \Delta^1 \times \mathcal{X} \to \mathcal{Y}$ such that $Hd_0 = f$ and $Hd_1 = g$. We denote by $\pi(\mathcal{X}, \mathcal{Y})$ the set $Hom_{sPSh(T)}(\mathcal{X}, \mathcal{Y})/\sim$ where $\sim$ is the equivalence relation generated by simplicial homotopies.

Theorem 1.2.3. *Let $\mathcal{X}, \mathcal{Y} \in sPSh(T)$ with $\mathcal{Y}$ fibrant. There is a canonical bijection*

$$Hom_{\mathcal{H}(T)}(\mathcal{X}, \mathcal{Y}) \cong \pi(\mathcal{X}, \mathcal{Y}).$$

This is follows from the fact that every object in $sPSh(T)$ is cofibrant and the equivalence relation defining the Hom sets in $\mathcal{H}(T)$ is precisely the simplcial homotopy equivalence relation [Jar87, Sec 3].

Corollary 1.2.4. *Given fibrant presheaves $\mathcal{X}, \mathcal{Y} \in sPSh(T)$, a morphism $f : \mathcal{X} \to \mathcal{Y}$ is a weak equivalence if and only if there exists $g : \mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are simplicially homotopic to the identity. We call f a* simplicial homotopy equivalence*.*

Proof. Every simplicial homotopy equivalence is a sectionwise homotopy equivalence and hence a sectionwise weak equivalence. The image of f in $\mathcal{H}(T)$ is an isomorphism and hence has an inverse $g \in Hom_{\mathcal{H}(T)}(\mathcal{Y}, \mathcal{X})$. As $\mathcal{X}$ is fibrant, g has a lift to a map of presheaves $\mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are both equal to identity in the homotopy category. As both $\mathcal{X}$ and $\mathcal{Y}$ are fibrant, the result follows from the fact that simplicial homotopy is already an equivalence relation when $\mathcal{Y}$ is fibrant [Qui67]. □

The category of simplicial presheaves $sPSh(T)$ is a closed symmetric monoidal category. The internal Hom $(\mathcal{X}, \mathcal{Y}) \mapsto \underline{Hom}_T(\mathcal{X}, \mathcal{Y})$ is given by

$$\underline{Hom}_T(\mathcal{X}, \mathcal{Y})(U)_n = Hom_{sPSh(T/U)}(\Delta^n \times \mathcal{X}_{|U}, \mathcal{Y}_{|U})$$

where Δ^n is the constant simplicial presheaf corresponding to the simplicial set Δ^n. $\underline{Hom}_T$ is right adjoint to taking product of presheaves. For any a pair of sheaves $\mathcal{X}, \mathcal{Y}$, the Hom functor

$$U \mapsto Hom_{PSh(T/U)}(\mathcal{X}_{|U}, \mathcal{Y}_{|U}) = Hom_{Sh(T/U)}(\mathcal{X}_{|U}, \mathcal{Y}_{|U})$$

is a sheaf. Therefore the closed symmetric monoidal structure restricts to $sSh(T)$. We also have a simplicial Hom functor $\mathbf{Hom}_T : sPSh(T) \times sPSh(T) \to \mathbf{sSet}$ given by $\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y}) = Hom_{sPSh(T)}(\Delta^n \times \mathcal{X}, \mathcal{Y})$. If T has a terminal object $*$, then $\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y}) = \underline{Hom}_T(\mathcal{X}, \mathcal{Y})(*)$.

Lemma 1.2.5. *Let $sPSh(T)$ be the category of presheaves on a site T.*

1. *For any pair of cofibrations $(i : \mathcal{X} \to \mathcal{Y}, j : \mathcal{W} \to \mathcal{Z})$, the induced morphism*
$$(\mathcal{X} \times \mathcal{Z}) \coprod_{\mathcal{X}\times\mathcal{W}} (\mathcal{W} \times \mathcal{Y}) \to \mathcal{Y} \times \mathcal{Z}$$
is a cofibration which is acyclic if either i or j is acyclic.

2. *For any pair of morphisms $(i : \mathcal{X} \to \mathcal{Y}, p : \mathcal{W} \to \mathcal{Z})$ where i is a cofibration and p is a fibration, the induced morphism*
$$\underline{Hom}_T(\mathcal{Y}, \mathcal{W}) \to \underline{Hom}_T(\mathcal{X}, \mathcal{W}) \times_{\underline{Hom}_T(\mathcal{X},\mathcal{Z})} \underline{Hom}_T(\mathcal{Y}, \mathcal{Z})$$
is a fibration which is acyclic if either i or p is acyclic.

3. *For any pair of morphisms $(i : \mathcal{X} \to \mathcal{Y}, p : \mathcal{W} \to \mathcal{Z})$ where i is a cofibration and p is a fibration, the induced map of simplicial sets*
$$\mathbf{Hom}_T(\mathcal{Y}, \mathcal{W}) \to \mathbf{Hom}_T(\mathcal{X}, \mathcal{W}) \times_{\mathbf{Hom}_T(\mathcal{X},\mathcal{Z})} \mathbf{Hom}_T(\mathcal{Y}, \mathcal{Z})$$
is a Kan fibration which is acyclic if either i or p is acyclic.

The lemma follows from the analogous statement for simplicial sets after taking points in T. Such a model category is called a *simplicial model category* [GJ09, II.3]. We will also need the model structure on *pointed* simplicial presheaves.

Theorem 1.2.6. *The category of pointed simplicial presheaves $sPSh_\bullet(T)$ has a model structure where the cofibrations are monomorphisms and weak equivalences are maps $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ whose underlying morphism of presheaves is a local weak equivalence. The smash product $(\mathcal{X}, x) \wedge (\mathcal{Y}, y)$ and the pointed Hom presheaves, $(\underline{Hom}_T((\mathcal{X}, x), (\mathcal{Y}, y)), y)$, induce a simplicial model category structure on $sPSh_\bullet(T)$.*

The adjoint pair $(-)_+ : sPSh(T) \leftrightarrows sPSh_\bullet(T) : F$ is a Quillen adjunction. Using 1.2.5 and 1.2.3, we can give a more explicit description of the homotopy groups of fibrant objects. Given a pointed simplicial presheaf $(\mathcal{Y}, y)$ we denote by $\Omega^n_y\mathcal{Y}$ the pointed simplicial presheaf $\underline{Hom}_T((S^n, s_0), (\mathcal{Y}, y))$.

Lemma 1.2.7. *For any fibrant simplicial presheaf $\mathcal{Y}$ and any point $y : \Delta^0 \to \mathcal{Y}$, $\Omega^n_y\mathcal{Y}$ is fibrant.*

This follows from applying 1.2.5 to $(\Delta^0 \xrightarrow{s_0} S^n, \mathcal{Y} \to \Delta^0)$.

Theorem 1.2.8. *Let $\mathcal{X}, \mathcal{Y} \in sPSh(T)$ with $\mathcal{Y}$ fibrant. For any $y : \Delta^0 \to \mathcal{Y}$,*
$$\pi_n(\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y}), y) \cong \pi(\mathcal{X}, \Omega^n_y\mathcal{Y}) \cong Hom_{\mathcal{H}(T)}(\mathcal{X}, \Omega^n_y\mathcal{Y})$$
where y in the left corresponds to the map $\mathcal{X} \to \Delta^0 \to \mathcal{Y}$. In particular when $\mathcal{X}$ is a representable presheaf $Hom_T(-, X)$ we get $\pi_n(\mathcal{Y}(X), y) \cong Hom_{\mathcal{H}(T)}(X, \Omega^n_y\mathcal{Y})$.

Proof. By 1.2.5, $\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y})$ is a Kan complex and hence by the above discussion,
$$\pi_n(\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y}), f) \cong \pi((S^n, s_0), (\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y}), f)).$$

For any simplicial presheaf $(\mathcal{X}, x)$, $Hom_{sPSh_\bullet(T)}((S^n, s_0), (\mathcal{X}, x))$ is the fiber of the map $Hom_{sPSh(T)}(\Delta^n, \mathcal{X}) \to Hom_{sPSh(T)}(\partial\Delta^n, \mathcal{X})$ over x. From the definition of $\mathbf{Hom}_T(\mathcal{X}, \mathcal{Y})$ it follows that $Hom_{sPSh_\bullet(T)}((S^n, s_0), (\underline{Hom}_T(\mathcal{X}, \mathcal{Y}), y))$ is the fiber of the map

$$Hom_{sPSh(T)}(\Delta^n \times \mathcal{X}, \mathcal{Y}) \to Hom_{sPSh(T)}(\partial\Delta^n \times \mathcal{X}, \mathcal{Y})$$

over $\partial\Delta^n \times \mathcal{X} \to \mathcal{X} \xrightarrow{y} \mathcal{Y}$. By adjointness we get that the fiber of this map is

$$Hom_{sPSh(T)}(\mathcal{X}, \underline{Hom}_T((S^n, s_0), (\mathcal{Y}, y))) \cong Hom_{sPSh(T)}(\mathcal{X}, \Omega^n_y \mathcal{Y}).$$

As all these maps are functorial they are compatible with simplicial homotopies and hence we get the desired results. □

Remark 1.2.2. Let us denote by $R\Omega^n$ the right derived functor of $\Omega^n : sPSh_\bullet(T) \to sPSh_\bullet(T)$ given by $R\Omega^n \mathcal{X} = \Omega^n \circ a\mathcal{X}_f$, where a is the associated sheaf functor and $(-)_f$ a choice of a fibrant replacement.

$$\pi_n(a\mathcal{Y}_f(X), y) \cong \pi(X, R\Omega^n_y \mathcal{Y}) \cong Hom_{\mathcal{H}(T)}(X, R\Omega^n_y \mathcal{Y}) \cong Hom_{\mathcal{H}(T)}(X, \Omega^n_y \mathcal{Y})$$

Further, by the adjunction between pointed and unpointed simplicial presheaves we get

$$\pi(X, R\Omega^n_y \mathcal{Y}) \cong \pi(X_+, (R\Omega^n_y \mathcal{Y}, y)) \cong \pi((S^n, s_0) \wedge X_+, (a\mathcal{Y}_f, y))$$

and hence $Hom_{\mathcal{H}(T)}((S^n, s_0) \wedge X_+, (\mathcal{Y}, y)) \cong \pi_n(a\mathcal{Y}_f(X), y)$.

1.3 Nisnevich topology

Let S be a quasi-compact quasi-separated(qcqs) scheme.

Definition 1.3.1 (Nisnevich topology)**.** A *Nisnevich cover* of $X \in Sm_S$ is a finite family of étale morphisms in Sm_S, $\{i_\alpha : U_\alpha \to X\}_\alpha$, such that for any point $x \in X$ there is an α and a point $u \in U_\alpha$ such that the induced map of residue fields $k(x) \to k(u)$ is an isomorphism. We call the topology induced by these covers the *Nisnevich topology* and denote the site Sm_S with the Nisnevich topology by $(Sm_S)_{Nis}$.

From the definition it follows that the images $i_\alpha(U_\alpha)$ form an open cover of X. In general every open cover is a Nisnevich cover and every Nisnevich cover is an étale cover so we have $\tau_{Zar} \subset \tau_{Nis} \subset \tau_{\acute{e}t}$ (1.2.1). All three of these topologies are subcanonical i.e, every representable presheaf is a sheaf (cf.[Mil80, I.2.17]).

Definition 1.3.2. Let $F : (Sm_S)^{op}_{Nish} \to \mathbf{Set}$ be a presheaf and $I \to Sm_S$ a cofiltered diagram of schemes such that each transition map is affine. Let $X = \lim_I X_i$ be the limit scheme of the diagram. We define $F(X)$ to be the set $\operatorname{colim}_I F(X_i)$.

Given a point $x \in X$ where $X \in Sm_S$, the local ring $Spec(\mathcal{O}_{X,x}) = \lim_{x \in U} U$ is such a limit scheme. The henselization $\mathcal{O}^h_{X,x} = \operatorname{colim}_{(S,q)} S$ where $(R, \mathfrak{m}_x) \to (S, q)$ is an étale map of local rings, is a filtered colimit of rings and hence we get a filtered limit of the corresponding schemes. Note from these examples that the limit scheme need not be smooth (or even of finite type).

Remark 1.3.1. Let $\mathbf{Sch}_S$ be the (large) category of all S-schemes and let $X = \lim_I X_i$ be the filtered limit of smooth S-schemes where every transition map is affine. Given any smooth S-scheme $Y \to S$, we have canonical bijections,

$$\operatorname{colim}_I Hom_{Sm_S}(X_i, Y) \to \operatorname{colim}_I Hom_{\mathbf{Sch}_S}(X_i, Y) \to Hom_{\mathbf{Sch}_S}(\lim_I X_i, Y).$$

The first map is a bijection as Sm_S is a full subcategory of $\mathbf{Sch}_S$. The second map is a bijection by [Sta18, Tag 01ZC].

The set of maps $F \mapsto F(\mathcal{O}^h_{X,x})$ for pairs $X \in Sm_S$ and $x \in X$ forms a conservative family of points in $(Sm_S)_{Nis}$ [AGV72, IV .6.5].

Definition 1.3.3. A distinguished square in $(Sm_S)_{Nis}$ is a cartesian square of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

such that p is étale, i is an open embedding and $p^{-1}(X - U) \to X - U$ is an isomorphism (with the reduced closed subscheme structure). The pair $\{i : U \to X, p : V \to X\}$ gives a Nisnevich covering of X.

Lemma 1.3.1. *For any distinguished square above, the canonical morphism of presheaves $V/U \times_X V \to X/U$ is an isomorphism after sheafification.*

Proof. This follows from the fact that the open immersion $i : U \to X$ is a monomorphism of presheaves and $U \coprod V \to X$ induces a surjection of k-points for any field k. □

Lemma 1.3.2. *Let $F : (Sm_S)^{op}_{Nis} \to$ **Set** be a presheaf. F is a sheaf in the Nisnevich topology if and only if $F(\emptyset) = *$ and for every distinguished square in $(Sm_S)_{Nis}$, the induced diagram*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \downarrow & & \downarrow p \\ F(U) & \xrightarrow{i} & F(U \times_X V) \end{array}$$

is a cartesian square.

This is proved in [MV99, 3.1.4] when S is regular Noetherian of finite dimension. Using Hoyois' remark [Hoy16] the proof extends to qcqs schemes. We will denote the simplicial presheaf categories by $sPSh(Sm_S)_{Nis}$ and $sPSh_\bullet(Sm_S)_{Nis}$ to make the model structure clear and we denote the corresponding homotopy categories by $H^s(S)$ and $H^s_\bullet(S)$ respectively.

Lemma 1.3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial presheaves in $sPSh(Sm_S)_{Nis}$. f is a local weak equivalence if and only if the induced map $\mathcal{X}(\mathcal{O}^h_{X,x}) \to \mathcal{Y}(\mathcal{O}^h_{X,x})$ is a weak equivalence of simplicial sets for all points (X, x) in $(Sm_S)_{Nis}$.*

Proof. This follows from the fact that $F \mapsto F(\mathcal{O}^h_{X,x})$ is a conservative set of points in $(Sm_S)_{Nis}$. □

Definition 1.3.4 (Nisnevich excision)**.** Let $\mathcal{X} \in sPSh((Sm_S)_{Nis})$ be a simplicial presheaf. We say that $\mathcal{X}$ satisfies *Nisnevich excision* if for every distinguished square 1.3.3, the commutative square of simplicial sets

$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{X}(U) & \xrightarrow{i} & \mathcal{X}(U \times_X V) \end{array}$$

is a homotopy cartesian diagram of simplicial sets and $\mathcal{X}(\emptyset)$ is contractible.

Lemma 1.3.4. *Every fibrant simplicial sheaf satisfies Nisnevich excision.*

Proof. First note that the sheaf condition implies that $\mathcal{X}(\emptyset) = \Delta^0$. Given any fibrant sheaf $\mathcal{X}$, 1.3.2 implies that for any distinguished square, the induced square

$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{X}(U) & \xrightarrow{i} & \mathcal{X}(U \times_X V) \end{array}$$

is cartesian. The result will then follow if we show that for any open embedding $V \to U$, the map $\mathcal{X}(U) \to \mathcal{X}(V)$ is a Kan fibration. This is equivalent to showing every diagram of the form

$$\begin{array}{ccc} \Lambda^n_k \times U \coprod_{\Lambda^n_k \times V} \Delta^n \times V & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \Delta^n \times U & \longrightarrow & * \end{array}$$

has a lift. But the left arrow is an acyclic cofibration, by 1.2.5 and the fact that $V \to U$ is a monomorphism of (pre)sheaves, and hence the lift exists. ☐

Theorem 1.3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a weak equivalence of simplicial presheaves. If both $\mathcal{X}$ and $\mathcal{Y}$ satisfy Nisnevich excision then f is a sectionwise equivalence.*

This is [MV99, 3.1.18].

Corollary 1.3.6. *Let $\mathcal{X} \in sPSh(Sm_S)_{Nis}$ satisfy Nisnevich excision. Every fibrant replacement $\mathcal{X} \to a\mathcal{X}_f$ of the associated sheaf is a sectionwise weak equivalence. In particular we have*

$$\pi_n(\mathcal{X}(U)) \cong \pi_n(a\mathcal{X}_f(U)) \cong Hom_{H^s_\bullet(S)}((S^n, s_0) \wedge U_+, \mathcal{X})$$

Let $f : S \to T$ be a morphism of schemes. f induces a morphism $f^{-1} : Sm_T \to Sm_S$ given by pullbacks. This is well defined as smooth morphisms of finite type are closed under pullbacks. f^{-1} induces a continuous map of sites $f : (Sm_S)_{Nis} \to (Sm_T)_{Nis}$ (cf.[MV99, 2.1.42]). We then have adjoint pairs

$$f^* : sPSh(Sm_T) \leftrightarrows sPSh(Sm_S) : f_*$$

$$f^* : sSh(Sm_T) \leftrightarrows sSh(Sm_S) : f_*$$

None of these functors preserve weak equivalences, but they do preserve simplicial homotopies. The functor $f_* \circ a(-)_f$ therefore preserves weak equivalences and we denote by Rf_* the associated right derived functor,

$$Rf_* : H^s(S) \to H^s(T).$$

We can repeat this construction for pointed presheaves and get

$$Rf_* : H^s_\bullet(S) \to H^s_\bullet(T).$$

Theorem 1.3.7. *For any morphism of schemes $f : S \to T$, there exists a functor $ad_f : sPSh(Sm_T) \to sPSh(Sm_T)$ and a natural transformation $ad_f \to id$ such that*

1. *$f^* \circ ad_f : sPSh(Sm_T) \to sPSh(Sm_S)$ preserves weak equivalences;*
2. *$f^*(\mathcal{X}) \to f^* \circ ad_f(\mathcal{X})$ is a weak equivalence for all representable sheaves $\mathcal{X} \cong Hom_{sPSh(Sm_S)}(-, X)$;*
3. *the induced map $Lf^* : H^s(T) \to H^s(S)$ is left adjoint to $Rf_* : H^s(S) \to H^s(T)$;*
4. *For any pair of composable morphisms of schemes f, g, there are canonical isomorphisms*
$$R(g \circ f)_* \cong Rg_* \circ Rf_*$$
$$L(g \circ f)^* \cong Lf^* \circ Lg^*$$

Proof. The central idea is that while f_* might not preserve fibrant objects, it does preserve presheaves with Nisnevich excision. Morel and Voevodsky prove this for the category of sheaves [MV99, Prop.3.1.20]. The result for presheaves follows as Lf^* only needs to exist at the level of the homotopy category and our choice of Rf_* factors through the category of sheaves. □

If f^* commutes with finite limits then by [MV99, 2.1.47] it preserves weak equivalences. In particular this is true when f^* has a left adjoint.

Theorem 1.3.8. *For any smooth morphism $f : S \to T$ there exists a functor $f_\# : PSh(Sm_S) \to PSh(Sm_T)$ left adjoint to f^* that satisfies:*

1. *for any $(U \to S) \in Sm_T$, $f_\#(U) = U \to S \to T$ is a smooth scheme over T.*
2. *when f is étale then $f_\#$ preserves weak equivalences.*

Proof. This is a combination of [MV99, Prop.1.23] and [MV99, Prop.1.26]. □

In this case $f^* \cong Lf^* : H^s(T) \to H^s(S)$ has a left adjoint given by $Lf_\#$.

1.4 $\mathbb{A}^1$-homotopy theory

As before let S be a qcqs scheme. As usual $\mathbb{A}^n_S = \mathbb{A}^n \times S$ is the affine n-space over S which is $Spec(R[x_1, x_2, \dots, x_n])$ on an affine S-scheme $Spec(R)$.

Definition 1.4.1. 1. A simplicial presheaf $\mathcal{X} \in sPSh(Sm_S)_{Nis}$ is called $\mathbb{A}^1$-*local* if for all $\mathcal{Z} \in sPSh(Sm_S)_{Nis}$, the canonical map

$$Hom_{H^s(S)}(\mathcal{Z}, \mathcal{X}) \to Hom_{H^s(S)}(\mathbb{A}^1 \times \mathcal{Z}, \mathcal{X})$$

induced by the projection, is a bijection.

2. A map $\mathcal{X} \to \mathcal{Y}$ is called an $\mathbb{A}^1$-*weak equivalence* (or just $\mathbb{A}^1$-equivalence) if for any $\mathbb{A}^1$-local object $\mathcal{Z}$, the map

$$Hom_{H^s(S)}(\mathcal{Y}, \mathcal{Z}) \to Hom_{H^s(S)}(\mathcal{X}, \mathcal{Z})$$

is a bijection. We denote the class of $\mathbb{A}^1$-weak equivalences by $W_{\mathbb{A}^1}$.

Theorem 1.4.1. *Let $(sPSh(Sm_S), W_{Nis}, C, F_{Nis})$ denote the Nisnevich local model structure on $sPSh(Sm_S)$ discussed above. Let $F_{\mathbb{A}^1}$ denote the class of all maps which satisfy the right lifting property with respect to $W_{\mathbb{A}^1} \cap C$. Then,*

1. *the tuple $(sPSh(Sm_S), W_{\mathbb{A}^1}, C, F_{\mathbb{A}^1})$ defines a proper model category;*
2. *the identity functor induces a Quillen adjunction*

$$(sPSh(Sm_S), W_{Nis}, C, F_{Nis}) \leftrightarrows (sPSh(Sm_S), W_{\mathbb{A}^1}, C, F_{\mathbb{A}^1})$$

We call this the $\mathbb{A}^1$-local model structure. This is shown in [MV99, Sec.3] for simplicial sheaves, the same proof extends to presheaves. We will denote $(sPSh(Sm_S), W_{\mathbb{A}^1}, C, F_{\mathbb{A}^1})$ by $\mathbf{Spc}(S)$ and the corresponding homotopy category by $H(S)$. By the construction of the right derived functor, the Quillen adjunction above gives us the following corollary.

Corollary 1.4.2. *Let $\mathcal{Y}$ be fibrant in the $\mathbb{A}^1$-local model structure ($\mathbb{A}^1$-fibrant for short). For any $\mathcal{X} \in \mathbf{Spc}(S)$ we have a canonical bijection*

$$Hom_{H^s(S)}(\mathcal{X}, \mathcal{Y}) \cong Hom_{H(S)}(\mathcal{X}, \mathcal{Y}).$$

The underlying category of $\mathbf{Spc}(S)$ is just $sPSh(Sm_S)$ but $\mathbf{Spc}(S)$ gives us a more succinct notation. We call elements of $\mathbf{Spc}(S)$ *motivic spaces.*

Remark 1.4.1. We can also define the $\mathbb{A}^1$-local model structure on pointed presheaves giving us $\mathbf{Spc}_\bullet(S)$ with homotopy category $H_\bullet(S)$.

We will recall some of the important results about the $\mathbb{A}^1$-local model structure below. We call a simplicial presheaf $\mathcal{X}$ $\mathbb{A}^1$-*invariant* if the canonical map $\mathcal{X}(-) \to \mathcal{X}(\mathbb{A}^1 \times -)$ is a sectionwise weak equivalence.

Lemma 1.4.3. *Let $\mathcal{X}$ be fibrant in the Nisnevich topology. The following are then equivalent*

1. *$\mathcal{X}$ is $\mathbb{A}^1$-local.*

2. $\mathcal{X}$ *is* $\mathbb{A}^1$*-fibrant.*

3. $\mathcal{X}$ *is* $\mathbb{A}^1$*-invariant, i.e,* $\mathcal{X}(U) \xrightarrow{\sim} \mathcal{X}(\mathbb{A}^1 \times U)$ *for all* $U \in Sm_S$*.*

This result is collected at [MV99, Lem.2.2.8] for localizing w.r.t an arbitrary set of morphisms A. The result is given for sheaves but nothing changes for presheaves as every presheaf is weak equivalent to it's shaefification

Remark 1.4.2. Using the above result we can represent the $\mathbb{A}^1$-homotopy category as the full subcategory of $\mathbb{A}^1$-local objects in $H^s(S)$.

Definition 1.4.2 ($\mathbb{A}^1$-homotopy)**.** Let $\mathcal{X}, \mathcal{Y} \in \mathbf{Spc}(S)$. Given $f, g : \mathcal{X} \to \mathcal{Y}$, an $\mathbb{A}^1$*-homotopy* from f to g is a morphism $H : \mathbb{A}^1 \times \mathcal{X} \to \mathcal{Y}$ such that $H(i_0 \times id) = f$ and $H(i_1 \times id) = g$ where $i_j : S \to \mathbb{A}^1$ is the map $R[x] \to R[x]/(x - j)$ on affine schemes. A map $f : \mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$-homotopy equivalence if there exists a map $g : \mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are $\mathbb{A}^1$-homotopic to $id_{\mathcal{X}}$ and $id_{\mathcal{Y}}$ respectively.

For any scheme S, let $\Delta_S^\bullet$ be the cosimplicial object in the category of S-schemes given by

$$\Delta_S^n = S \times Spec(\mathbb{Z}[x_0, \ldots, x_n]/(x_0 + x_1 \ldots + x_n - 1))$$

The coface and codegeneracy maps are defined along the lines of the face and degeneracy maps of the standard topological n-simplices.

Definition 1.4.3 ($Sing_*^{\mathbb{A}^1}$ construction)**.** The functor $Sing_*^{\mathbb{A}^1} : PSh(Sm_S) \to sPSh(Sm_S)$ is defined to be $Sing_*^{\mathbb{A}^1}(\mathcal{X})_n = \mathcal{X}(\Delta_S^n \times_S -)$. We extend this to get $Sing_*^{\mathbb{A}^1} : sPSh(Sm_S)_{Nis} \to sPSh(Sm_S)_{Nis}$ by taking the diagonal of the resulting bisimplicial presheaf. There is a natural transformation $id \Rightarrow Sing_*^{\mathbb{A}^1}$ induced by the projections $s : \Delta_S^n \times_S U \to U$.

The $Sing_*^{\mathbb{A}^1}$ functor has several important properties.

Theorem 1.4.4. *For any base scheme* S*,* $(Sing_*^{\mathbb{A}^1}, s)$ *satisfies the following.*

1. *For any* $\mathcal{X} \in \mathbf{Spc}(S)$*,* $Sing_*^{\mathbb{A}^1}(\mathcal{X})$ *is* $\mathbb{A}^1$*-invariant.*

2. $Sing_*^{\mathbb{A}^1}$ *commutes with limits,*

3. $Sing_*^{\mathbb{A}^1}$ *takes* $i : S \to \mathbb{A}^1$ *to a simplicial homotopy equivalence,*

4. *for any* $\mathcal{X}$*, the morphism* $s_{\mathcal{X}} : \mathcal{X} \to Sing_*^{\mathbb{A}^1}(\mathcal{X})$ *is a monomorphism and an* $\mathbb{A}^1$*-weak equivalence,*

5. $Sing_*^{\mathbb{A}^1}$ *takes* $F_{\mathbb{A}^1}$ *to* $F_{\mathbb{A}^1}$*.*

This is proved in [MV99] for an arbitrary site with an interval.

Theorem 1.4.5. *Let* $f, g : \mathcal{X} \to \mathcal{Y}$ *be two morphisms. An* $\mathbb{A}^1$*-homotopy from* f *to* g*, induces a simplicial homotopy from* $Sing_*^{\mathbb{A}^1}(f)$ *to* $Sing_*^{\mathbb{A}^1}(g)$*. This implies every* $\mathbb{A}^1$*-homotopy equivalence is an* $\mathbb{A}^1$*-weak equivalence.*

Proof. Let $H : \mathbb{A}^1 \times \mathcal{X} \to \mathcal{Y}$ be an $\mathbb{A}^1$-homotopy from f to g. Applying the $Sing_*^{\mathbb{A}^1}$ functor we get $Sing_*^{\mathbb{A}^1}(H) : Sing_*^{\mathbb{A}^1}(\mathbb{A}^1) \times Sing_*^{\mathbb{A}^1}(\mathcal{X}) \to Sing_*^{\mathbb{A}^1}(\mathcal{Y})$. The problem then reduces to showing that $Sing_*^{\mathbb{A}^1}(i_0)$ and $Sing_*^{\mathbb{A}^1}(i_1)$ are simplicially homotopic as maps $S = Sing_*^{\mathbb{A}^1}(S) \to Sing_*^{\mathbb{A}^1}(\mathbb{A}^1)$. Giving such a homotopy is equivalent to giving an element of $Sing_1^{\mathbb{A}^1}(\mathbb{A}^1)(S)$ whose 0-simplices are i_0 and i_1. The identity in $Hom(\mathbb{A}^1, \mathbb{A}^1) = Sing_1^{\mathbb{A}^1}(\mathbb{A}^1)(S)$ gives us the required map. Given an $\mathbb{A}^1$-homotopy equivalence $f : \mathcal{X} \to \mathcal{Y}$, $Sing_*^{\mathbb{A}^1}(f)$ is a simplicial homotopy equivalence. As $\mathbb{A}^1$-weak equivalences satisfy 2-out of-3 property and every Nisnevich equivalence is an $\mathbb{A}^1$-weak equivalence we get that f is an $\mathbb{A}^1$-weak equivalence. □

The above proof also gives us 1.4.4(3). $\mathbb{A}^1$-homotopy equivalences give us an easy to find class of $\mathbb{A}^1$-weak equivalences.

Example 1.4.1 ($\mathbb{A}^1$-weak equivalences)**.** For any $X \in Sm_S$, $\mathbb{A}^n \times X \to X$ is an $\mathbb{A}^1$-weak equivalence (we can check by induction). In general the structure map of any vector bundle $V \to X$ is an $\mathbb{A}^1$-homotopy equivalence with the zero section $z : X \to V$ giving an $\mathbb{A}^1$-homotopy inverse and the desired homotopy $H : \mathbb{A}^1 \times V \to V$ is the structure map of the associated $\mathcal{O}_X$-module, $\mathcal{O}_X \times V \to V$.

Definition 1.4.4. Let the pair (Ex, θ) denote a choice of Nisnevich fibrant sheaf replacement $Ex : \mathbf{Spc}(S) \to \mathbf{Spc}(S)$ and a natural transformation $\theta : id \Rightarrow Ex$ such that $\theta_{\mathcal{X}} : \mathcal{X} \to Ex(\mathcal{X})$ is an acyclic cofibration for each $\mathcal{X}$. The *$\mathbb{A}^1$-resolution functor* $Ex_{\mathbb{A}^1} : \mathbf{Spc} \to \mathbf{Spc}$ is defined to be,

$$Ex_{\mathbb{A}^1}(-) = Ex \circ (\operatorname*{colim}_n (Ex \circ Sing_*^{\mathbb{A}^1})^n) \circ Ex$$

Lemma 1.4.6. *The functor $\mathcal{X} \mapsto Ex_{\mathbb{A}^1}(\mathcal{X})$, along with the natural transformation $\mathcal{X} \to Ex_{\mathbb{A}^1}(\mathcal{X})$ induced by composing θ and s, is an $\mathbb{A}^1$-fibrant replacement. In particular $Ex_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-local.*

This is [MV99, Lem.3.2.6].

Theorem 1.4.7 (Representability)**.** *Let $\mathcal{F} : (Sm_S) \to$ **sSet** be a motivic space satisfying Nisnevich excision and $\mathbb{A}^1$-homotopy invariance. For any $X \in Sm_S$ and $f \in \mathcal{F}(X)_0$, we have a natural isomorphism*

$$\pi_n(\mathcal{F}(X), f) \xrightarrow{\sim} Hom_{H_\bullet(S)}(S^n \wedge X_+, (\mathcal{F}, f))$$

where $a(-)$ and $(-)_f$ are the Nisnevich sheafification and Nisnevich fibrant replacement functor respectively.

This proof is clearly laid out in [Hor05, Th.3.1].

Proof. By 1.3.6 we have $\pi_n(\mathcal{F}(X), f) \xrightarrow{\sim} Hom_{H^s_\bullet(S)}(S^n \wedge X_+, (\mathcal{F}, f))$. The Nisnevich fibrant sheaf $a\mathcal{F}_f$ is $\mathbb{A}^1$-local as $\mathcal{F} \to a\mathcal{F}_f$ is a sectionwise weak equivalence and $\mathcal{F}$ is $\mathbb{A}^1$-invariant. By the localization adjunction we have

$$Hom_{H^s_\bullet(S)}(S^n \wedge X_+, (a\mathcal{F}_f, a(f)_f)) \cong Hom_{H_\bullet(S)}(S^n \wedge X_+, (a\mathcal{F}_f, a(f)_f))$$

The left hand side is isomorphic to $Hom_{H^s_\bullet(S)}(S^n \wedge X_+, (\mathcal{F}, f))$. As all Nisnevich local weak equivalences are $\mathbb{A}^1$-weak equivalences, the right hand side is isomorphic to $Hom_{H_\bullet(S)}(S^n \wedge X_+, (\mathcal{F}, f))$. □

Along the lines of 1.3.7, given a morphism of schemes $f : S \to T$ we have $f^*(U \times \mathbb{A}^1) = f^*(U) \times \mathbb{A}^1$ for any $U \in Sm_T$. The functor Lf^* therefore preserves $\mathbb{A}^1$-weak equivalences and hence induces $L_{\mathbb{A}^1} f^* : H(T) \to H(S)$. Rf_* preserves $\mathbb{A}^1$-local objects and hence we have the induced functor $R^{\mathbb{A}^1} f_* : H(S) \to H(T)$. We then have results analogous to the simplicial case.

Theorem 1.4.8. *For any morphism $f : S \to T$, we have an adjunction*

$$L_{\mathbb{A}^1} f^* : H(T) \leftrightarrows H(S) : R^{\mathbb{A}^1} f_*$$

Theorem 1.4.9. *Let $f : T \to S$ be a smooth morphism. The functor $Lf_\#$ preserves $\mathbb{A}^1$-weak equivalences and left adjoint to $L_{\mathbb{A}^1} f^* \cong f^*$. In addition, f^* preserves $\mathbb{A}^1$-local objects.*

1.5 Homotopy limits and colimits in $\mathbb{A}^1$-homotopy theory

The homotopy category of a model category usually does not have limits or colimits. To rectify this we introduce the notion of homotopy limits and colimits.

Theorem 1.5.1 ([Lur09, Prop.A.2.8.2])**.** *Let M be a combinatorial model category and I a small category. The diagram category M^I has two model structures where the weak equivalences are the objectwise weak equivalences. The* projective model structure *on M^I is generated by objectwise weak equivalences and objectwise fibrations, we denote it by M^I_{proj}. The* injective model structure *on M^I is generated by objectwise weak equivalences and objectwise cofibrations, we denote it by M^I_{inj}.*

The constant diagram functor $\Delta : M \to M^I$ has left and right adjoint functors given by the limit and colimits functor respectively. It follows that we have Quillen adjunctions

$$\Delta : M \leftrightarrows M^I_{inj} : \lim_I$$

$$\operatorname{colim}_I : M^I_{proj} \leftrightarrows M : \Delta$$

We call the right derived functor associated to $\lim_I$, the *homotopy limit* and denote it by holim_I and we call the left derived functor associated to colim_I, the *homotopy colimit* and denote it by $\operatorname{hocolim}_I$.

Corollary 1.5.2. *Let (P, U, ϕ) be Quillen adjunction between combinatorial model categories M and M' and I a small category.*

1. *For any diagram $I \to M$, there is an isomorphism*

$$LP(\operatorname{hocolim}_I m_i) \cong \operatorname{hocolim}_I LP(m_i)$$

in $\operatorname{Ho}M'$.

2. *For any diagram $I \to M'$, there is an isomorphism*

$$RU(\operatorname{holim}_I m_i) \cong \operatorname{holim}_I RU(m_i)$$

in $\operatorname{Ho}M$.

In particular for any morphism of schemes $f : T \to S$, $L_{\mathbb{A}^1} f^*$ and $R^{\mathbb{A}^1} f_*$ preserve homotopy colimits and homotopy limits respectively.

This is proved in [AE16, Prop.4.10] using the fact that a Quillen adjunction on the underlying categories induces a Quillen adjunction between the corresponding diagram categories. Applying these constructions to **sSet** we get the usual homotopy theory of simplicial sets.

Remark 1.5.1. Note that the above definition applied to the diagram category $\mathbf{sSet}^I = sPSh(I)$ gives us the *global* injective and projective model structures as opposed to the *local* injective and projective model structures defined for sites above.

When the model category is $sPSh(T)$ there is a general construction of $\operatorname{hocolim}_I$ and holim_I functors.

$$\operatorname*{holim}_I : sPSh(T)^I \to sPSh(T)$$

$$\operatorname*{hocolim}_I : sPSh(T)^I \to sPSh(T)$$

which preserve weak equivalences [BK72].

Lemma 1.5.3. *Let $sPSh(T)$ be the category of simplicial presheaves on a site T. For any small category I and any diagram $I \to sPSh(T)$ there are canonical isomorphisms of presheaves*

$$\underline{Hom}_T(\operatorname*{hocolim}_I \mathcal{X}_i, \mathcal{Y}) \cong \operatorname*{holim}_{I^{op}} \underline{Hom}_T(\mathcal{X}_i, \mathcal{Y})$$

$$\underline{Hom}_T(\mathcal{Y}, \operatorname*{holim}_I \mathcal{X}_i) \cong \operatorname*{holim}_I \underline{Hom}_T(\mathcal{Y}, \mathcal{X}_i).$$

In particular we have isomorphisms of simplicial sets,

$$\mathbf{Hom}_T(\operatorname*{hocolim}_I \mathcal{X}_i, \mathcal{Y}) \cong \operatorname*{holim}_{I^{op}} \mathbf{Hom}_T(\mathcal{X}_i, \mathcal{Y})$$

$$\mathbf{Hom}_T(\mathcal{Y}, \operatorname*{holim}_I \mathcal{X}_i) \cong \operatorname*{holim}_I \mathbf{Hom}_T(\mathcal{Y}, \mathcal{X}_i).$$

Lemma 1.5.4. *If I is a right filtering category then the canonical map $\operatorname{hocolim}_I \mathcal{X}_i \to \operatorname{colim}_I \mathcal{X}$ is a weak equivalence.*

Definition 1.5.1. For any simplicial model category M with a zero object $*$ and any object X in M, we define the suspension ΣX and loop space ΩX as the homotopy pushout and pullback,

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array} \qquad \begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

The holim and holim functors are well behaved with respect to localization. Let A be a set of morphisms in $sPSh(T)$ where T is a site. The following lemma is [MV99, Lem.2.2.12].

Lemma 1.5.5. *For any small category I and any diagram $I \to sPSh(T)$, given a natural transformation $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}, \mathcal{Y} \in sPSh(T)$ such that f_U is in W_A for all U in T. Then the morphism $\operatorname{hocolim}_I \mathcal{X} \to \operatorname{hocolim}_I \mathcal{Y}$ in in W_A.*

Using 1.5.4 we get the following corollary.

Corollary 1.5.6. *Let I by a right filtering category and $f : \mathcal{X} \to \mathcal{Y}$ as above.*

1. *The canonical morphism $\operatorname{colim}_I \mathcal{X} \to \operatorname{colim}_I \mathcal{Y}$ is in W_A.*

2. *In particular, if each $\mathcal{X}_i \to \mathcal{X}_j$ is in W_A then $\mathcal{X}_i \to \operatorname{colim}_I \mathcal{X}$ is in W_A.*

Theorem 1.5.7. *Let $\mathcal{X} \in sPSh(T)$ be a presheaf on a site T. We then have a canonical local weak equivalence*

$$\operatorname*{hocolim}_{\Delta^{op}} \mathcal{X}_n \xrightarrow{\sim} \mathcal{X}$$

This is proved in [BK72, XII.3.4].

Theorem 1.5.8. *Let $f : Y \to X$ be a morphism in Sm_S. If there exists a Nisnevich cover $\{U_\alpha \to X\}_{\alpha \in I}$ such that for each α, $f^*U_\alpha \to U_\alpha$ is the projection $U_\alpha \times \mathbb{A}^{n_\alpha}$ for some n_α, then f is an $\mathbb{A}^1$-equivalence.*

Proof. We prove this using the Čech nerve construction. Given the cover $\{U_\alpha \to X\}_{\alpha \in I}$, let $\check{C}(U)_\bullet$ be the simplicial sheaf defined as,

$$\check{C}(U)_n = \coprod_{\substack{I' \subset I \\ |I'| = n+1}} U_{I'}$$

where $U_{I'} = U_{\alpha_1} \times_X U_{\alpha_1} \ldots \times_X U_{\alpha_n}$ with $I' = \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$. There is a canonical map $\check{C}(U)_\bullet \to X$ induced by $U_\alpha \to X$. Taking stalks we can see that $\check{C}(U)_\bullet(\mathcal{O}^h_{Z,z}) \to X(\mathcal{O}^h_{Z,z})$ is a trivial Kan fibration for every point (Z, z)(follows from the property of Nisnevich covers). Therefore $\check{C}(U)_\bullet \to X$ is a Nisnevich local weak equivalence. Similarly $f^*\check{C}(U)_\bullet \cong \check{C}(f^*U)_\bullet \to Y$ is also a Nisnevich weak equivalence. The map $\check{C}(f^*U)_\bullet \to \check{C}(U)_\bullet$ is given in each degree n by a disjoint union of maps $f^*U_{I'} \to U_{I'}$. Each of these maps is a projection $U_{I'} \times \mathbb{A}^{n'_I} \to U_{I'}$ as projections are preserved under pullbacks. We are done by 1.5.7 and 1.5.5. □

Example 1.5.1. Let $f : X \to Y$ in Sm_S. We call f an *affine bundle* if there exists an open cover $\{U_i \to X\}$ such that $f^*U_i \xrightarrow{\sim} \mathbb{A}^n \times U_i$ for a fixed n. It follows from the above theorem that the structure map of any affine bundle is an $\mathbb{A}^1$-equivalence.

Fibrations in the injective model structure are hard to describe and manipulate. It is easier to handle fiber sequences.

Definition 1.5.2 (Homotopy fiber sequences)**.** Let S be any scheme. A sequence of morphisms $F \to E \to B$ in $\mathbf{Spc}_\bullet(S)$ is called a simplicial fiber sequence if the map from F to the homotopy pullback of the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ S & \longrightarrow & B \end{array}$$

is a local weak equivalence in the simplicial model structure ($S \to B$ is the base point). We call $F \to E \to B$ an $\mathbb{A}^1$-local fiber sequence if F is $\mathbb{A}^1$-equivalent to the homotopy pullback in the $\mathbb{A}^1$-local model structure.

The above definition can be generalised to fiber sequences in any model category. It follows that given any simplicial ($\mathbb{A}^1$-local) fiber sequence $F \to E \to B$ and any simplicial ($\mathbb{A}^1$-) weak equivalence $B' \to B$, the pullback sequence $F' \to E \times_B B' \to B'$ is a simplicial ($\mathbb{A}^1$-local) fiber sequence which is simplicially ($\mathbb{A}^1$-) equivalent to the original sequence.

1.6 Classifying spaces of groups and monoids

Recall that for any monoid M, the classifying space $BM \in \mathbf{sSet}$ is the nerve of M considered as a category with one object $*$ and $Hom(*, *) = M$. When the monoid is a group G, there is a G-bundle $EG \to BG$ (as simplicial sets or as topological spaces after geometric realization) given by $EG_n = G^{\times n}$ with the face and degeneracy maps are given by projections and inclusions $(g_1, g_2, \ldots, g_n) \mapsto (g_0, g_1, \ldots, 0, \ldots, g_n)$ respectively. The G-action on EG is given by $g(g_0, g_1, \ldots, g_n) \mapsto (gg_0, gg_1, \ldots, gg_n)$ and further $EG/G \cong BG$. The geometric realization $|EG| \to |BG|$ is the universal covering map of $|BG|$. In particular $|EG|$ is contractible and $|EG| \to |BG|$ is a fibration. We will extend BM to sheaves of simplicial monoids. Let $Mon(sSh(T))$ and $Grp(sSh(T))$ be the categories of simplicial sheaf of monoids and groups respectively. Composing with the group completion functor $+ : \mathbf{Mon} \to \mathbf{Grp}$ induces $+ : Mon(sSh(T)) \to Grp(sSh(T))$ which is left adjoint to the forgetful functor $Grp(sSh(T)) \to Mon(sSh(T))$.

Lemma 1.6.1. *Let $M : T^{op} \to \mathbf{sMon}$ be a simplicial sheaf of monoids. If for each n, M_n is a free monoid on some sheaf of sets, the canonical morphism $BM \to BM^+$ is a local weak equivalence and there is a canonical isomorphism*

$$M^+ \cong R\Omega_s BM$$

in $\mathcal{H}_\bullet(T)$ where M is pointed at its identity element.

This follows from the analogous result for simplicial monoids after taking stalks.

Theorem 1.6.2. *There exists a functor $\Phi_{Mon} : Mon(sSh(T)) \to Mon(sSh(T))$ and a natural transformation $\Phi_{Mon} \to id$ such that,*

1. *for any $M \in Mon(sSh(T))$, $\Phi_{Mon}(M)_n$ is the free sheaf of monoids on a direct sum of representable sheaves.*
2. *The map $\Phi_{Mon}(M) \to M$ is an acyclic local fibration.*

In particular for any sheaf of monoids M, there exists a weakly equivalent monoid $\tilde{M} \in Mon(sSh(T))$ such that $\tilde{M}^+ \cong R\Omega^1_s BM$ in $\mathcal{H}(T)$.

This is given in [MV99, Lem.4.1.1]. The functor $M \mapsto R\Omega^1_s BM$ preserves $\mathbb{A}^1$-weak equivalences.

Lemma 1.6.3. *Let $f : M_1 \to M_2$ be a morphism of simplicial sheaves of monoids over $(Sm_S)_{Nis}$. If f is an $\mathbb{A}^1$-equivalence then so is*

$$R\Omega^1_s BM_1 \to R\Omega^1_s BM_2.$$

This is proved in [MV99, Lem.2.2.35]. Note that the right derived functor is taken with respect to the Nisnevich topology and not $\mathbb{A}^1$-locally. For any simplicial sheaf of monoids $\mathcal{G} : T^{op} \to \mathbf{sGrp}$, on a site T, a left $\mathcal{G}$-action on a simplicial sheaf $\mathcal{X}$ is a morphism $\mathcal{G}\times\mathcal{X} \to \mathcal{X}$ where the usual diagrams commute.

Definition 1.6.1. A principal $\mathcal{G}$-bundle (also called a $\mathcal{G}$-torsor) over a simplicial sheaf $\mathcal{X}$ is a morphism $\mathcal{P} \to \mathcal{X}$ together with a (left) $\mathcal{G}$-action on $\mathcal{P}$ over $\mathcal{X}$ such that,

1. $\mathcal{G} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ given by $(g, v) \mapsto (g(p), p)$ is a monomorphism of sheaves.
2. The canonical morphism $\mathcal{P}/\mathcal{G} \to \mathcal{X}$, where $\mathcal{P}/\mathcal{G}$ is the coequalizer of the projection and group action maps $\mathcal{P} \times \mathcal{G} \rightrightarrows \mathcal{P}$, is an isomorphism of sheaves.

The projection $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$ is a principal $\mathcal{G}$-bundle and is called the trivial principal $\mathcal{G}$-bundle.

A principal $\mathcal{G}$-bundle $\mathcal{P} \to \mathcal{X}$ is trivial if and only if it has a section $s : \mathcal{X} \to \mathcal{P}$. This is because we have a map $\mathcal{G} \times \mathcal{X} \to \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ which induces a bijection at all stalks. When $\mathcal{X} := Hom_{T,\tau}(, X)$ is representable, 1.6.1(2) is equivalent to the map $\mathcal{V} \to Hom_{T,\tau}(-, X)$ being τ-locally split, i.e, there exists a τ-cover $\{U_\alpha \to X\}_\alpha$ such that $\mathcal{V} \times_{U_\alpha} X \to U_\alpha$ is a trivial bundle for all α. Given a principal $\mathcal{G}$-bundle $\mathcal{V} \to \mathcal{X}$ and a morphism of sheaves $\mathcal{Y} \to \mathcal{X}$, the pullback $\mathcal{V} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is also a principal $\mathcal{G}$-bundle. This implies $\mathcal{X} \mapsto P(\mathcal{X}, \mathcal{G})$ is a functor, where $P(\mathcal{X}, \mathcal{G})$ is the set of equivalence classes of principal $\mathcal{G}$-bundles over $\mathcal{X}$. For any sheaf of simplicial groups $\mathcal{G}$, the classifying space $B\mathcal{G}$ is the simplicial sheaf $n \mapsto (B\mathcal{G}_n)_n$ (this is the diagonal of the bisimplicial sheaf $(m, n) \mapsto (B\mathcal{G}_m)_n$). Consider the simplicial sheaf $E\mathcal{G}$ defined as $E\mathcal{G}(U)_n := E(\mathcal{G}_n(U))_n$. The canonical map $E\mathcal{G} \to B\mathcal{G}$ is a principal $\mathcal{G}$-bundle.

Theorem 1.6.4 ([MV99, Prop.4.1.15])**.** *Let $\mathcal{G}$ be a simplicial sheaf of groups of dimension zero over a site T. There is an isomorphism of functors $sPSh(T)^{op} \to$* **Set***,*

$$P(-, \mathcal{G}) \cong Hom_{\mathcal{H}(T)}(-, B\mathcal{G})$$

which sends $1_{B\mathcal{G}} \in Hom_{H_s(T)}(B\mathcal{G}, B\mathcal{G})$ to $E\mathcal{G} \to B\mathcal{G}$.

By dimension zero we mean it has no non-degenerate elements in positive dimensions. As for each $U \in T$, $E\mathcal{G}(U)$ is contractible, the fiber sequence

$$\mathcal{G} \to E\mathcal{G} \to B\mathcal{G}$$

induces a map $\mathcal{G} \to \Omega_s B\mathcal{G}$ which is a local weak equivalence as it is a weak equivalence at stalks. This follows from the analogous result for simplicial sets after reducing to the case of stalks. Let S be a quasi-compact quasi-separated scheme. Let G be an *affine group S-scheme*, i.e, G is a group scheme over S such that the structure map $G \to S$ is affine. By a G-torsor over an S-scheme X we will mean a G-torsor in the above sense with respect to $(Sch_S)_{fppf}$, the category of S-schemes of finite type with the fppf topology. This topology is finer than all the topologies we need and we have the following lemma [AHW18, Lem.2.3.3].

Lemma 1.6.5. *Suppose $G, X \in Sch_S$ with G an affine group S-scheme. Any G-torsor $\mathcal{P} \to X$ over the site $(Sch_S)_{fppf}$ is representable. If $G \to S$ is finitely presented, flat or smooth, then so is $\mathcal{P} \to X$.*

Corollary 1.6.6. *Let $G \in Sm_S$ be a smooth affine group S-scheme of finite type. Let $(Sm_S)_\tau$ be the site with étale, Nisnevich or Zariski topology. For any $X \in Sm_S$ and any G-torsor $\mathcal{P} : (Sm_S)_\tau \to$ **sSet** over X, $\mathcal{P}$ is representable by an object in Sm_S.*

Proof. The corollary follows from the fact that every étale cover is fppf [Sta18, 021N]. ☐

Example 1.6.1. For any scheme S and any n, the general linear group $GL_n \to S$ is the open subscheme of $\mathbb{A}^{n^2} \to S$ defined by inverting the determinant. It follows that GL_n is a smooth affine group scheme. For any $X \in Sm_S$, there is a bijection between vector bundles of rank n and GL_n-torsors over X in the Zarsiki topology [AHW18, Ex.2.3.4]. In fact every étale locally trivial GL_n-torsor is Zariski locally trivial [MV99, Lem.3.6]. We therefore have

$$Hom_{H^s(S)}(X, BGL_n) \cong P_{Nis}(X, GL_n) \cong P_{Zar}(X, GL_n) \cong Vect_n(X).$$

1.7 Motivic spheres and Thom spaces

Let $\mathbf{Spc}_\bullet(S) = sPSh_\bullet(Sm_S)_{Nis}$ over a scheme S.

Definition 1.7.1.
1. We denote by S^1_s, the pointed constant simplicial presheaf given by the simplicial set $\Delta^1/\partial\Delta^1$, pointed by the image of $\partial\Delta^1$.
2. We denote by S^1_t, the scheme $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$ pointed by $i : pt \xrightarrow{x=1} \mathbb{G}_m$.
3. We denote by T, the quotient presheaf $\mathbb{A}^1/\mathbb{A}^1 - \{0\}$ pointed by the image of $\mathbb{A}^1 - \{0\}$.

Recall that given pointed simplicial presheaves (X, x) and (Y, y), the smash product $(X, x) \wedge (Y, y)$ is the pushout of the diagram $* \leftarrow (X, x) \vee (Y, y) \to (X, x) \times (Y, y)$. As $(X, x) \vee (Y, y) \to (X, x) \times (Y, y)$ is a monomorphism, $(X, x) \wedge (Y, y)$ is the homotopy pushout. We denote $(S^1_s)^{\wedge n}$, $(S^1_t)^{\wedge n}$ and $T^{\wedge n}$, by S^n_s, S^n_t and T^n respectively.

Definition 1.7.2 (Motivic sphere)**.** We call an object in $\mathbf{Spc}_\bullet(S)$ a *motivic sphere* if it is isomorphic to $S^n_s \wedge S^m_t$ in $H_\bullet(S)$ for some $n, m \in \mathbb{N}$. We denote by $S^{p,q}$ the motivic sphere $(S^1_s)^{p-q} \wedge (S^1_t)^q$ when $p \geq q \geq 0$.

When $p, q = 0$ the simplicial sphere S^0 is just the constant presheaf associated to the two point set $\{0, 1\}$, which is representable by $S \coprod S = S_+$.

Definition 1.7.3. Let $E \to X$ be a vector bundle over $X \in Sm_S$. We define the *motivic Thom space* to be the quotient $Th(E) = Th(E/X) = E/(E - z(X))$ where $z : X \to E$ is the zero section which is a closed embedding.

For any vector bundle E, let $\mathbb{P}(E) \to X$ be the associated projective bundle. We have the following properties of Thom spaces [MV99, Prop.2.17].

Theorem 1.7.1.
1. *Let E_1, E_2 be vector bundles on $X_1, X_2 \in Sm_S$. There is a canonical isomorphism of pointed presheaves $Th(E_1 \times E_2/X_1 \times X_2) = Th(E_1/X_1) \wedge Th(E_2/X_2)$.*

2. *Let $E \to X$ be a vector bundle and $\mathbb{P}(E) \to \mathbb{P}(E \oplus \mathcal{O})$ be the closed embedding at infinity. The canonical morphism $\mathbb{P}(E \oplus \mathcal{O}) \to Th(E)$ induces an isomorphism $\mathbb{P}(E \oplus \mathcal{O})/\mathbb{P}(E) \to Th(E)$ in $H_\bullet(S)$.*

When E is a trivial vector bundle $\mathbb{A}^n \times X$, $Th(\mathbb{A}^n \times X/X) = \mathbb{A}^n \times X/(\mathbb{A}^n - 0 \times X) \cong \mathbb{A}^n/(\mathbb{A}^n - 0) \wedge X_+$. The above theorem implies $\mathbb{A}^n \times S/(\mathbb{A}^n - 0) \times S \cong T^n$ and $T^n \cong \mathbb{P}^n/\mathbb{P}^{n-1}$ in $H_bullet(S)$.

Lemma 1.7.2. *We have the following canonical isomorphisms in $H_\bullet(S)$.*

1. $S^{2n,n} \cong S_s^n \wedge S_t^n \cong T^n \cong (\mathbb{P}^1, \infty)^{\wedge n}$.
2. $\mathbb{P}^n/\mathbb{P}^{n-1} \cong T^n$.
3. $\mathbb{A}^n - \{0\} \cong S_s^{n-1} \wedge S_t^n \cong S^{2n-1,n}$.

The Thom space construction preserves $\mathbb{A}^1$-weak equivalences for sufficiently well behaved schemes. Let S be a scheme which is ind-smooth over a Dedekind ring k with perfect residue fields.

Theorem 1.7.3. *Let $X \in Sm_S$ be a smooth S-scheme and $E \to X$ a vector bundle of constant rank. Suppose that X has a point $x : S \to X$. For any $\mathbb{A}^1$-equivalence of pointed schemes $f : (Y, y) \to (X, x)$ and any vector bundle $E \to X$ of constant rank n, the induced map of Thom spaces $Th(f^*E) \to Th(E)$ is an $\mathbb{A}^1$-equivalence.*

To prove this we need the following lemma.

Lemma 1.7.4. *Let $E \to B$ be a principal GL_n-bundle over S. Given a point $b : S \to B$, the diagram*

$$GL_n \to E \to B$$

coming from the pullback

$$\begin{array}{ccc} GL_n & \longrightarrow & E \\ \downarrow & & \downarrow \\ S & \xrightarrow{b} & B \end{array}$$

is an $\mathbb{A}^1$-local fiber sequence. Furthermore, for any scheme F with a GL_n-action $\sigma : GL_n \times F \to F$ the induced diagram

$$F \to E \times_\sigma F \to B$$

is an $\mathbb{A}^1$-local fiber sequence.

Proof. Note that for any locally trivial bundle $P \to X$ with fiber F and any point x in X the pullback diagram

$$F \to P \to X$$

is a simplicial fiber sequence (taking stalks gives a fiber sequence of simplicial sets). For any smooth S-scheme B there is a bijection

$$Vect_n(B) \cong P_{Nis}(B, GL_n) \cong P_{Nis}(B, GL_n) \cong Hom_{H^s(S)}(B, BGL_n)$$

from 1.6.1. This implies that the map $E \to B$ is a pullback of the GL_n-bundle $E_{Nis}GL_n \to B_{Nis}GL_n$ where $B_{Nis}GL_n$ is a Nisnevich fibrant replacement.

As every vector bundle (and hence every GL_n-torsor) is Zariski locally trivial, BGL_n satisfies Nisnevich descent and by 1.3.6 for any $X \in Sm_S$ we have

$$\pi_0(B_{Nis}GL_n(X)) \cong \pi_0(BGL_n(X)).$$

By [AHW18, Th.5.2.3], the set of rank n vector bundles $Vect_n(-)$ is $\mathbb{A}^1$-invariant for affine schemes over S and hence we have

$$Vect_n(X) \cong \pi_0(B_{Nis}GL_n(X)) \cong \pi_0(BGL_n(X))$$

for X affine. By [AHW18, Th.2.2.5]

$$G \to E_{Nis}G \to B_{Nis}G$$

is an $\mathbb{A}^1$-local fiber sequence hence by [Wen11, Pro.2.3]

$$GL_n \to E \to B$$

is an $\mathbb{A}^1$-local fiber sequence. For any scheme F with a GL_n-action we can show that

$$F \to E_{Nis}(GL_n) \times_\sigma F \to B_{Nis}(GL_n)$$

is an $\mathbb{A}^1$-local fiber sequence along the lines of [Wen11, Prop. 5.1]. The simplicial fiber sequence

$$F \to E \times_\sigma F \to B$$

is a pullback of the universal sequence and hence is also $\mathbb{A}^1$-local. □

Proof of theorem. Given any vector bundle $E \to X$ of rank n, the 2 out of 3 property implies that an $\mathbb{A}^1$-weak equivalence $f : Y \to X$ induces an $\mathbb{A}^1$-weak equivalence $f^*E \to E$. The complement of the zero section $E - X \to X$ is a locally trivial bundle with fiber $\mathbb{A}^n - 0$. The fiber sequence

$$\mathbb{A}^n - 0 \to E - X \to X$$

is obtained by twisting the GL_n-torsor associated to the vector bundle $E \to X$ by the standard GL_n-action on $\mathbb{A}^n - 0$ and is hence an $\mathbb{A}^1$-local fiber sequence. The pullback of $E - X$ along an $\mathbb{A}^1$-equivalence $f : Y \to X$ induces an $\mathbb{A}^1$-equivalence $f^*E - Y \to E - X$. We therefore have an equivalence of cofibration sequences

$$\begin{array}{ccccc} f^*E - Y & \longrightarrow & f^*E & \longrightarrow & Th(f^*E) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ E - X & \longrightarrow & E & \longrightarrow & Th(E) \end{array}$$

giving the desired $\mathbb{A}^1$-equivalence of Thom spaces. □

1.8 Stable homotopy theory

The standard reference for this section in [Jar00]. Let S be any scheme. The category of pointed motivic spaces $\mathbf{Spc}_\bullet(S)$ over S is a category of pointed simplicial presheaves and hence the smash product $\wedge$ gives us a symmetric monoidal category structure on $\mathbf{Spc}_\bullet(S)$.

Definition 1.8.1 (Spectra)**.** Let $P \in \mathbf{Spc}_\bullet(S)$ be a pointed motivic space. A *P-spectrum* $\mathbf{E}$ is a sequence $\mathbf{E} = \{E_i\}_{i\geq 0}$ of pointed motivic spaces together with assembly morphisms $e_i : P \wedge E_i \to E_{i+1}$. A morphism $\mathbf{E} \to \mathbf{F}$ is a collection of morphisms $E_i \to F_i$ which are compatible with the assembly morphisms. We denote the category of P-spectra by $\mathbf{Spt}_P(S)$.

Given any $X \in \mathbf{Spc}_\bullet(S)$, we can define functors $X\wedge : \mathbf{Spt}_P(S) \to \mathbf{Spt}_P(S)$ given by $(X \wedge \mathbf{E})_i = X \wedge E_i$ with assembly maps given by

$$P \wedge X \wedge E_i \xrightarrow{\sim} X \wedge P \wedge E_i \to X \wedge E_{i+1}.$$

For any $n \in \mathbb{Z}$ we also have shift functors $[n] : \mathbf{Spt}_P(S) \to \mathbf{Spt}_P(S)$ given by $\mathbf{E}[n]_i = E_{i+n}$, where we set $E_i = pt$ for all $i \leq 0$. When $n < 0$, the underlying sequence is of the form $(pt, \ldots, E_0, E_1, \ldots)$ and the assembly maps $e[n]_i : P \wedge \mathbf{E}[n]_i \to \mathbf{E}[n]_{i+1}$ are the zero maps as $X \wedge S \cong S$ for any pointed motivic space X.

Example 1.8.1 (Suspension spectrum)**.** Let $X \in \mathbf{Spc}_\bullet(S)$. We define the *suspension spectrum* $\sum_P^\infty X$ to be $(\sum_P^\infty X)_i = P^i \wedge X$ with $P \wedge P^i \wedge X \to P^{i+1} \wedge X$ the identity morphism. It follows that

$$Hom_{\mathbf{Spt}_P(S)}(\Sigma_P^\infty X, \mathbf{E}) \cong Hom_{\mathbf{Spc}_\bullet(S)}(X, E_0)$$

The suspension construction $\sum_P^\infty$ gives us a functor $\sum_P^\infty : \mathbf{Spc}(S) \to \mathbf{Spt}_P(S)$. We will construct the stable motivic homotopy category $SH(S)$ as the homotopy category associated to a model structure on $\mathbf{Spc}_T(S)$, where T is the Tate circle defined before. To do this we need the notion of compact motivic spaces introduced in [Jar00, Sec.2.2].

Definition 1.8.2. Let $X \in \mathbf{Spc}(S)$. We say X is *flasque* if,

1. $X(U)$ is a Kan complex for all $U \in Sm_S$;
2. for any $Y \in \mathbf{Spc}(S)$ and any point x in X, the canonical map

   $$\pi(Y, \Omega_x^n X) \to Hom_{H^s(S)}(Y, \Omega_x^n X) \quad n \geq 0$$

   is an bijection.

We say X is *motivic flasque* if it is flasque and $\mathbb{A}^1$-invariant.

Definition 1.8.3. We call a pointed motivic space X *compact* if it satisfies the following conditions,

1. Any filtered colimit of pointed motivic spaces $\operatorname{colim}_I Y_i$ induces an isomorphism of pointed simplicial Homs

   $$\mathbf{Hom}_*(X, \operatorname*{colim}_I Y_i) \cong \operatorname*{colim}_I \mathbf{Hom}_*(X, Y_i);$$

2. for any motivic flasque Y, $\mathbf{Hom}_*(X, \operatorname{colim}_I Y_i)$ is also flasque and $\mathbf{Hom}_*(X, -)$ preserves sectionwise weak equivalences of motivic flasque spaces.

We denote the category of compact spaces by $\mathbf{Spc}_\bullet^c(S)$.

The following lemma is [Jar00, Lem.2.2].

Lemma 1.8.1. *1. Every representable pointed presheaf is compact.*

2. Every finite pointed simplicial set is compact as a constant presheaf.

3. If $X \hookrightarrow Y$ is an inclusion of schemes, then the quotient Y/X is compact.

4. If $X_1, X_2 \in \mathbf{Spc}^c_\bullet(S)$, then $X_1 \vee X_2, X_1 \wedge X_2 \in \mathbf{Spc}^c_\bullet(S)$.

In particular every motivic sphere $S^{p,q}$ is compact. Let $P \in \mathbf{Spc}^c(S)$. For any P-spectrum $\mathbf{E}$ we have a family of functors $E^n : \mathbf{Spc}^c_\bullet(S) \to \mathbf{sSet}$, $n \in \mathbb{Z}$ given by

$$E^n(X, x) = \operatorname*{colim}_{i+n\geq 0} Hom_{H_\bullet(S)}(P^{\wedge i} \wedge (X, x), E_{i+n})$$

where the maps in the colimit are induced by the assembly maps. When $i < 0$, we define the Hom set to be trivial. Given a morphism of P-spectra $\mathbf{E} \to \mathbf{F}$ such that each $E_i \to F_i$ is an $\mathbb{A}^1$-weak equivalence, the induced map of presheaves

$$Hom_{H_\bullet(S)}(P^{\wedge i} \wedge -, E_{i+n}) \to Hom_{H_\bullet(S)}(P^{\wedge i} \wedge -, \wedge F_{i+n})$$

is an isomorphism. In particular the induced map $E^n(-) \to F^n(-)$ is an isomorphism. We now have the required machinery to define weak equivalences in $\mathbf{Spt}(S)$.

Definition 1.8.4. Let $P \in \mathbf{Spc}^c_\bullet(S)$. A morphism of P-spectra $f : \mathbf{E} \to \mathbf{F}$ is called a *stable equivalence* if the induced map of presheaves $f^n : E^n(-) \to F^n(-)$ is an isomorphism for all $n \in \mathbb{Z}$.

Every morphism $f : \mathbf{E} \to \mathbf{F}$ which is a levelwise $\mathbb{A}^1$-weak equivalence is a stable equivalence.

Lemma 1.8.2. *Let $\mathbf{E}$ be a P-spectrum and $n \geq 0$. There are stable equivalences $s_n : \mathbf{E}[n][-n] \to \mathbf{E}$ and $P^{\wedge n} \wedge \mathbf{E} \to \mathbf{E}[n]$.*

Proof. The morphism $s_n : \mathbf{E}[n][-n] \to \mathbf{E}$ is given by

$$(s_n)_i = \begin{cases} id_{E_i} & i \geq n \\ * & i < n \end{cases}$$

where $*$ is the zero map $pt \to E_i$. The induced morphism of presheaves $E[-n][n]^i(-) \to E^i(-)$ is an isomorphism as the terms in the colimit are equal for after the first n terms. The map $P^{\wedge n} \wedge \mathbf{E} \to \mathbf{E}[n]$ is given levelwise by $P^{\wedge n} \wedge E_i \to E_{i+n}$ which is constructed by composing assembly maps and by definition of $E^i(-)$ it follows that this map is a stable equivalence. □

These two stable equivalences imply that if the localization of $\mathbf{Spt}(S)$ with respect to stable equivalences (denoted by $Ho\mathbf{Spt}(S)$) exists, then $P\wedge$ induces an isomorphism on $Ho\mathbf{Spt}(S)$ with the shift functor $[-1]$ giving an inverse. To define our desired model structure

Theorem 1.8.3. *Let $P \in \mathbf{Spc}^c_\bullet(S)$. There exists a proper closed simplicial model structure on $\mathbf{Spt}_P(S)$ where weak equivalences are stable equivalences and cofibrations are morphisms $\mathbf{E} \to \mathbf{F}$ which have the left lifting property with respect to levelwise acyclic fibrations.*

This theorem follows from [Jar00, Th.2.9] after noting that our definition of stable equivalence is equivalent to the one given by Jardine after taking stable fibrant replacement. We denote the homotopy category of $\mathbf{Spt}_P(S)$ by $SH(S)_P$. From this point onward P will denote a compact motivic space. Note that $\mathbf{Spt}_P(S)$ has all homotopy limits and colimits.

Theorem 1.8.4 (Change of suspension)**.** *Given an isomorphism $P \cong Q$ in $H_\bullet(S)$ of compact motivic spaces we have an isomorphism of stable homotopy categories $SH(S)_P \cong SH(S)_Q$.*

This is given in [Jar00, Prop.2.13] when we have an $\mathbb{A}^1$-weak equivalence. Of particular interest to us will be changing the suspension to the corresponding $\mathbb{A}^1$-cone.

Lemma 1.8.5. *For any $(P,p) \in \mathbf{Spc}^c_\bullet(S)$, denote by P^+ the pushout of $\mathbb{A}^1 \xleftarrow{0} S \xrightarrow{p} P$ pointed at $1 : pt \to \mathbb{A}^1$. The projection $P^+ \to P$ is an $\mathbb{A}^1$-equivalence and for any pair of pointed schemes (X,x) and (Y,y) in $(Sm_S)_*$, giving a map $P^+ \wedge (X,x) \to (Y,y)$ is equivalent to giving a maps $f : (P,p) \times (X,x) \to (Y,y)$ and $h : \mathbb{A}^1 \times (X,x) \to (Y,y)$ such that $f_{x\times Y}$ is the zero map of pointed schemes and $h_{X\times 0} = f_{X\times y}$ and $h_{X\times 1}$ is the zero map.*

In other words giving a map $P^+ \wedge (X,x) \to (Y,y)$ is equivalent to giving a map $P^+ \times (X,x) \to (Y,y)$ which is constant up to naive $\mathbb{A}^1$-homotopy on $P \wedge (X,x)$. Clearly $P \cong P^+$ in $H_\bullet(S)$.

Theorem 1.8.6. *For any compact motivic space (X,x) and any P-spectrum $\mathbf{E}$ we have a canonical isomorphism*

$$Hom_{SH(S)_P}(\Sigma_P^\infty(X,x), \mathbf{E}) \cong \operatorname*{colim}_n Hom_{H_\bullet(S)}(P^{\wedge n} \wedge (X,x), E_n)$$

In particular we have

$$Hom_{SH(S)_P}(\Sigma_P^\infty(X,x), \Sigma_P^\infty(Y,x)) \cong \operatorname*{colim}_n Hom_{H_\bullet(S)}(P^{\wedge n}\wedge(X,x), P^{\wedge n}\wedge(Y,y)).$$

For any $(X,x) \in \mathbf{Spc}_\bullet(S)$, let $\Omega_P E$ be the pointed motivic space which is the fiber

$$\Omega_P E = \underline{Hom}_{Sm_S}(P,(X,x)) \to \underline{Hom}_{Sm_S}(P,X) \to \underline{Hom}_{Sm_S}(pt,X).$$

The loop space functor Ω_P is right adjoint to $P\wedge$. We can recursively define $\Omega_P^n = \Omega_P(\Omega_P^{n-1})$. Given a P-spectrum $\mathbf{E}$, we have the associated infinite loop space $\Omega_P^\infty \mathbf{E} = \operatorname{colim}_i \Omega_P^i E_i$.

Definition 1.8.5. Let $\mathbf{E}$ be a P-spectrum. We call $\mathbf{E}$ a Ω_P-spectrum if E_i is $\mathbb{A}^1$-fibrant for all i and the maps $E_i \to \Omega_P E_{i+1}$, which are adjoint to the assembly maps, are $\mathbb{A}^1$-weak equivalences.

For any Ω_P-spectrum $\mathbf{E}$ 1.8.6 implies the following result.

Corollary 1.8.7. *Let $\mathbf{E}$ be a Ω_P-spectrum. For any compact space (X,x) we have a canonical isomorphism*

$$Hom_{SH(S)_P}(\Sigma_P^\infty(X,x), \mathbf{E}) \cong Hom_{H_\bullet(S)}((X,x), E_0).$$

Proof. By 1.8.6 we have an isomorphism

$$Hom_{SH(S)_P}(\Sigma^\infty_P(X,x),\mathbf{E}) \cong \operatorname*{colim}_n Hom_{H_\bullet(S)}(P^{\wedge n}\wedge(X,x),E_n).$$

As E_n is $\mathbb{A}^1$-fibrant we have by 1.2.3 and 1.4.2 we have,

$$\operatorname*{colim}_n Hom_{H_\bullet(S)}(P^{\wedge n}\wedge(X,x),E_n) \cong \operatorname*{colim}_n \pi(P^{\wedge n}\wedge(X,x),E_n).$$

The simplicial homotopy class is compatible with the adjunction

$$\pi(P^{\wedge n}\wedge(X,x),E_n) \cong \pi((X,x),\Omega^n_P E_n) \cong \pi((X,x),E_0).$$

The last isomorphism follows from the fact that $\mathbf{E}$ is an Ω_P-spectrum. □

We now have enough machinery to define the stable motivic homotopy theory.

Definition 1.8.6 (Stable motivic homotopy)**.** Let $T = S^1_t\wedge S^1_s \cong \mathbb{P}^1$. The stable motivic homotopy category $SH(S)$ is defined to be $SH(S)_T$. The category of motivic spectra is defined to be $\mathbf{Spt}(S) = \mathbf{Spt}_T(S)$.

By 1.7.2 we have an isomorphism $T \cong \mathbb{P}^1 \cong \mathbb{A}^1/(\mathbb{A}^1-0)$ in $H_\bullet(S)$. Therefore we have isomorphism of homotopy categories $SH(S)_T \cong SH(S)_{\mathbb{P}^1} \cong SH(S)_{\mathbb{A}^1/(\mathbb{A}^1-0}$. Let Σ^∞ denote the suspension with respect to T. For the rest of this section we will use the model $S^1_s\wedge S^1_t$. We can also define $SH(S)$ using the category of bigraded spectra $\{E_{a,b}\}_{a,b}$ with assembly maps $S^1_s\wedge E_{a,b}\to E_{a+1,b}$ and $S^1_t\wedge E_{a,b}\to E_{a,b+1}$. This gives an equivalent definition of the stable homotopy category [DLR+07]. As $S^1_s\wedge S^1_t\wedge$ induces an isomorphism on $SH(S)$, so do $S^1_s\wedge$ and $S^1_t\wedge$. In fact $\mathbf{E}\mapsto S^1_t\wedge\mathbf{E}[-1]$ is the inverse of $S^1_s\wedge$ in the homotopy category and vice versa. We therefore make the following definition

Definition 1.8.7. For any $p,q\in\mathbb{Z}$, we define the functor $\Sigma^{p,q} : \mathbf{Spc}(S)\to\mathbf{Spc}(S)$ to be

$$\Sigma^{p,q}\mathbf{E} = \begin{cases} S^{p-q}_s\wedge S^q_t\wedge\mathbf{E} & p\ge q\ge 0\\ S^{-q}_s\wedge S^{q-p}_t\wedge\mathbf{E}[p] & p\le q\le 0\\ S^{p-2q}_s\wedge\mathbf{E}[q] & p\ge q, q\le 0\\ S^{2q-p}_t\wedge\mathbf{E}[p-q] & p\le q, 0\le q\end{cases}$$

There are several other definitions of $\Sigma^{p,q}$ all of which are equivalent as functors $SH(S)\to SH(S)$ and satisfy $\Sigma^{p,q}\Sigma^{r,s}(-)\cong\Sigma^{p+r,q+s}$. Let $\mathbb{S}$ be the suspension of the point $\Sigma^\infty S^0$(remember that $S^0 = \partial\Delta^1 = S\coprod S)$. We call $\mathbb{S}$ the *motivic sphere spectrum.* When $p\ge q\ge 0$ we have $\Sigma^{p,q}\mathbb{S}\cong\Sigma^\infty S^{p,q}$. For any $p,q\in\mathbb{Z}$, we write $S^{p,q}$ for $\Sigma^{p,q}\mathbb{S}$. For any $\mathbf{E}$, we can use the levelwise acyclic cofibration $E_i\to\Delta^1\wedge E_i$ to show that $S^1_s\wedge\mathbf{E}$ is the homotopy pushout

$$\begin{array}{ccc} \mathbf{E} & \longrightarrow & pt\\ \downarrow & & \downarrow\\ pt & \longrightarrow & S^1_s\wedge\mathbf{E}\end{array}$$

Using the simplicial circle S^1_s and the fold map $S^1_s\wedge S^1_s\to S^1_s$, we can give an additive category structure on $SH(S)$ along the lines of topological spectra.

Theorem 1.8.8. *$SH(S)$ is an additive category with the direct sum given by $(\mathbf{E}\oplus\mathbf{F})_i = E_i \vee F_i$ and assembly maps given by the isomorphism $T\wedge(E_i\vee F_i) \xrightarrow{\sim} (T\wedge E_i)\vee(T\wedge F_i)$.*

In particular $Hom_{SH(S)}(\mathbf{E},\mathbf{F})$ is an abelian group for any pair of motivic spectra. For any object $\mathbf{E}$ in $SH(S)$ we have a family of functors $E^{p,q}(-) : \mathbf{Spc}_\bullet^{op}(S) \to \mathbf{Ab}$ and $E_{p,q}(-) : \mathbf{Spc}_\bullet(S) \to \mathbf{Ab}$ given by

$$E^{p,q}(X,x) = Hom_{SH(S)}(\Sigma^\infty(X,x), S^{p,q}\wedge\mathbf{E})$$

$$E_{p,q}(X,x) = Hom_{SH(S)}(S^{p,q}, \mathbf{E}\wedge\Sigma^\infty(X,x)).$$

for all $p,q\in\mathbb{Z}$. In this notation, $E^{2n,n}$ is equal to E^n defined before.

Remark 1.8.1. Note that for $\mathbf{E} = \{E_n\}$ we have canonical maps $\Sigma^\infty E_n[-n] \to \mathbf{E}$ for all $n\geq 0$. In fact $\mathbf{E} = \operatorname{colim}_n \Sigma^\infty E_n[-n]$ and as this is a filtered colimit we get that

$$\mathbf{E} = \operatorname*{hocolim}_n \Sigma^{-2n,-n}\Sigma^\infty E_n$$

Given any morphism $f : S_1 \to S_2$ of schemes, the pullback

$$f^* : sPSh(Sm_{S_2}) \to sPSh(Sm_{S_1})$$

satisfies $f^*(U) = f^*(U)\times_{S_2} S_1$ for any S_2-scheme $U\to S_2$. In particular this is true when $U = \mathbb{A}^1_{S_2}, \mathbb{P}^1_{S_2}$. As f^* preserves colimits so we also have

$$f^*(\mathbb{A}^1_{S_2}/\mathbb{A}^1_{S_2}-0) \cong \mathbb{A}^1_{S_1}/\mathbb{A}^1_{S_1}-0$$

This implies f^* extends to the category of spectra

$$f^* : \mathbf{Spt}(S_2) \to \mathbf{Spt}(S_1).$$

Ayoub in [Ayo07, Sec.4] was able to construct the stable analogue of the left derived functor Lf^*.

Theorem 1.8.9. *Let $f : S_1 \to S_2$ be any morphism of schemes.*

1. *There exists a functor $Lf^* : SH(S_2) \to SH(S_1)$ such that for any scheme $X \in Sm_{S_1}$,*

$$Lf^*(\Sigma^{p,q}\Sigma^\infty X_+) \cong \Sigma^{p,q} f^*(\Sigma^\infty X_+) \cong \Sigma^{p,q}\Sigma^\infty f^* X_+$$

   in $SH(S_1)$.

2. *Lf^* preserves homotopy colimits.*

3. *Lf^* has a right adjoint $Rf_* : SH(S_1) \to SH(S_2)$.*

4. *When f is smooth, Lf^* has a left adjoint $Lf_\# : SH(S_1) \to SH(S_2)$.*

Finally we need the concept of a cellular spectrum.

Definition 1.8.8. For any scheme S, let $\mathbf{Spt}_{cell}(S)$ be the smallest full subcategory of $\mathbf{Spt}(S)$ such that

1. $S^{p,q} \in \mathbf{Spt}_{cell}(S)$ for all $p,q\in\mathbb{Z}$;

2. if $\mathbf{F}$ is stably equivalent to $\mathbf{E}$ for some $\mathbf{E} \in \mathbf{Spt}_{cell}(S)$ then $\mathbf{F} \in \mathbf{Spt}_{cell}(S)$;

3. For any diagram $D \to \mathbf{Spt}_{cell}(S)$, $\operatorname{hocolim} D$ is in $\mathbf{Spt}_{cell}(S)$.

As $\mathbf{Spt}_{cell}(S)$ is closed under stable equivalences, it defines a subcategory $SH(S)_{cell}$ of $SH(S)$. We call elements of $\mathbf{Spt}_{cell}(S)$ *cellular spectra.* Given a morphism $f : S_1 \to S_2$, we have the following.

Lemma 1.8.10. *For any morphism of schemes $f : S_1 \to S_2$, $Lf^* : SH(S_2) \to SH(S_1)$ restricts to a morphism of cellular objects $Lf^* : SH(S_2)_{cell} \to SH(S_1)_{cell}$.*

Proof. This follows from the fact that Lf^* preserves all $S^{p,q}$ and homotopy colimits 1.8.9. □

In the last part of this thesis we will need some properties of cellular spectra which we will state below.

Lemma 1.8.11. *If $\mathbf{E} \in \mathbf{Spt}_{cell}(S)$ then $\Sigma^{p,q}\mathbf{E} \in \mathbf{Spt}_{cell}(S)$ for any $p, q \in \mathbb{Z}$.*

This is a special case of [DI05, 3.4].

Lemma 1.8.12. *Given any cofibration sequence $\mathbf{E} \to \mathbf{F} \to \mathbf{G}$ of motivic spectra, if any two are cellular then the third is as well.*

This is [DI05, 2.5].

Chapter 2

Algebraic and Hermitian K-theory

2.1 Algebraic K-theory

We first recall some important results about algebraic K-theory. There are several books that cover these results thoroughly, see for example [Wei13] and [FG05]. For a reference more focused on rings and schemes see [Sri91]. The constructions of algebriac K-theory we are interested in are the $S^{-1}S$-construction for symmetric monoidal categories [Qui73] and the $wS_\bullet$-construction for Waldhausen categories [Wal85].

For any (small) symmetric monoidal category $(\mathcal{C}, \square, e)$, the functor $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces a map of topological spaces $B(\square) : B(\mathcal{C}) \times B(\mathcal{C}) \cong B(\mathcal{C} \times \mathcal{C}) \to B(\mathcal{C})$. The natural isomorphisms induced by the identity object $e \square c \cong c \cong c \square e$ imply that the map $B(\square)(e, -)$ is homotopic to the identity map. Therefore, $(B\mathcal{C}, B(\square))$ is an H-space. In [Gra76] Quillen gave a construction of a monoidal category $\mathcal{C}^{-1}\mathcal{C}$ with the property that whenever every morphism in $\mathcal{C}$ is an isomorphism (i.e, when $\mathcal{C}$ is a groupoid) $B\mathcal{C}^{-1}\mathcal{C}$ is the *group completion* of the H-space $B\mathcal{C}$.

Definition 2.1.1. Let $(\mathcal{C}, \square, e)$ be a (small) symmetric monoidal category. We define $\mathcal{C}^{-1}\mathcal{C}$ to be the category with objects pairs (c, d) of objects of $\mathcal{C}$ and morphisms between two objects (c, d) and (c', d') are given by equivalence classes of pairs $(s \square c \to c', s \square d \to d')$ where s is any object in $\mathcal{C}$. Two such pairs, $(s \square c \to c', s \square d \to d')$ and $(s' \square c \to c', s' \square d \to d')$ are equivalent if there is an isomorphism $\phi : s \xrightarrow{\sim} s'$ such that the relevant triangles commute.

$\mathcal{C}^{-1}\mathcal{C}$ is a symmetric monoidal category with $(c, d) \square (c', d') = (c \square c', d \square d')$ and the functor $\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$ sending c to (c, e) is monoidal.

Theorem 2.1.1. *Let $(\mathcal{C}, \square, e)$ be a symmetric monoidal groupoid. The H-space $B\mathcal{C}^{-1}\mathcal{C}$ is a group completion of $B\mathcal{C}$ under the induced map $B\mathcal{C} \to B\mathcal{C}^{-1}\mathcal{C}$.*

This is a special case of the theorem in [Gra76, pg. 5].

Definition 2.1.2. Let $(\mathcal{C}, \square, e)$ be a (small) symmetric monoidal groupoid. The algebraic K-theory space of $\mathcal{C}$ is defined to be $K^\square(\mathcal{C}) = B\mathcal{C}^{-1}\mathcal{C}$. The K-groups are defined to be the homotopy groups $K_i^\square(\mathcal{C}) = \pi_i(K^\square(\mathcal{C}))$.

Theorem 2.1.2 ([Wei13, Th.4.4.3]). *Let $g : X \to Y$ be an H-space group completion such that $\pi_0(X)$ is either countable or contains a countable cofinal submonoid. Then for any other group completion $f : X \to Z$ there exists a homotopy equivalence $h : Y \to Z$ such that hg and f induce equal maps on homotopy groups $\pi_i(X, x_0) \to \pi_i(Y, y_0)$, $i \geq 0$.*

Remark 2.1.1. In most of the cases we consider the above condition will be satisfied and therefore the algebraic K-theory space can be taken to mean *any* group completion of $B\mathcal{C}$.

Example 2.1.1.

1. Let $\mathbf{P}(R)$ be the category of finitely generated projective modules over R and $i\mathbf{P}(R)$ the associated groupoid of isomorphisms. The direct sum $\oplus$ gives a symmetric monoidal structure on $\mathbf{P}(R)$. As $\pi_0(i\mathbf{P}(R))$ is the set of isomorphism classes of finitely generated projective modules, $\pi_0 K^{\oplus}(i\mathbf{P}(R)))$ is isomorphic to $K_0(R)$, Grothendieck's original K-group of a ring.

2. Let $\mathbf{F}(R)$ be the full subcategory of $\mathbf{P}(R)$ generated by free modules. $\mathbf{F}(R)$ is closed under $\oplus$ and for any object $P \in \mathbf{P}(R)$, there exists another object $Q \in \mathbf{P}(R)$ such that $P \oplus Q \cong R^{\oplus n}$. This property is called *cofinality.* Further, $i\mathbf{F}(R)$ is equivalent to the disjoint union of categories $\coprod_n GL_n(R)$ and therefore $K_0^{\oplus}(i\mathbf{F}(R)) \cong \mathbb{Z}$. As $B(\coprod_n GL_n(R)) \cong \coprod_n BGL_n(R)$ is a topological monoid, the loop space $\Omega B(\coprod_n BGL_n(R))$ is a model for the group completion of $\coprod_n BGL_n(R)$ (cf.[Ada78]) and therefore $K^{\oplus}(\mathbf{F}(R)) \cong \Omega B(\coprod_n BGL_n(R))$.

Definition 2.1.3. The algebraic K-theory space of a ring R, denoted by $K(R)$, is the algebraic K-theory space $K^{\oplus}(i\mathbf{P}(R))$. The higher K-groups of R are $K_n(R) = \pi_n K(R)$.

Theorem 2.1.3 (Cofinality theorem [Qui73]). *Let $f : S \to T$ be a cofinal monoidal functor and $K(f)$ the induced map of topological spaces $K(S) \to K(T)$. Then, $\pi_n K(f) : K_n(S) \xrightarrow{\sim} K_n(T)$ for all $n \geq 1$.*

Applied to the functor $\mathbf{F}(R) \to \mathbf{P}(R)$, we get $K_n(i\mathbf{F}(R)) \xrightarrow{\sim} K_n(R)$ for $n \geq 1$. The $S^{-1}S$ construction can be extended to an arbitrary scheme S by replacing $\mathbf{P}(R)$ with the category of vector bundles $Vect(S)$. The $S^{-1}S$ construction however has major drawbacks. $K^{\oplus}(Vect(S))$ does not have good geometric properties, for example it does not satisfy Zariski excision or extend in a natural way to relative K-groups and in the non-affine case, the $S^{-1}S$-construction does not capture the exact category structure (cf.[Qui73]) of $Vect(S)$. These problems can be solved using Waldhausen's $wS_\bullet$-construction first introduced in [Wal85].

Definition 2.1.4 (Waldhausen categories). A Waldhausen category is a tuple $(\mathcal{C}, 0, cof, w)$, where $\mathcal{C}$ is any category, 0 is a choice of a zero object in $\mathcal{C}$, cof and w are subcategories of $\mathcal{C}$ such that

1. Every isomorphism is in both cof and w, in particular every object is in both cof and w.

2. The unique map $0 \to A$ belongs to cof for every object A in $\mathcal{C}$.

3. For any morphism $A \rightarrowtail B$ in cof and any morphism $A \to C$, the pushout $C \to B \cup_A C$ exists and is in cof. We denote the pushout of $A \rightarrowtail B$ along $A \to 0$ by the quotient B/A.

4. Denoting morphisms in w by $\xrightarrow{\sim}$, for every commutative diagram of the form

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array}$$

the induced morphism of pushouts $C \cup_A B \to C' \cup'_A B'$ is in w.

Morphisms in cof and w are called cofibrations and weak equivalences respectively. Most Waldhausen categories we consider will also satisfy the *2 out of 3* property, that is, given any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow \quad \nearrow & \\ & C & \end{array}$$

such that, if two of the three morphisms are weak equivalences, then the third is also a weak equivalence. Applying axiom 3 to the diagram

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \\ 0 & & \end{array}$$

we see that the pushout B/A exists. We will denote the canonical map $B \to B/A$ by $\twoheadrightarrow$ and call the sequences of the form $A \rightarrowtail B \twoheadrightarrow B/A$ *cofibration sequences.* It follows that the unique map to the zero object always fits into a cofibration sequence $A \xrightarrow{=} A \twoheadrightarrow 0$. Given any pair of elements A and B in $\mathcal{C}$, they fit into a cofibration sequence $A \rightarrowtail A \coprod B \twoheadrightarrow B$ given by the pushout diagram

$$\begin{array}{ccc} 0 & \rightarrowtail & A \\ \downarrow & & \downarrow \\ B & \rightarrowtail & A \coprod B. \end{array} \quad .$$

An *exact functor* $F : (\mathcal{C}, 0, cof, w) \to (\mathcal{D}, 0, cof, w)$ between Waldhausen categories is a functor $F : \mathcal{C} \to \mathcal{D}$ which preserves cofibrations, weak equivalences, zero objects and pushouts along cofibrations. In particular, F must preserve cofibration sequences. We denote by **Wald** the category of small Waldhausen categories and exact functors.

Example 2.1.2. 1. For any scheme S, the category of vector bundles $Vect(S)$ is a Waldhausen category with weak equivalences being isomorphisms and cofibrations being a monomorphisms $A \rightarrowtail B$ of $\mathcal{O}_S-$modules such that B/A is also a vector bundle. More generally, every exact category $\mathcal{E}$ has a

Waldhausen category structure with isomorphisms being the weak equivalences and admissible monomorphisms being the cofibrations. Cofibration sequences in $\mathcal{E}$ are exactly the exact sequences. A functor between exact categories is exact in the sense of Waldhausen categories if and only if it is an exact functor in the sense of exact categories.

2. For any exact category $\mathcal{E}$, let $Ch(\mathcal{E})$ be the corresponding category of chain complexes with q and cof the subcategories of quasi-isomorphisms and levelwise admissible monomorphisms respectively. There is a Waldhausen category structure on $Ch(\mathcal{E})$ with q and cof the categories of weak equivalences and cofibrations respectively. The category of bounded chain complexes $Ch^b(\mathcal{E})$ is a Waldhausen subcategory of $Ch(\mathcal{E})$.

The above examples can be generalised.

Definition 2.1.5 (Exact category with weak equivalences)**.** An *exact category with weak equivalences* is a pair $(\mathcal{E}, w)$ where $\mathcal{E}$ is an exact category and w is a subcategory of $\mathcal{E}$, called the category of weak equivalences, such that $(\mathcal{E}, 0, mono, w)$ and $(\mathcal{E}^{op}, 0, epi^{op}, w^{op})$ are both Waldhausen categories. Here *mono* and *epi* are the subcategories of admissible monomorphisms and admissible epimorphisms in $\mathcal{E}$ respectively.

$(Ch(\mathcal{E}), q)$ and $(Ch^b(\mathcal{E}), q)$ are both exact categories with weak equivalences. All the Waldhausen categories we will be interested in are exact categories with weak equivalences.

Definition 2.1.6 ($wS_n(-)$)**.** Let $(\mathcal{C}, 0, cof, w)$ be a Waldhausen category. $wS_n\mathcal{C}$ is the category whose objects are sequences of cofibrations

$$0 = A_{0,0} \rightarrowtail A_{1,0} \rightarrowtail \ldots \rightarrowtail A_{n,0}$$

along with choices of quotients $A_{ij} = A_{i,0}/A_{j,0}$. These choices are compatible in the sense that all the composite maps $A_{ij} \to A_{kl}$ ($i \leq k$ and $j \leq l$), for fixed i, j, k, l, are equal. We use $A..$ to denote the objects of $wS_n\mathcal{C}$. A morphism $A.. \to B..$ in $wS_n\mathcal{C}$, is a collection of weak equivalences $A_{ij} \xrightarrow{\sim} B_{ij}$ which commute with the structure maps.

These $wS_n(\mathcal{C})$ together form a simplicial category.

Definition 2.1.7. For $0 < n$ and $0 \leq i \leq n$, we define $d_i : wS_n(\mathcal{C}) \to wS_{n-1}(\mathcal{C})$ to be the functor which omits the $A_{i.}$ row and $s_i : wS_{n-1}(\mathcal{C}) \to wS_n(\mathcal{C})$ the functor which duplicates the $A_{i.}$ row. $wS_n(\mathcal{C})$ together form a simplicial category $wS_\bullet(\mathcal{C})$, with face and degeneracy maps given by d_i and s_i respectively.

Definition 2.1.8. Let $(\mathcal{C}, 0, cof, w)$ be a Waldhausen category. Its algebraic K-theory space $K(\mathcal{C}, w)$ is the loop space $\Omega|wS_\bullet(\mathcal{C})|$, where $|wS_\bullet(\mathcal{C})|$ is the geometric realisation of the bisimplicial set $([n], [m]) \mapsto N(wS_n(\mathcal{C}))_m$.

An exact functor of Waldhausen categories $F : (\mathcal{C}, 0, cof, w) \to (\mathcal{D}, 0, cof, w)$ induces a map of simplicial sets $K(F) : K(\mathcal{C}, w) \to K(\mathcal{D}, w)$. Whenever F is an equivalence of categories, $K(F)$ is a weak equivalence.

Theorem 2.1.4. *Let $(\mathcal{C}, 0, cof, w)$ be a Waldhausen category. $K_0(\mathcal{C}, w)$ has a presentation with generators $[C]$ for each object C in $\mathcal{C}$ and relations*

1. $[C] = [C']$ *whenever there is a weak equivalence* $C \xrightarrow{\sim} C'$,

2. $[B] = [A] + [B/A]$ *for each cofibration sequence* $A \rightarrowtail B \twoheadrightarrow B/A$.

Proof. As $|wS_\bullet(\mathcal{C})|_0 = N(wS_0(\mathcal{C}))_0$ is a point, the fundamental group $\pi_1(|wS_\bullet(\mathcal{C})|)$ is generated by the elements of $|wS_\bullet(\mathcal{C})|_1$ modulo the relations $d_1(x) = d_0(x)d_2(x)$ for each $x \in |wS_\bullet(\mathcal{C})|_2$. $N(wS_1(\mathcal{C}))_1$ is the set of weak equivalences $C \xrightarrow{\sim} C'$(we suppress the 0s). We will denote by $[C]$ the identity element $C \xrightarrow{=} C$. Elements of $|wS_\bullet(\mathcal{C})|_2$ are diagrams

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ A'' & \rightarrowtail & B'' & \twoheadrightarrow & B''/A''. \end{array}$$

If we denote this diagram by D then $d_0(D) = B/A \xrightarrow{\sim} B''/A''$, $d_1(D) = B' \xrightarrow{\sim} B''$ and $d_2(D) = A \xrightarrow{\sim} A'$. Taking the case where all the weak equivalences are identities we get $[A] = [B][B/A]$ for each cofibration sequence $A \rightarrowtail B \twoheadrightarrow B/A$. For any pair of objects C and C' in $\mathcal{C}$, we have cofibration sequences $C \rightarrowtail C \coprod C' \twoheadrightarrow C'$ and $C \rightarrowtail C' \coprod C' \twoheadrightarrow C$. This gives us $[C][C'] = [C \coprod C'] = [C'][C]$. Hence the group is abelian and we denote the group action by $+$. From the diagram

$$\begin{array}{ccccc} C & \xrightarrow{=} & C & \twoheadrightarrow & 0 \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow \\ C' & \xrightarrow{=} & C' & \twoheadrightarrow & 0 \\ \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle=} & & \downarrow \\ C' & \xrightarrow{=} & C' & \twoheadrightarrow & 0 \end{array}$$

we get $[C'] = [C] + [0] = [C]$. We can see that all other diagrams can be built from these and hence we have all the desired relations. □

Remark 2.1.2. The loop space $\Omega|wS_\bullet(\mathcal{C})|$ is in general only defined as a topological space. However, for a pointed Kan complex $(K, *)$, the geometric realization of simplicial loop space $Hom_{\mathbf{sSet}_*}(\Delta^1/\partial\Delta^1, (K, *))$ is weakly equivalent to the topological loop space. As dealing with simplicial sets is more convenient we replace $\Omega|wS_\bullet(\mathcal{C})|$ with the weakly equivalent simplicial loop space,

$$\Omega_s|wS_\bullet(\mathcal{C})| = Hom_{\mathbf{sSet}_*}((\Delta^1/\partial\Delta^1, *), (Ex^\infty|wS_\bullet(\mathcal{C})|, 0)).$$

where 0 is the unique element in $wS_0(\mathcal{C})$ and Ex^∞ is the Kan fibrant replacement functor [GJ09, III.4].

Fixing a model for loop spaces as above, $K(-)$ gives a functor from the category of small Waldhausen categories to simplicial sets

$$K(-) : \mathbf{Wald} \to \mathbf{sSet}.$$

The product of finitely many Waldhausen categories $\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \mathcal{C}_r$ is again a Waldhausen category with $K(\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \mathcal{C}_n) \cong K(\mathcal{C}_1) \times K(\mathcal{C}_2) \times \ldots K(\mathcal{C}_n)$. For any scheme S, we denote $K(Vect(S), iso)$ (given in 2.1.2) by $K(S)$ and when $S \cong Spec(R)$, we denote $K(S)$ by $K(R)$. $K_0(S)$ is then the classical K-group of vector bundles introduced by Grothendieck [Wei13, Ch.2]. This construction agrees with $K^{\oplus}(-)$ on affine schemes.

Theorem 2.1.5. *For any commutative ring R, there is a zigzag of weak equivalences between simplicial sets $K^{\oplus}(R)$ and $K(R)$.*

Waldhausen in his original paper shows that for an exact category $\mathcal{E}$, $|iS_\bullet\mathcal{E}|$ is weakly equivalent to Quillen's Q space $Q\mathcal{E}$ [Wal85, Sec.1.9]. The loop space $\Omega Q\mathbf{P}(R)$ is weakly equivalent to $K^{\oplus}(R)$ [Qui73].

Definition 2.1.9 (Category of big vector bundles)**.** Fix a small category of schemes $\mathcal{V}$. Given a scheme $X \in Ob(\mathcal{V})$, a *big vector bundle* over X is a choice of vector bundles E_Y over Y, for each $Y \to X$ in $\mathcal{V}$, with fixed isomorphisms $f^*E_Y \xrightarrow{\sim} E_Z$ for each morphism $f : Z \to Y$ over X satisfying the following conditions.

1. For each identity map $id_Y : Y \to Y$, the associated $id_Y^* E_Y \to E_Y$ is the identity map.
2. For each pair of composable maps, $W \xrightarrow{f} Z \xrightarrow{g} Y$ over X, the composition $f^*g^*E_Y \to f^*E_Z \to E_W$ is equal to the map associated to $g \circ f : W \to Y$.

The category of big vector bundles over X, $Vect_{\mathcal{V}}(X)$ (with morphisms completely described by morphisms between vector bundles over X), is an exact category and the forgetful functor $Vect_{\mathcal{V}}(X) \to Vect(X)$ is an equivalence of Waldhausen categories inducing a weak equivalence $K(Vect(X)) \cong K(Vect_{\mathcal{V}}(X))$. Objects in $Vect_{\mathcal{V}}(X)$ have fixed choices for pullbacks f^*E and hence given any morphism of schemes $f : X \to Y$ in $\mathcal{V}$, taking pullbacks of vector bundles induces an exact functor $f^* : Vect_{\mathcal{V}}(Y) \to Vect_{\mathcal{V}}(X)$. The space $K(S)$ captures the exact category structure of $Vect_{Sm_S}(S)$ and satisfies the desired geometric properties. The following two theorems are due to Thomason and Trobaugh [TT90] and Quillen [Qui73] respectively.

Theorem 2.1.6 (Nisnevich Excision)**.** *Let S be a regular Noetherian scheme of finite dimension and Sm_S the category of smooth S-schemes of finite type. Given any Nisnevich square in Sm_S,*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

The corresponding square of K-theory spaces

$$\begin{array}{ccc} K(X) & \longrightarrow & K(V) \\ \downarrow & & \downarrow \\ K(U) & \longrightarrow & K(U \times_X V) \end{array}$$

is homotopy cartesian.

Theorem 2.1.7. *($\mathbb{A}^1$-invariance) For any $X \in Sm_S$, the map of spaces $K(X) \to K(X \times \mathbb{A}^1)$, induced by the projection $X \times \mathbb{A}^1 \to X$, is a weak equivalence.*

The proofs of these theorems are in [Sri91, Ch.5]. It follows from 1.4.7 that the K-theory presheaf $K(-) : Sm_S^{op} \to \mathbf{sSet}$ is representable in the unstable homotopy category $H_\bullet(S)$.

Theorem 2.1.8. *For all $X \in Sm_S$ and all $n \geq 0$, we have natural isomorphisms*

$$\mathrm{Hom}_{H_\bullet(S)}(S^n \wedge X_+, K) \cong K_n(X)$$

For a pointed S-scheme (X, x), we denote by $K_i(X, x)$ the kernel of the map $K_i(X) \to K_i(pt)$ ($pt = S$) and for a pair of pointed S-schemes (X, x) and (Y, y), we denote by $K_i((X, x) \wedge (Y, y))$ the kernel of the map $K_i(X \times Y) \to K_i(X) \oplus K_i(Y)$ induced by the inclusions $X \times \{y\} \subset X \times Y$ and $\{x\} \times Y \subset X \times Y$. There is a more topological definition of these groups using homotopy fibers.

Definition 2.1.10. Let $f : Y \to X$ be a monomorphism of S-schemes. The *relative K-theory space* $K(X, Y)$ is the homotopy fiber of the map $K(X) \to K(Y)$ and relative K-groups are defined to be the homotopy groups $K_n(X, Y) = \pi_n K(X, Y)$.

This is compatible with the definitions of $K_n(X, x)$ and $K_n((X, x) \wedge (Y, y))$ as the underlying maps have left inverses. They are also consistent with the definition of $(X, x) \wedge (Y, y) \cong X \times Y/(X, x) \vee (Y, y) \in \mathbf{Spc}$. More precisely,

Corollary 2.1.9. *For any pair of pointed schemes (X, x) and (Y, y), there are natural isomorphisms*

$$Hom_{H_\bullet(S)}(S^n \wedge (X, x), K) \cong K_n(X)$$

$$Hom_{H_\bullet(S)}(S^n \wedge (X, x) \wedge (Y, y), K) \cong K_n((X, x) \wedge (Y, y))$$

for all $n \geq 0$.

Let E be a rank $n + 1$ vector bundle over $X \in Sm_S$. Tensoring with line bundles $\mathcal{O}(-i)$, $0 \leq i \leq n$, over the projective bundle $\mathbb{P}(E)$ induces functors $u_i : Vect_{Sm_S}(X) \to Vect_{Sm_S}(\mathbb{P}(E))$ given by $E \mapsto \pi^* E \otimes \mathcal{O}(-i)$.

Theorem 2.1.10 (Projective bundle theorem)**.** *Let $\mathbb{P}(E)$ be a projective bundle over $X \in Sm_S$. The u_i above induce a weak equivalence $K(X)^{n+1} \cong K(\mathbb{P}(E))$.*

A proof of this result can be found in [TT90, 4.1]. Fixing a vector bundle on the base scheme $E \to S$, the u_i extend to a natural transformation $u_i : Vect_{Sm_S}(-) \to Vect_{Sm_S}(\mathbb{P}(V) \times_S -)$. The theorem then implies that there is a natural isomorphism $K(-)^{n+1} \cong K(\mathbb{P}(E) \times -)$ of simplicial presheaves. For the rest of this section we will suppress the subscript Sm_S in $Vect_{Sm_S}(X)$ for simplicity.

Corollary 2.1.11. *Let $\mathbb{P}^1$ be the projective line over the base scheme S. For every $X \in Sm_S$, we have an isomorphism $K_n(X, x) \cong K_n((\mathbb{P}^1, \infty) \wedge (X, x))$ for every point $x : S \to X$ and for all n.*

Proof. Applying 2.1.10 to $E = \mathcal{O}^{\oplus 2}$, we get isomorphisms $u_0 + u_1 : K_n(X) \oplus K_n(X) \cong K_n(\mathbb{P}^1 \times X)$. For any point $x : S \to X$, the desired isomorphism will be obtained by constructing a commutative diagram of exact categories,

$$\begin{array}{ccc} Vect(X) \times Vect(X) & \longrightarrow & Vect(X) \times Vect(S) \\ \downarrow & & \downarrow \\ Vect(\mathbb{P}^1 \times X) & \longrightarrow & Vect(X) \times Vect(\mathbb{P}^1) \end{array}$$

The functor on the left induces the isomorphism of K-groups. The top functor is induced by the pullback x^*. The bottom functor is induced by the inclusion $X \times \{\infty\} \cup \{x\} \times \mathbb{P}^1 \hookrightarrow X \times \mathbb{P}^1$. The functor $Vect(X) \times Vect(S) \to Vect(X)$ on the right sends (V, W) to $V \oplus p_X^* W$, where $p_X : X \to S$ is the structure map. The functor $Vect(X) \times Vect(S) \to Vect(\mathbb{P}^1)$ sends (V, W) to the direct sum of V and W pulled back along $\{x\} \times \mathbb{P}^1 \to \mathbb{P}^1 \times X \to X$ and $\mathbb{P}^1 \to S$ respectively. It is easily checked that the diagram commutes gives us a commutative square of K groups,

$$\begin{array}{ccc} K_n(X) \oplus K_n(X) & \longrightarrow & K_n(X) \oplus K_n(S) \\ \downarrow{\sim} & & \downarrow \\ K_n(\mathbb{P}^1 \times X) & \longrightarrow & K_n(X) \oplus K_n(\mathbb{P}^1) \end{array}$$

inducing a map between the kernels $K_n(X, x) \to K_n((\mathbb{P}^1, \infty) \wedge (X, x))$. We have a map $K_n((\mathbb{P}^1, \infty) \wedge (X, x)) \to K_n(X) \to K_n(X)$ induced by the isomorphism. Projecting onto the first component $K_n((\mathbb{P}^1, \infty) \wedge (X, x)) \to K_n(X)$ is zero as it is the same map as the bottom map onto $K_n(X)$. Composing with the right map we get $K_n((\mathbb{P}^1, \infty) \wedge (X, x)) \to K_n(S)$ which is zero after composing with the map $K_n(S) \to K_n(\mathbb{P}^1)$. Applying 2.1.10 to $\mathbb{P}^1 \to S$, the map $K_n(S) \to K_n(\mathbb{P}^1)$ is a monomorphism and hence $K_n((\mathbb{P}^1, \infty) \wedge (X, x)) \to K_n(S)$ is zero. Hence we get the inverse map to the kernel $K_n(X, x)$. □

Theorem 2.1.12. *Let the K-theory presheaf K be pointed by the constant* 0 *vector bundle giving* $(K, 0)$. *For any scheme S, there exists an isomorphism* $(\mathbb{P}^1, \infty) \wedge (K, 0) \xrightarrow{\sim} (K, 0)$ *in* $H_\bullet(S)$ *giving us a* $(2,1)$*-periodic* $\mathbb{P}^1$*-spectrum* $\mathbf{K} = (K, K, \ldots) \in \mathbf{Spt}_{\mathbb{P}^1}$.

Proof. By the PBT, we have a simplicial weak equivalence of presheaves $K(-) \times K(-) \xrightarrow{\sim} K(\mathbb{P}^1 \times -)$ by the above discussion. Giving a map of pointed presheaves $(\mathbb{P}^1, \infty) \wedge (K, 0) \to (K, 0)$ is equivalent to giving a map $(K, 0) \to \Omega_{\mathbb{P}^1} K$. The loop space $\Omega_{\mathbb{P}^1} K$ is the homotopy fiber of the map $\mathrm{Hom}(\mathbb{P}^1, K)(-) \xrightarrow{\infty} \mathrm{Hom}(S, K)(-)$. For any scheme X, $\mathrm{Hom}(X, K)(-) := K(X \times_S -)$ and hence $\Omega_{\mathbb{P}^1} K$ is the homotopy fiber of the map $K(\mathbb{P}^1 \times -) \to K(-)$. We then have a commutative square

$$\begin{array}{ccc} K(-) \times K(-) & \xrightarrow{\sim} & K(\mathbb{P}^1 \times -) \\ \downarrow{+} & & \downarrow \\ K(-) & \xrightarrow{=} & K(-) \end{array}$$

where $+$ is induced by $(A, B) \mapsto A \oplus B$. Let $s \in K(S)$ such that $[s] = -[\mathcal{O}]$ in $K_0(S)$. There is a map $K(-) \times K(-) \to K(-) \times K(-)$ given by $(x, y) \mapsto (x, y + s \times x)$ where $\times$ is the map induced by $\otimes : Vect_{Sm_S} \times Vect_{Sm_S} \to Vect_{Sm_S}$. This map is a homotopy inverse of the map $(x, y) \mapsto (x, x + y)$ and hence is a homotopy equivalence. The composite map is homotopic to the projection $(x, y) \mapsto y$ and hence the homotopy fiber is equivalent to $K(X)$ for each $X \to S$. This gives us a levelwise weak equivalence $K(-) \xrightarrow{\sim} \Omega_{\mathbb{P}^1,\infty} K(-)$. $\square$

This theorem is the motivic analog of Bott periodicity for topological K-theory. By 1.8.6 this gives us the representability of K-theory in the stable category first shown in [Voe98]

$$Hom_{SH(S)}(\Sigma^\infty X_+, \mathbf{K}) \cong Hom_{H_\bullet(S)}(X_+, K) \cong K_n(X).$$

2.2 Bilinear forms

In this section we will discuss the properties of bilinear forms. The main reference for this section will be [Lam06]. Suppose we have a (commutative) ring R and an R-module M. A bilinear form on M is an R-module homomorphism $\phi : M \otimes M \to R$. By adjointness, this is equivalent to giving a homomorphism $M \to M^\vee = \mathrm{Hom}(M, R)$, where ϕ gives us $\hat{\phi} : M \to \mathrm{Hom}(M, R)$ defined as $\hat{\phi}(x) := \phi(- \otimes x)$. When $M \cong R^{\oplus n}$, every bilinear form on M can be uniquely determined (for a fixed choice of basis) by an $n \times n$ matrix A_ϕ, where $\phi(x \otimes y) = x^T A_\phi y$. We will describe some important classes of bilinear forms below.

Definition 2.2.1. Let M be an R-module and $\sigma : M \otimes M \xrightarrow{\sim} M \otimes M$ be the switch map $\sigma(m \otimes n) = n \otimes m$.

1. A *symmetric* form ϕ on M is a bilinear form such that $\phi \circ \sigma = \phi$.
2. A *skew-symmetric* form ψ on M is a bilinear form such that $\psi \circ \sigma = -\psi$ (equivalently $\psi \circ \sigma + \psi = 0$).
3. An *alternating* form ψ on M is a bilinear form such that $\psi(m \otimes m) = 0$ for all $m \in M$.

An *isometry* between two R-modules with bilinear forms, (M, ϕ) and (N, ψ), is an R-module isomorphism $f : M \xrightarrow{\sim} N$ such that $f^*\psi = \phi$, where $f^*\psi(x \otimes y) = \psi(f(x) \otimes f(y))$.

Remark 2.2.1. Every alternating form is skew-symmetric as $\psi((x+y) \otimes (x+y)) = 0$ implies $\psi(x \otimes x) + \psi(x \otimes y) + \psi(y \otimes x) + \psi(y \otimes y) = 0$ and $\psi(x \otimes x) = \psi(y \otimes y) = 0$ gives us $\psi(x \otimes y) + \psi(y \otimes x) = 0$. But, a skew-symmetric form need not be alternating as, if $2 = 0$ in R then the symmetric form $R \otimes R \to R$ given by multiplication is also skew-symmetric (as $1 = -1$), but not alternating. However, when 2 is a non-zero divisor in R these two properties are equivalent as we have $\phi(x \otimes x) + \phi(x \otimes x) = 0$ for skew symmetric forms giving $2\phi(x \otimes x) = 0$ and hence $\phi(x \otimes x) = 0$. In particular every skew-symmetric form on $\mathbb{Z}$ is an alternating form.

From here on, we will denote $\phi(x \otimes y)$ by $\phi(x, y)$. When P is a finitely generated projective R-module, the natural transformation $\eta_P : P \to P^{\vee\vee}$ given by, $\eta_P(p)(f) = f(p)$ is an isomorphism.

Definition 2.2.2. A bilinear form ϕ on a projective module P is called *non-degenerate*, if the adjoint map $\hat{\phi} : P \to \mathrm{Hom}(P, R)$ is an isomorphism. Let (P, ϕ) be a finitely generated projective module with a non-degenerate bilinear form.

1. We call (P, ϕ) a *symmetric space* when ϕ is a non-degenerate symmetric form on P.

2. We call (P, ϕ) a *skew-symmetric space* when ϕ is a non-degenerate skew-symmetric form on P.

3. We call (P, ϕ) a *symplectic space* when ϕ is a non-degenerate alternating form on P.

Example 2.2.1.
1. For any ring R and any $n \in \mathbb{N}$, bilinear forms on $R^{\oplus n}$ are in one to one correspondence with square matrices of size n. Every square matrix $A \in M_{n\times n}(R)$ defines a bilinear form on $R^{\oplus n}$ by $(\vec{x}, \vec{y}) \mapsto \vec{x}^T A\vec{y}$. This bilinear form is symmetric or skew-symmetric if and only if the corresponding matrix is. The form is alternating if and only if the corresponding matrix is skew-symmetric with strictly zeros on the diagonal.

2. *Hyperbolic forms:* Given any projective module P, $P \oplus P^\vee$ has natural symmetric and symplectic space structures, $H_+(P) = (P \oplus P^\vee, \begin{pmatrix} 0 & 1 \\ \eta_P & 0 \end{pmatrix})$ and $H_-(P) = (P \oplus P^\vee, \begin{pmatrix} 0 & 1 \\ -\eta_P & 0 \end{pmatrix})$.

Definition 2.2.3. (Orthogonal sum) Given two modules with bilinear forms (M, ϕ) and (N, ψ) we can define a bilinear form (denote it by $\phi\perp\psi$) on $M \oplus N$ where $\phi\perp\psi((m+n)\otimes(m'+n')) = \phi(m \otimes m') + \psi(n \otimes n')$, called the orthogonal sum. We also use $M\perp N$ when the actual forms are understood.

Let $N \subset P$ be a submodule of a symmetric or skew-symmetric space (P, ψ). Define $N^\perp = \{p \in P |\, \psi(n \otimes p) = 0, \forall n \in N\}$ to be the orthogonal complement of N.

Theorem 2.2.1. *Suppose we have a symmetric or skew-symmetric space (P, ψ) and a submodule N such that ψ restricted to N is non-degenerate. Then, $\psi_{|N^\perp}$ is also non-degenerate, $N \cap N^\perp = \{0\}$ and $N \perp N^\perp = P$. In particular both N and $N^\perp$ are symmetric (resp. skew-symmetric) spaces.*

This is proved in [Lam06, Lem1.3], where symmetric spaces are called inner product spaces. The above result is very useful for decomposing symplectic and symmetric spaces into orthogonal sums. Let H_- be the symplectic space $(R^{\oplus 2}, (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$.

Theorem 2.2.2. *Let (P, ψ) be a symplectic space over a ring R. There is a decomposition $(P, \psi) \cong H_-^{\oplus n} \perp Q$ such that $\psi(a, b) \in R - R^\times$, for all $a, b \in Q$.*

Proof. Suppose $\psi(a,b) \in R - R^\times$ for all $a, b \in P$. The statement then holds with $P = Q$. Otherwise, there exist $a, b \in P$ such that $\psi(a,b) = u \in R^\times$. Replacing b with $v \cdot b$, where $vu = 1$, we can assume $\psi(a,b) = 1$. As ψ is symplectic, $a \neq b$, $\psi(b,a) = -1$ and a, b are linearly independent. Therefore, the submodule N of P generated by a, b is free of rank 2, and ψ restricted to N is isometric to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This gives us $P \cong H_- \perp N^\perp$. We can apply the arguments again for $N^\perp$, and the result follows by induction on the rank of P. □

From the above proof, it is easy to derive the following corollary.

Corollary 2.2.3. *For any ring R, any symplectic space with underlying module $R^{\oplus 2}$ is isometric to H_-. Further, if every projective module over R is free, then every symplectic space is isometric to $H_-^{\oplus n}$ for some n.*

In particular, this applies to local rings. For any ring R and any symplectic space (P, ϕ), given an element $f \in R$ let us denote by (P_f, ϕ_f) the symplectic space over $R_f = R[\frac{1}{f}]$ obtained by tensoring with $\otimes R_f$.

Corollary 2.2.4. *Let (P, ψ) be a symplectic space over a ring R, there exist elements $f_1, f_2 \dots, f_m \in R$ which generate R such that $(P_{f_i}, \psi_{f_i}) \cong H_-^{\oplus n_i}$ for all $i \leq m$.*

Proof. We know that for any projective module P, there exist $g_1, \dots, g_p \in R$ which generate R and $P_{g_i} \cong R^{\oplus n_i}$ [Wei13, 2.2.2]. Therefore, it is enough to prove the theorem when the underlying module is free. Let $P \cong (R^{\oplus n}, \phi)$ be a symplectic space. By the isomorphism $\hat{\phi} : R^{\oplus n} \xrightarrow{\sim} Hom_{R-mod}(R^{\oplus n}, R)$, there exists an element $a \in R^{\oplus n}$ such that $\phi(a, e_1) = 1$, where $e_1 = (1, 0, 0, \dots, 0)$. For any pair of elements r_1 and r_2 in R, $\phi(a, r_1 a + r_2 e_1) = r_2$ and $\phi(e_1, r_1 a + r_2 e_1) = -r_1$. Hence a and e_1 are linearly independent and $(aR \oplus e_1 R, \phi_{|aR \oplus e_1 R}) \cong H_-$. This implies $(P, \phi) \cong H_- \perp Q$, where Q is the orthogonal compliment of $aR \oplus e_1 R$. (Q, ϕ_Q) is again a symplectic space and we can find $(f_1, \dots, f_k) = R$, where $Q_{f_j} \cong R^{m_j}$ and $m_j + 2 = n$. Hence the theorem follows by induction on the rank of P. □

There is a similar but weaker decomposition result for symmetric spaces. For $a \in R^\times$, let $< a >$ denote the symmetric space whose underlying module is R, and the bilinear form is $(x, y) \mapsto xay$. It follows that $< a > \cong < b >$ if and only if $a = br^2$, with $r \in R^\times$.

Theorem 2.2.5. *Let (M, ϕ) be a symmetric space over a ring R. There is a decomposition $M \cong < a_1 > \perp < a_2 > \perp \dots \perp < a_n > \perp N$, such that $\phi(n,n) \in R - R^\times$ for all $n \in N$.*

Proof. Let $m \in M$ be an element of M such that $\phi(m, m) \in R^\times$. Setting $\phi(m, m) = a$ we see that the subspace generated by m is isometric to $< a >$. Therefore we have $M = < a > \perp < a >^\perp$. The result then follows from induction on the rank of M. □

We can generalise the definition of these classes of forms to arbitrary schemes.

Definition 2.2.4. Let S be a scheme and $\mathcal{F}$ a coherent $\mathcal{O}_S$-module. A bilinear form on $\mathcal{F}$ is a morphism of $\mathcal{O}_S$-modules $\phi : \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}_S$ (equivalently a morphism $\mathcal{F} \to \mathcal{F}^\vee$).

1. A *symmetric form* is a bilinear form such that $\phi \circ \tau = \phi$, where $\tau : \mathcal{F} \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{F} \otimes \mathcal{F}$ is the switch map. It is called a *symmetric space* if $\mathcal{F}$ is a vector bundle and $\hat{\phi} : \mathcal{F} \to Hom(\mathcal{F}, \mathcal{O}_S)$ is an isomorphism.

2. A *skew-symmetric form* is a bilinear form such that $\phi \circ \tau = -\phi$, where $\tau : \mathcal{F} \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{F} \otimes \mathcal{F}$ is the switch map. It is called a skew-symmetric space if $\mathcal{F}$ is a vector bundle and $\hat{\phi} : \mathcal{F} \to Hom(\mathcal{F}, \mathcal{O}_S)$ is an isomorphism.

3. An *alternating form* is a bilinear form such that $\psi(x \hat{\otimes} x) = 0$ where $\mathcal{F} \hat{\otimes} \mathcal{F}$ is the presheaf tensor product with the sheafification map $\mathcal{F} \hat{\otimes} \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$. Similar to the affine case, such a pair $(\mathcal{F}, \psi)$ is called a *symplectic space* if $\mathcal{F}$ is a vector bundle and $\hat{\psi} : \mathcal{F} \to Hom(\mathcal{F}, \mathcal{O}_S)$ is an isomorphism.

We extend the definitions of the hyperbolic forms $H_+^{\oplus n}$ and $H_-^{\oplus n}$ to arbitrary schemes. Recall that every vector bundle $\mathcal{F}$ on a scheme is representable (as a sheaf) by an S-scheme $V(\mathcal{F}) \to S$. The form $\phi : \mathcal{F} \to \mathcal{F}^\vee$ then induces a morphism of S-schemes $V(\phi) : V(\mathcal{F}) \to V(\mathcal{F}^\vee)$. Given any two vector bundles with bilinear forms (V, ϕ) and (W, ψ), $(V \oplus W, \phi \perp \psi)$ is defined along the same lines as in the affine case.

Given a morphism of schemes $f : S \to T$ and a vector bundle equipped with a form (V, ϕ) on T, the pullback $f^*\phi : f^*V \to f^*V^\vee$ is a form on S which is a symmetric or a symplectic space whenever ϕ is.

Remark 2.2.2. For the remainder of this thesis we will use the structure map of a bilinear form $\phi : V \otimes V \to \mathcal{O}_X$ and it's dual $\hat{\phi} : V \to V^\vee$ interchangeably.

Definition 2.2.5. Given two vector bundles with bilinear forms (V, ϕ) and (W, ψ), we define $\phi \otimes \psi$ to be the bilinear form on $V \otimes W$ given by the map $V \otimes W \xrightarrow{\phi \otimes \psi} V^\vee \otimes W^\vee \xrightarrow{\sim} (V \otimes W)^\vee$.

Sometimes we will suppress the forms and write $V \otimes W$ for $(V \otimes W, \phi \otimes \psi)$. When the underlying vector bundles are trivial, the matrix corresponding to $\phi \otimes \psi$ is the tensor product of the matrices corresponding to ϕ and ψ after choosing compatible bases. From this definition it follows that $\phi \otimes \psi$ is non-degenerate whenever ϕ and ψ are.

Theorem 2.2.6. *Let (V, ϕ) and (W, ψ) be vector bundles with bilinear forms.*

1. *If (V, ϕ) and (W, ψ) are symmetric spaces, then $(V \otimes W, \phi \otimes \psi)$ is again a symmetric space. In particular $H_+ \otimes H_+ \cong H_+^{\oplus 2}$.*

2. *If (V, ϕ) and (W, ψ) are symplectic spaces, then $\phi \otimes \psi$ is a symmetric space. In particular $H_- \otimes H_- \cong H_+^{\oplus 2}$.*

3. *If (V, ϕ) is a symmetric space and (W, ψ) a symplectic space, then $\phi \otimes \psi$ is a symplectic space. In particular $H_- \otimes H_+ \cong H_-^{\oplus 2}$.*

Proof. It is enough to show this is true when the base scheme is affine and the module is free as these properties can be checked locally. As $(M \otimes N)^T = M^T \otimes N^T$ for any pair of matrices, it follows that if $M = \pm M^T$ and $N = \pm N^T$, then $M \otimes N = M^T \otimes N^T = (M \otimes N)^T$. Similarly if $M = -M^T$ and $N = N^T$ then $M \otimes N = -(M \otimes N)^T$. Further if N is alternating and $\sum_i x_i \otimes y_i$,

$$(\sum_i x_i \otimes y_i)^T (M \otimes N)(\sum_i x_i \otimes y_i) = \sum_{ij} (x_i^T M x_j) \otimes (y_i^T N y_j) = 0.$$

The underlying forms of $H_+ \otimes H_+$, $H_+ \otimes H_-$ and $H_- \otimes H_-$ are given by the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Let P_{13} denote the matrix which permutes the first and third rows and $P_{13}(-1)$ the signed permutation matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We get the required isometries using P_{13} for A and B and $P_{13}(-1)$ for C. □

Inductively we can show that $H_-^{\oplus n} \otimes H_-^{\oplus m} \cong H_+^{\oplus 2(m+n)}$ and similar results for the other two cases.

2.3 Symplectic and Orthogonal groups

Algebraic K-theory is closely connected with the infinite general linear group $GL(R) = colim_n GL_n(R)$. In fact, Quillen's first construction of the algebraic K theory space used the classifying space BGL [Qui73]. Similarly hermitian K-theory is closely related to the classifying spaces of the symplectic and orthogonal groups.

Definition 2.3.1. Let P be a projective R-module and $Aut(P)$ the group of R-module automorphisms $P \xrightarrow{\sim} P$.

1. Given a non-degenerate symmetric bilinear form $\phi : P \to P^\vee$, the associated orthogonal group $O(P, \phi)$ is the subgroup of $Aut(P)$ defined by
$$O(P, \phi) = \{p \in Aut(P) |\, p^\vee \phi p = \phi\}$$
2. Given a non-degenerate symplectic bilinear form $\psi : P \to P^\vee$, the associated symplectic group $Sp(P, \psi)$ is the subgroup of $Aut(P)$ defined by
$$Sp(P, \psi) = \{p \in Aut(P) |\, p^\vee \psi p = \psi\}$$

We use the adjoint map $\hat{\phi}$ as it makes other definitions more convenient but using $\phi : P \otimes P \to R$ works equally well. When $(P, \psi) = H_-$, we denote $Sp(P, \psi)$ by $Sp_{2n}(R)$. The inclusions $H_-^{\oplus n} \to H_-^{\oplus n+1}$ give us inclusions of groups $Sp_{2n}(R) \to Sp_{2n+2}(R)$. The colimit $Sp_\infty = colim_n Sp_{2n}$ is called the infinite symplectic group. Similarly $O_\infty = colim_n O(H_+^n)$ is called the infinite orthogonal group.

Remark 2.3.1. $Sp_{2n}(R)$ is sometimes defined to be the symplectic group associated to the space $(R^{\oplus 2n}, \Omega_n)$, where

$$\Omega_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

This space is isometric to $H_-^{\oplus n}$ by permuting the bases.

The definitions of symplectic and orthogonal groups can be extended to arbitrary schemes.

Definition 2.3.2. Let $V \to S$ be a vector bundle over a scheme S and $\phi : V \to V^\vee$ a bilinear form on V. Let $Aut(V, \phi)$ be the closed subscheme of $GL(V)$ given by $Aut(V, \phi)(U) = \{f \in GL(V)(U) | f^*\phi f = \phi\}$. When (V, ϕ) is a symmetric or symplectic space we denote $Aut(V, \phi)$ by $O(V, \phi)$ and $Sp(V, \phi)$ respectively. $O(V, \phi)$ and $Sp(V, \phi)$ are the *orthogonal* and *symplectic group schemes.*

These are closed subschemes as they are locally given by solutions to polynomial equations. Therefore they are affine group schemes. We denote $Sp(H_+^{\oplus n})$ by Sp_{2n}. Using 2.2.4 we can give a local description of symplectic spaces similar to the case of vector bundles.

Lemma 2.3.1. *Let $(V, \phi) \to X$ be a symplectic space of constant rank $2n$. There exists an open cover $\bigcup_i U_i = X$ such that there are isomorphisms T_i : $(V, \phi) \times_X U_i \xrightarrow{\sim} \mathbb{A}^{2n}_{U_i}$ and the transition functions $T_{i|U_i \cap U_j} \circ T^{-1}_{j|U_{ij}} \in GL_{2n}(U_i \cap U_j)$ belong to the subgroup $Sp_{2n}(U_i \cap U_j)$. In particular, there is an Sp_{2n}-action on $(V, \phi) \to X$.*

The converse is also true. That is, symplectic spaces of constant rank are uniquely described by the above data. This follows from the fact that every representable presheaf is a Zariski sheaf. This allows us to give an equivalence between Sp_{2n}-torsors and symplectic spaces of constant rank $2n$. Given any symplectic space $(V, \phi) \to X$ of constant rank $2n$, let $\mathcal{P}_V : Sm_X^{op} \to \mathbf{Set}$ be the (Zariski) subsheaf of $Hom(\mathcal{O}_X^{\oplus 2n}, V)$ given by

$$\mathcal{P}_V(U) = \{f : \mathcal{O}_U^{\oplus 2n} \to V_{|U} | \ f^*\phi_U f \in Sp_{2n}(U_i \cap U_j)\},$$

$\mathcal{P}_V$ has a (right) Sp_{2n}-action given by composition. As (V, ϕ) is locally isometric to $H_-^{\oplus n}$, there exists an open cover $\cup_i U_i = X$ such that $\mathcal{P}_V(U_i) \cong Sp_{2n}(U_i)$. This implies that $\mathcal{P}_V$ is an Sp_{2n}-torsor.

Lemma 2.3.2. *For any symplectic bundle $(V, \phi) \to S$, let $\mathcal{P}_V$ be the sheaf with Sp_{2n}-action over $(Sm_S)_{Zar}$ constructed above. $\mathcal{P}_V$ is an Sp_{2n}-torsor.*

For the other direction, given an Sp_{2n}-torsor $\mathcal{P} : (Sm_S^{op})_{Zar} \to \mathbf{Set}$, the sheaf $\mathcal{P} \times \mathbb{A}^{2n}$ has a natural $Sp_{2n} \times Sp_{2n}$-action. Using the diagonal inclusion $\Delta : Sp_{2n} \to Sp_{2n} \times Sp_{2n}$, we can construct the quotient sheaf $(\mathcal{P} \times \mathbb{A}^{2n})/Sp_{2n}$.

Lemma 2.3.3. *For any Sp_{2n}-torsor $\mathcal{P} : (Sm_S^{op})_{Zar} \to \mathbf{Set}$, the quotient sheaf $(\mathcal{P} \times \mathcal{O}^{\oplus 2n})/Sp_{2n}$ is representable by a vector bundle and the induced Sp_{2n}-action turns it into a symplectic space.*

Proof. As $\mathcal{P}$ is a principal Sp_{2n}-bundle, $(\mathcal{P} \times \mathcal{O}^{\oplus 2n})/Sp_{2n}$ is locally isomorphic to the sheaf $\mathcal{O}_S^{\oplus 2n}$. If $\cup_i U_i = X$ is an open cover with a local trivialization of $\mathcal{P}$, $\phi_i : \mathcal{P}(U_i) \xrightarrow{\sim} Sp_{2n}(U_i)$, we have transition functions $\phi_j \circ \phi_i^{-1} : Sp_{2n}(U_i \cap U_j) \to Sp_{2n}(U_i \cap U_j)$. The transition functions $\phi_j \circ \phi_i^{-1}$ correspond to a unique element in $Sp_{2n}(U_i \cap U_j)$ by evaluating at the identity. These same elements give us the transition functions

$$\mathcal{O}^{\oplus 2n}_{U_i \cap U_j} \xrightarrow{\sim} (\mathcal{P} \times \mathcal{O}^{\oplus 2n}/Sp_{2n})(U_i \cap U_j) \xrightarrow{\sim} (\mathcal{P} \times \mathcal{O}^{\oplus 2n}/Sp_{2n})(U_i \cap U_j) \xrightarrow{\sim} \mathcal{O}^{2n}_{U_i \cap U_j}.$$

This implies that $\mathcal{P} \times \mathcal{O}^{\oplus 2n}/Sp_{2n}$ is representable by a vector bundle. As the transition functions are in Sp_{2n}, the hyperbolic forms on $\mathcal{O}_{U_i}^{\oplus 2n} \xrightarrow{\sim} \mathcal{P} \times (\mathcal{O}^{\oplus 2n}/Sp_{2n})_{U_i}$ can be glued to give a symplectic space structure on $(\mathcal{P} \times \mathcal{O}^{\oplus 2n}/Sp_{2n})_{U_i}$. □

From 2.3.1 and 2.3.2 we get a bijection between symplectic spaces of a fixed finite rank and Sp_{2n}-torsors. Let G be $Sp(V, \phi)$ or $O(V, \phi)$ over S, the representable presheaf $G(-) := Hom_{Sch_S}(-, G) : Sch_S^{op} \to \mathbf{Set}$ is a presheaf of groups. As every representable presheaf is a sheaf in the étale (Nisnevich, Zariski) topology, 1.6.4 implies that BG classifies G-torsors over (Sm_S, τ) where τ is the étale, Nisnevich or Zariski site. We then have the following theorem.

Theorem 2.3.4. *Let S be any scheme. For any $X \in Sm_S$, let $Symp_{2n}(X)$ denote the set of isometry classes of symplectic spaces over X of constant rank $2n$. We have bijections*

$$Symp_{2n}(X) \cong P_{Sm_S}(X, Sp_{2n}) \cong Hom_{H^s(S)}(X, BSp_{2n})$$

Recall that for GL_n we know that every étale torsor is already a Zariski torsor 1.6.1. This together with 2.3.2 implies that every étale Sp_{2n}-torsor is a Zariski (and hence a Nisnevich) torsor. If we denote by $B_{et}Sp_{2n}$, the étale fibrant replacement of BSp_{2n}, we have the following corollary,

Theorem 2.3.5. *Let S be any scheme. We have a canonical isomorphism $BSp_{2n} \cong B_{et}Sp_{2n}$ in $H^s(S)$.*

A similar but weaker statement is true for the orthogonal groups. $O(V, \phi)$ is not Zariski locally trivial. $O(V, \phi)$ is however étale locally trivial over schemes where 2 is invertible. In this case we get we get an analogous result involving the étale classifying space $B_{et}O(H_+)$ [ST15].

2.4 Hermitian K-theory

Let $\mathbf{Sym}(X)$ and $\mathbf{Symp}(X)$ denote the categories of symmetric and symplectic spaces over a scheme X respectively, where a morphism $f : (V, \phi) \to (W, \psi)$ is a morphism of vector bundles $f : V \to W$ such that $f^*\psi f = \phi$. The orthogonal sum $\perp$ turns $\mathbf{Sym}(X)$ and $\mathbf{Symp}(X)$ into (essentially small) symmetric monoidal categories. We will use 2.1.2 to define the orthogonal and symplectic K-theory spaces.

Definition 2.4.1. Let R be a commutative ring.

1. The *symplectic K-theory space* $KSp(R)$ is the algebraic K-theory space $K^\perp(i\mathbf{Symp}(Spec(R)))$ and $KSp_n(R) = \pi_n KSp(R)$ are the symplectic K-groups.

2. The *orthogonal K-theory space* $KO(R)$ is the algebraic K-theory space $K^\perp(i\mathbf{Sym}(Spec(R)))$ and $KO_n(R) = \pi_n KO(R)$ the orthogonal K-groups.

As is the case with algebraic K-theory, we only care about the homotopy type of $KO(R)$ and $KSp(R)$. The zeroth orthogonal K-group $KO_0(R)$ is equal to the classical Grothendieck-Witt group $GW(R)$(called the *Witt-Grothendieck group* in [Lam05, Def.1.1]). When $\frac{1}{2} \in R$, $KO(R)$ is the *hermitian K-theory* space in [Hor05] and $KSp(R)$ is the hermitian K-theory space. For any commutative ring R, $\coprod_n O(H_+^n)(R)$ and $\coprod_n Sp_{2n}(R)$ are symmetric monoidal subgroupoids of $\coprod_n GL_n(R)$. We have monoidal functors $\coprod_n O(H_+^n)(R) \to \mathbf{Sym}(Spec(R))$ and $\coprod_n Sp_{2n}(R) \to \mathbf{Symp}(Spec(R))$ given by $n \mapsto H_+^{\oplus n}$ and $n \mapsto H_-^{\oplus n}$ respectively. These then induce maps of spaces

$$\Omega_s B(\coprod_n BO(H_+^n))(R) \to KO(R)$$

$$\Omega_s B(\coprod_n BSp_{2n})(R) \to KSp(R)$$

respectively. There are several ways to extend $KO(R)$ and $KSp(R)$ to arbitrary schemes. Firstly we note that for any scheme S, KO and KSp define functors $(Sch_S^{aff})^{op} \to \mathbf{sSet}$ from affine S-schemes to simplicial sets. The naive way to extend this to arbitrary schemes is to use the $\mathbf{Symp}(X)$ category for non-affine schemes as well. For any $X \in Sm_S$, we define the categories of big symmetric and big symplectic spaces, $\mathbf{Symp}_{Sm_S}(X)$ and $\mathbf{Sym}_{Sm_S}(X)$ respectively, along the same lines as the category of big vector bundles (i,e. we fix a choice of pullback for each form). We can then define simplicial presheaves

$$KO^{\perp} : Sm_S^{op} \to \mathbf{sSet}$$

$$KSp^{\perp} : Sm_S^{op} \to \mathbf{sSet}$$

extending $KO(R)$ and $KSp(R)$ respectively.

Definition 2.4.2. Let S be any scheme.

1. $KO^{\perp} : Sm_S^{op} \to \mathbf{sSet}$ is the simplicial presheaf given by $KO^{\perp}(X) = K^{\perp}(i\mathbf{Sym}_{Sm_S}(X))$.
2. $KSp^{\perp} : Sm_S^{op} \to \mathbf{sSet}$ is the simplicial presheaf given by $KSp^{\perp}(X) = K^{\perp}(i\mathbf{Symp}_{Sm_S}(X))$.

Theorem 2.4.1. *The morphism $\coprod_n Sp_{2n}(-) \to i\mathbf{Symp}_{Sm_S^{op}}(-)$ in $Fun(Sm_S^{op}, \mathbf{Cat})$ given by $n \mapsto H_-^{\oplus n}$ induces a weak equivalence of simplicial presheaves*

$$\coprod_n BSp_{2n} \to B(i\mathbf{Symp}_{Sm_S^{op}})$$

over the Zariski site. In particular they are isomorphic as objects in $H(S)$.

Proof. By 2.2.3 over a local ring R, every symplectic space is isometric to some $H_-^{\oplus n}$. This implies that $\coprod Sp_{2n}(R) \to i\mathbf{Symp}(R)$ is an equivalence of groupoids and hence induces a weak equivalence of simplicial sets

$$\coprod_n BSp_{2n}(R) \to Bi\mathbf{Symp}(R).$$

As BSp_{2n} is a degreewise representable simplicial sheaf we have by 1.3.1 that the stalks $\coprod_n BSp_{2n}(\mathcal{O}_{U,u})$ are just the evaluations of the simplicial presheaf at $Spec(\mathcal{O}_{U,u})$. 2.2.3 also implies that the canonical map $\operatorname{colim}_{x\in U} Bi\mathbf{Symp}(U) \to Bi\mathbf{Symp}(\mathcal{O}_{U,u})$ is an isomorphism. This implies that we have an equivalence $\coprod_n BSp_{2n}(\mathcal{O}_{U,u}) \to Bi\mathbf{Symp}(\mathcal{O}_{U,u})$ of stalks in the Zariski topology and so we are done. □

This result implies that the induced map of objectwise group completions is also a weak equivalence.

Corollary 2.4.2. *For any scheme S, there is an isomorphism in $H_\bullet(S)$*

$$\Omega^1_s B(\coprod_n BSp_{2n}) \xrightarrow{\sim} K^\perp(\coprod_n Sp_n) \xrightarrow{\sim} KSp^\perp.$$

Proof. The above theorem implies $K^\perp(\coprod_n Sp_n(\mathcal{O}_{U,u})) \xrightarrow{\sim} KSp^\perp(\mathcal{O}_{U.u})$ for any point (U, u). Therefore it is enough to show that these spaces are weakly equivalent to the corresponding stalks. This follows from the analagous result for the classifying spaces and the construction of the group completion given in 2.1.1. □

Remark 2.4.1. Currently there is some ambiguity about what is the correct definition of symplectic K-theory over an arbitrary scheme. In general the hermitian K-theory for categories with duality gives us higher Grothendieck-Witt groups $GW_n(X)$ and $GW_n^-(X)$ ([Sch10a]) for any scheme X that gives us all the desired properties for regular Noetherian schemes with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. We will describe this theory below.

Definition 2.4.3. A category with duality is a triple $(\mathcal{C}, *, \eta)$, where $\mathcal{C}$ is a category, $*$ is a functor $* : \mathcal{C} \to \mathcal{C}^{op}$ and η is a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow **^{op}$. A category with *strict* duality is a category with duality such that η is a natural isomorphism.

We can define symmetric forms in any category with duality.

Definition 2.4.4 (Symmetric forms and symmetric spaces)**.** Let $(\mathcal{C}, *, \eta)$ be an exact category with duality and A an object in $\mathcal{C}$. A *symmetric form* on A is a morphism $\phi : A \to A^*$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A^* \\ & \searrow^{\eta_A} & \uparrow{\scriptstyle \phi^*} \\ & & A^{**} \end{array}$$

commutes. A symmetric form $\phi : A \to A^*$ is called *non-degenerate* if it is an isomorphism. A *symmetric space* is a pair (A, ϕ) where A is an object in $\mathcal{C}$ and $\phi : A \to A^*$ is a non-degenerate symmetric form.

We denote by $\mathcal{C}_h$ the *category of symmetric forms* in $(\mathcal{C}, *, \eta)$, the objects in $\mathcal{C}_h$ are symmetric forms (A, ϕ) and morphisms $(A, \phi) \to (A', \phi')$ are morphisms $f : A \to A'$ such that $f^* \circ \phi' \circ f = \phi$. There is a forgetful functor $F : \mathcal{C}_h \to \mathcal{C}$ sending (A, ϕ) to A

Definition 2.4.5. An *exact category with duality* is a category with strict duality $(\mathcal{E}, *, \eta)$ such that

1. $\mathcal{E}$ (and hence $\mathcal{E}^{op}$) is an exact category.

2. $* : \mathcal{E} \to \mathcal{E}^{op}$ is an exact functor.

3. $1_E = \eta^*_E \circ \eta_{E^*}$ for all objects E in $\mathcal{E}$.

Theorem 2.4.3. *Let S be a scheme, $\mathcal{L}$ a line bundle over S and $\varepsilon \in \{1, -1\}$. The triple $(Vect(S), Hom_{\mathcal{O}_S}(-, \mathcal{L}), \varepsilon ev)$ is an exact category with duality, where $Vect(S)$ is the category of vector bundles over S, $Hom_{\mathcal{O}_S}(-,-)$ is the sheaf of $\mathcal{O}_S$-module homomorphisms, and $ev_E : E \to Hom_{\mathcal{O}_S}(Hom_{\mathcal{O}_S}(E, \mathcal{L}), \mathcal{L})$ is the evaluation map.*

Proof. We need to show that $Hom_{\mathcal{O}_S}(-, \mathcal{L})$ is an exact functor of sheaves, ev is a natural isomorphism and $1_{E^*} = ev_E^* \circ ev_{E^*}$. Let $\bigcup_i Spec(R_i) = S$ be an affine cover of S, such that $\mathcal{L}_{|Spec(R_i)} \cong R_i$. Then $Hom_{\mathcal{O}_S}(-, \mathcal{L})_{|Spec(R_i)} \cong Hom_{\mathbf{P}(R_i)}(-, R_i)$ and hence exact. As exactness of sheaves can be checked locally, $Hom_{\mathcal{O}_S}(-, \mathcal{L})$ is exact. Similarly, ev restricted to $Spec(R_i)$ is the isomorphism $P \to Hom_{\mathbf{P}(R_i)}(Hom_{\mathbf{P}(R_i)}(P, R_i), R_i)$, which implies ev is an isomorphism. The case for $-ev$ is similar. □

Exact categories with dualities give us a broader framework to deal with KSp and KO. The symmetric forms in $(Vect(S), Hom_{\mathcal{O}_S}(-, \mathcal{O}_S), \varepsilon ev)$ are the usual symmetric forms for $\varepsilon = 1$ and the skew-symmetric forms for $\varepsilon = -1$. The additive category structure of the exact category gives us additional structure. Given an object E in an exact category with duality $(\mathcal{E}, *, \eta)$, the *hyperbolic symmetric space* $H(E)$ is the symmetric space with underlying object $E \oplus E^*$ and symmetric form $\begin{pmatrix} 0 & 1 \\ \eta_E & 0 \end{pmatrix}$ The hyperbolic form construction gives us a functor $\mathcal{E} \to \mathcal{E}_h$, $E \mapsto H(E)$.

Definition 2.4.6 (Orthogonal sum)**.** The orthogonal sum of symmetric forms (E, ϕ) and (E', ϕ') is the symmetric form on $E \oplus E'$ given by $\phi \oplus \phi' : E \oplus E' \to E^* \oplus E'^*$. We denote the orthogonal sum by $(E, \phi) \perp (E', \phi')$.

The orthogonal sum construction turns $\mathcal{E}_h$ into a symmetric monoidal category (identity is given by the unique form on the 0 object). We will also need the broader framework of Grothendieck-Witt space of an exact category with weak equivalences and duality developed in [Sch10b].

Definition 2.4.7. An *exact category with weak equivalences and duality* is a quadruple $(\mathcal{E}, *, \eta, w)$ where

1. $(\mathcal{E}, *, \eta)$ is category with duality
2. $(\mathcal{E}, w)$ is an exact category with weak equivalences
3. $* : (\mathcal{E}^{op}, w) \to (\mathcal{E}, w^{op})$ preserves weak equivalences and short exact sequences
4. $\eta : id \to **$ is a natural weak equivalence.

Note that η need not be an isomorphism as was the case for exact categories with dualities. This allows us to consider a broader class of duality functors. A symmetric form in $(\mathcal{E}, *, \eta, w)$ is just a symmetric form in $(\mathcal{E}, *, \eta)$ (however the duality is no longer strict). We call a symmetric form (E, ϕ) in $(\mathcal{E}, *, \eta, w)$ *non-degenerate* if ϕ is a weak equivalence. Every exact category with duality is an exact category with weak equivalences and duality with $w = i$ the set of isomorphisms.

Remark 2.4.2. Given an exact category with weak equivalences and duality $(\mathcal{E}, *, \eta, w)$, the quadruple $(\mathcal{E}, *, -\eta, w)$ is also an exact category with weak equivalences and duality. When $\mathcal{E}$ is a subcategory of $\mathcal{O}_X$-modules over some scheme X, the change from η to $-\eta$ is the change from symmetric to skew-symmetric forms.

Definition 2.4.8 (Form functor)**.** A *form functor* from one exact category with weak equivalences and duality $(\mathcal{E}, *, \eta, w)$ to another $(\mathcal{D}, *, \zeta, w)$ is a pair (F, α) where $F : \mathcal{E} \to \mathcal{D}$ is a functor and $\alpha : F* \to *F$ is a natural transformation such that for every object E in $\mathcal{E}$, $\alpha_E^* \zeta_{FE} = \alpha_E F(\eta_E)$. A form functor is called *exact* if F is an exact functor (preserves exact sequences) and sends weak equivalences to weak equivalences. When F is an isomorphism of exact category with weak equivalences and duality then we call (F, α) an isomorphism.

In cases where $\alpha = id$, we denote the form functor just by F.

Definition 2.4.9. The *Grothendieck Witt group* $GW_0(\mathcal{E}, *, \eta, w)$ of an exact category with weak equivalences and duality $(\mathcal{E}, *, \eta, w)$ is the quotient of the free abelian group, generated by isomorphism classes of symmetric spaces $[(E, \phi)]$ in $(\mathcal{E}, *, \eta, w)$, by the following relations

1. $[(E, \phi)] + [(F, \psi)] = [(E, \phi) \perp (F, \psi)]$;
2. $[(E, \phi)] = [(F, f^* \phi f)]$ for each weak equivalence $f : F \xrightarrow{\sim} E$;
3. for any commutative diagram of short exact sequences in $\mathcal{E}$ of the form

$$\begin{array}{ccccc} E_{-1} & \rightarrowtail & E_0 & \twoheadrightarrow & E_1 \\ {\scriptstyle \phi_{-1}}\downarrow{\scriptstyle \sim} & & {\scriptstyle \phi_0}\downarrow{\scriptstyle \sim} & & {\scriptstyle \phi_1}\downarrow{\scriptstyle \sim} \\ E_1^* & \rightarrowtail & E_0^* & \twoheadrightarrow & E_{-1}^* \end{array} \tag{2.4.1}$$

   where $(\phi_{-1}, \phi_0, \phi_1) = (\phi_1^* \eta, \phi_0^* \eta, \phi_{-1}^* \eta)$, we have

$$[E_0, \phi_0] = [E_{-1} \oplus E_1, \begin{pmatrix} 0 & \phi_{-1} \\ \phi_1 & 0 \end{pmatrix}].$$

Definition 2.4.10 (Witt group)**.** The Witt group $W_0(\mathcal{E}, *, \eta, w)$ of an exact category with weak equivalences and duality $(\mathcal{E}, *, \eta, w)$ is the quotient of $GW_0(\mathcal{E}, *, \eta, w)$ by the subgroup generated by forms $[E_0, \phi_0]$ that fit into a diagram of the form 2.4.1.

An exact form functor between two exact categories with weak equivalences and duality induces a homomorphism between the corresponding $GW_0(\mathcal{E}, *, \eta, w)$ and $W_0(\mathcal{E}, *, \eta, w)$.

We define the Grothendieck-Witt space $GW(\mathcal{E}, *, \eta, w)$ using the hermitian $S_\bullet$-construction [Sch10b, 2.6] which generalises 2.1.8. For any exact category with weak equivalences and duality $(\mathcal{E}, *, \eta, w)$, $S_n\mathcal{E}$ are also exact categories with weak equivalences and duality. This duality on $S_n\mathcal{E}$ is not compatible with the face and degeneracy maps. To rectify this we will use the edgewise subdivision functor.

Definition 2.4.11 (Edgewise subdivision)**.** Let Δ denote the usual simplex category. For each $n \in \mathbb{N}$, let $\underline{n}$ be the well ordered set $\underline{n} = \{n' < (n-1)' < \ldots 0' < 0 < 1 \ldots < n\}$ which is isomorphic to $[2n+1]$. $T : \Delta \to \Delta$ is the functor which is $[n] \mapsto \underline{n}$ on objects and for $\theta : [n] \to [m]$, $T(\theta)(i) = \theta(i)$ and $T(\theta)(i') = \theta(i)'$ for each $i \in [n]$. The *edgewise subdivision* functor $(-)^e : \mathbf{sSet} \to \mathbf{sSet}$ sends X to $X^e = X \circ T$.

By definition $X^e_n = X_{2n+1}$. The inclusion $[n] \to \underline{n}$ induces a natural transformation $()^e \to id$. The geometric realization $|X^e| \to |X|$ is a subdivision of the CW-complex [Seg73] and hence is a homeomorphism.

Consider $[n]$ as a poset category. $[n]$ has the structure of a category with strict duality given by $[n] \to [n]^{op}$, $i \mapsto n-i$ (in fact this structure is unique). This induces a category with duality structure on $S_n\mathcal{E} \subset Fun(Ar[n], \mathcal{E})$ given by $(E_{..})^*_{i,j} = E^*_{n-j,n-i}$. This duality is not compatible with the simplicial structure. For example the dual of $E_{..} = E_{1,0} \rightarrowtail E_{2,0} \rightarrowtail E_{3,0}$ is $E^*_{3,2} \rightarrowtail E^*_{3,1} \rightarrowtail E^*_{3,0}$. But $(d_1(E_{..}))^* = (E_{2,0} \rightarrowtail E_{3,0})^* = (E^*_{3,2} \rightarrowtail E^*_{3,0})$ which is not equal to $d_1(E^*_{..})$. To rectify this we replace $S_n\mathcal{E}$ with $S^e_n\mathcal{E} = S_{2n+1}\mathcal{E}$. We see that $n \mapsto S^e_n\mathcal{E}$ is a simplicial exact category with weak equivalences and duality (i.e, all the structures are compatible with the face and degeneracy maps) where the duality is given by $(E^*)_{ij} = E^*_{i'j'}$ and $(E^*)_{i'j'} = E^*_{ij}$ respectively.

Definition 2.4.12 (Grothendieck-Witt space)**.** Let $(\mathcal{E}, *, \eta, w)$ be an exact category with weak equivalences and duality. The assignment $n \mapsto S^e_n\mathcal{E}$ gives us a simplicial exact category with weak equivalences and duality $S^e_\bullet\mathcal{E}$. The subcategories of weak equivalences $wS^e_n\mathcal{E}$ in $S^e_n\mathcal{E}$ together give us a simplicial category with duality $wS^e_\bullet\mathcal{E}$. We denote by $(wS^e_\bullet\mathcal{E})_h$ the associated simplicial category of symmetric forms. The *Grothendieck Witt space* $GW(\mathcal{E}, *, \eta, w)$ is the homotopy fiber of the map

$$|(wS^e_\bullet\mathcal{E})_h| \to |wS_\bullet\mathcal{E}|$$

given by the composition $(wS^e_\bullet\mathcal{E})_h \to wS^e_\bullet\mathcal{E} \to wS_\bullet\mathcal{E}$. The *Grothendieck Witt groups* are the homotopy groups $GW_n(\mathcal{E}, *, \eta, w) = \pi_n GW(\mathcal{E}, *, \eta, w)$ for $n \geq 1$.

For our definition to be complete we want $\pi_0 GW(\mathcal{E}, *, \eta, w) \cong GW_0(\mathcal{E}, *, \eta, w)$ (note that $GW(\mathcal{E}, *, \eta, w)$ has a commutative H-space structure induced by $\perp$ on $S^e_\bullet\mathcal{E}$ and hence π_0 is a commutative monoid). This turns out to be true (cf.[Sch10b, Prop.3]) but we will not go into the details here.

Definition 2.4.13. Let X be a scheme with $\frac{1}{2} \in \Gamma(\mathcal{O}_X, X)$. Then, $(Vect(X), \vee, \pm\eta, iso)$ is a pair of exact categories with weak equivalences and duality, where the weak equivalences are isomorphisms. We define the symplectic and orthogonal K-theory spaces as

$$KO(X) = GW(Vect(X), \vee, \eta, iso)$$
$$KSp(X) = GW(Vect(X), \vee, -\eta, iso)$$

Theorem 2.4.4 (Affine scheme)**.** *Let R be a commutative ring with $\frac{1}{2} \in R$. We have isomorphisms in the homotopy category* Ho**sSet**.

$$KO^\perp(Spec(R)) \xrightarrow{\sim} KO(Spec(R))$$

$$KSp^\perp(Spec(R)) \xrightarrow{\sim} KSp(Spec(R))$$

This is constructed as a zigzag of weak equivalences utilizing the hermitian version of the Q-construction (cf.[Sch10b, Prop.2]) along the same lines as the algebraic K-theory case. This shows that over any base scheme S, the presheaf

$$KSp : Sm_S^{op} \to \mathbf{sSet} \quad X \mapsto KSp(X)$$

is weakly equivalent to the presheaf $KSp^{\perp}$ defined using symmetric monoidal structure when $\frac{1}{2} \in \Gamma(\mathcal{O}_S, S)$.

Definition 2.4.14. Let S be a scheme where 2 is not necessarily invertible. We define $KSp \in \mathbf{Spc}(S)$ to be the Nisnevich fibrant replacement of $KSp^{\perp}$.

Remark 2.4.3. For symplectic K-theory, the invertibility of 2 is crucial. Otherwise the category of symplectic and skew-symmetric spaces are distinct. When this happens, the symplectic spaces are no longer known to be the symmetric spaces is a suitable category with duality. To remedy this Schlichting recently introduced the notion of K-theory with quadratic functors in [Sch19]. We do know yet if this theory gives us all correct properties. In particular, we do not know if it satisfies the representability results below.

2.5 Hermitian K-theory of chain complexes

Let $(\mathcal{E}, *, \eta)$ be an exact category with (strict) duality. Recall that the pair $(Ch^b(\mathcal{E}), q)$, where $Ch^b(\mathcal{E})$ is the category of bounded chain complexes in $\mathcal{E}$ and q is the set of quasi-isomorphisms in $Ch(\mathcal{E})$, has the structure of an exact category with weak equivalences 2.1.2.

Definition 2.5.1 (Shifted dualities)**.** For $n \in \mathbb{Z}$, we define the functor $\eta^n : Ch^b(\mathcal{E}) \to Ch^b(\mathcal{E})^{op}$ to be given by $E. \mapsto (E.)^*[n]$. Where $(E.)^*$ is the chain complex $(E.)^*_i = E^*_{-i}$. That is,

$$(E^*, d^*) : \quad \dots E^*_{-i+1} \xrightarrow{d^*_{-i}} E^*_{-i} \xrightarrow{d^*_{-i-1}} E^*_{-i-1} \to \dots$$

$[n]$ is the usual shift functor $E[n]_i = E_{i-n}$. In particular $(d^{*^n})_i = (d_{-i-1-n})^*$. Let $\eta^n : (-) \Rightarrow (-)^{*^n *^n}$ be the natural transformation given by

$$(\eta^n_E)_i = (-1)^{\frac{n(n-1)}{2}} \eta_{E_i} \quad E. \in Ch^b(\mathcal{E})$$

The pairs $(*^n, \eta^n)$, give us an exact category with weak equivalences and duality

$$(Ch^b(\mathcal{E}), *^n, \eta^n, q)$$

for each $n \in \mathbb{Z}$. When $n = 0$, we drop the superscript and write $(*, \eta) = (*^0, \eta^0)$.

Lemma 2.5.1. *For any exact category with duality $(\mathcal{E}, *, \eta)$ and for all $n, k \in \mathbb{Z}$, there are isomorphisms of exact categories with weak equivalences and duality*

$$(Ch^b(\mathcal{E}), *^{n+4k+2}, \eta^{n+4k+2}, q) \xrightarrow{\sim} (Ch^b(\mathcal{E}), *^n, -\eta^n, q) \quad E. \mapsto E.[2k+1]$$

$$(Ch^b(\mathcal{E}), *^{n+4k}, \eta^{n+4k}, q) \xrightarrow{\sim} (Ch^b(\mathcal{E}), *^n, \eta^n, q) \quad E. \mapsto E.[2k]$$

Proof. The shift functors $[n]$ are isomorphisms which preserve weak equivalences. Therefore it is enough to show that these shift are compatible with duality. As $(E[k]^{*^n})_i = E^*[k]_{-i-n} = E^*_{-i-n-k} = E^*_{-i+k-n-2k} = (E^{*^{n+2k}})[k]_i$, we see that the dualities commute with the shift. Finally for the double dual identification, note that our choice of sign for η is compatible with the definition of $(\eta^n_E)_i = (-1)^{\frac{n(n-1)}{2}}\eta_{E_i}$. □

The functor $i_0 : \mathcal{E} \to Ch^b(\mathcal{E})$ sending an object to the chain complex concentrated in degree 0 induces a form functor $(\mathcal{E}, *, \eta, iso) \to (Ch^b(\mathcal{E}), *^0, \eta, q)$.

Theorem 2.5.2. *For any exact category with duality $(\mathcal{E}, *, \eta)$, the form functor $(\mathcal{E}, *, \eta, iso) \to (Ch^b(\mathcal{E}), *^0, \eta, q)$ induces a homotopy equivalence*

$$GW(\mathcal{E}, *, \eta, iso) \to GW(Ch^b(\mathcal{E}), *^0, \eta, q).$$

The isomorphism of groups $GW_0(Ch^b(\mathcal{E}), *^0, \eta, q) \xrightarrow{\sim} GW_0(\mathcal{E}, *, \eta, iso)$ is

$$[C., \phi.] \mapsto [C_0, \phi_0] + \sum_{i\geq 1}[H(C_i)].$$

This is [Sch10b, Prop.6]. This statement is analogous to [TT90, Th.1.11.7]. The hyperbolic forms enter the picture as $(C.\phi.)$ is a symmetric space implies $C_i \xrightarrow{\sim} C^*_{-i}$ and $[C, \phi] + [C, -\phi] = [H(C)] \in GW_0(\mathcal{E}, *, \eta, iso)$ [Sch10a, Lem.2.8]. We now have the required machinery to define symplectic and orthogonal K-theory for arbitrary schemes. Given a scheme X, the category of bounded chain complexes of vector bundles $Ch^b(Vect(X))$ and $\vee : Vect(X) \to Vect(X)$ the functor $E \mapsto Hom(E, \mathcal{O}_X)$, gives us a family of exact categories with weak equivalences and duality,

$$(Ch^b(Vect(X)), \vee^n, \pm\eta, q)$$

for each $n \in \mathbb{Z}$, where η is the usual double dual identification. Replacing $Vect(X)$ with an appropriate category of big vector bundles $Vect_{\mathcal{V}}(X)$, any morphism of functors $f : Y \to X$ induces an form functor

$$f^* : (Ch^b(Vect_{\mathcal{V}}(X)), \vee^n, \pm\eta, q) \to (Ch^b(Vect_{\mathcal{V}}(Y)), \vee^n, \pm\eta, q)$$

for all $n \in \mathbb{Z}$. Given an open subscheme $i : U \hookrightarrow X$, let q_U be the set of morphisms in $Ch^b(Vect_{\mathcal{V}}(X))$ which are sent to quasi-isomorphisms in $Ch^b(Vect_{\mathcal{V}}(U))$. It follows that $q \subset q_U$ and $f[n] \in q_U$ whenever $f \in q_U$. Let $Ch^b(Vect_{\mathcal{V}}(X))^{q_U}$ be the full subcategory of $Ch^b(Vect_{\mathcal{V}}(X))$, whose elements are all objects whose image in $Ch^b(Vect_{\mathcal{V}}(U))$ is acyclic. As the shift and duality functor both preserve acyclic complexes, $(\vee^n, \eta^n)$ restricts to a duality on $Ch^b(Vect_{\mathcal{V}}(X))^{q_U}$. We then get two related families of exact categories with weak equivalences and duality.

$$(Ch^b(Vect_{\mathcal{V}}(X)), \vee^n, \pm\eta, q_U)$$
$$(Ch^b(Vect_{\mathcal{V}}(X))^{q_U}, \vee^n, \pm\eta, q)$$

For simplicity we will use $Vect_{\mathcal{V}}(X)$ and $Vect(X)$ interchangeably as they give equivalent categories.

Definition 2.5.2. Let X be any scheme and $Ch^b(Vect(X))$ the associated category of chain complexes of vector bundles. We define the K-theory spaces $KO^{[n]}$, $KSp^{[n]}$ to be the Grothendieck-Witt spaces

$$KO^{[n]}(X) := GW(Ch^b(Vect(X)), \vee^n, \eta, q)$$

$$KSp^{[n]}(X) := GW(Ch^b(Vect(X)), \vee^n, -\eta, q)$$

Given an open subscheme $i : U \hookrightarrow X$, we define the relative K-theory spaces $KO^{[n]}(X, U)$ and $KSp^{[n]}(X, U)$ to be

$$KO^{[n]}(X, U) := GW(Ch^b(Vect(X))^{q_U}, \vee^n, \eta, q)$$

$$KSp^{[n]}(X, U) := GW(Ch^b(Vect(X))^{q_U}, \vee^n, -\eta, q)$$

we then have K-groups $KO_i^{[n]}(X) = \pi_i KO^{[n]}(X)$, $KSp_i^{[n]}(X) = \pi_i KSp^{[n]}(X)$, $KO_i^{[n]}(X, U) = \pi_i KO^{[n]}(X, U)$ and $KSp_i^{[n]}(X, U) = \pi_i KSp^{[n]}(X, U)$ for $i \geq 0$.

It follows immediately from 2.5.2 that we have isomorphisms

$$KO(X) \xrightarrow{\sim} KO^{[0]}(X)$$

$$KSp(X) \xrightarrow{\sim} KSp^{[0]}(X)$$

when $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. By 2.5.1, we have isomorphisms

$$KSp^{[n]}(X) \xrightarrow{\sim} KO^{[n+4k+2]}(X) \quad \text{and} \quad KSp^{[n]}(X, U) \xrightarrow{\sim} KO^{[n+4k+2]}(X, U) \tag{2.5.1}$$

for all $n, k \in \mathbb{Z}$ and for any open embedding $i : U \hookrightarrow X$. Schlichting in [Sch10b] showed that when X has an ample family of line bundles (in particular for regular Noetherian schemes) we have the following theorems.

Theorem 2.5.3 (Localization)**.** *For any quasi-compact open subscheme $U \hookrightarrow X$, the canonical pair of form functors*

$$(Ch^b(Vect(X))^{q_U}, \vee^n, \eta, q) \to (Ch^b(Vect(X)), \vee^n, \eta, q)$$

$$(Ch^b(Vect(X)), \vee^n, \eta, q) \to (Ch^b(Vect(X)), \vee^n, \eta, q_U)$$

induces homotopy fibration sequences

$$KO^{[n]}(X, U) \to KO^{[n]}(X) \to GW(Ch^b(Vect(X)), \vee^n, \eta, q_U).$$

Theorem 2.5.4 (Zariski excision)**.** *For any quasi-compact open subscheme $U \hookrightarrow X$ and a closed scheme $Z \hookrightarrow X$ with quasi-compact open complement, the form functor*

$$(Ch^b(Vect(X))^{q_{X \setminus Z}}, \vee^n, \eta, q_U) \to (Ch^b(Vect(U))^{q_{U \setminus (U \cap Z)}}, \vee^n, \eta, q)$$

induces weak equivalences between the corresponding Grothendieck-Witt spaces,

$$GW(Ch^b(Vect(X))^{q_{X \setminus Z}}, \vee^n, \eta, q_U) \xrightarrow{\sim} GW(Ch^b(Vect(U))^{q_{U \setminus (U \cap Z)}}, \vee^n, \eta, q)$$

In particular, when $Z = \emptyset$ we have an equivalence $GW(Ch^b(Vect(X)), \vee^n, \eta, q_U) \xrightarrow{\sim} KO^{[n]}(U)$.

Putting these two theorems together we get homotopy fibration sequences of K-groups

$$KO^{[n]}(X,U) \to KO^{[n]}(X) \to KO^{[n]}(U). \tag{2.5.2}$$

Using 2.5.4 in the case where $Z \subset U$ with $V = X \setminus Z$ we get

$$KO^{[n]}(X,V) \xrightarrow{\sim} KO^{[n]}(U, U \cap V). \tag{2.5.3}$$

This follows from the fact that $q = q_U$ in $Ch^b(Vect(X))^{q_V}$ because $Z \subset U$ implies $U \cup V = X$. Using the isomorphisms in 2.5.1, we then get

$$\begin{gathered} KSp^{[n]}(X,U) \to KSp^{[n]}(X) \to KSp^{[n]}(U) \\ KSp^{[n]}(X,V) \xrightarrow{\sim} KSp^{[n]}(U, U \cap V). \end{gathered} \tag{2.5.4}$$

From 2.5.2, 2.5.3 and 2.5.4, it follows that given any Zariski square,

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

The corresponding squares of K-theory spaces,

$$\begin{array}{ccc} KO^{[n]}(X) & \longrightarrow & KO^{[n]}(V) \\ \downarrow & & \downarrow \\ KO^{[n]}(U) & \longrightarrow & KO^{[n]}(U \cap V) \end{array} \qquad \begin{array}{ccc} KSp^{[n]}(X) & \longrightarrow & KSp^{[n]}(V) \\ \downarrow & & \downarrow \\ KSp^{[n]}(U) & \longrightarrow & KSp^{[n]}(U \cap V) \end{array}$$

are homotopy cartesian. Stronger results are known in the case when $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$.

Theorem 2.5.5 (Unstable representability [PW18, Th.5.1])**.** *Let X be any regular Noetherian scheme of finite dimension with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Then we have for any $n \geq 0$.*

1. *(Nisnevich excision) For any elementary Nisnevich square,*

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

   the induced square

$$\begin{array}{ccc} KO^{[n]}(X) & \longrightarrow & KO^{[n]}(U) \\ \downarrow & & \downarrow \\ KO^{[n]}(V) & \longrightarrow & KO^{[n]}(W) \end{array}$$

   is homotopy cartesian.

2. *The map $KO^{[n]}(X) \to KO^{[n]}(\mathbb{A}^1 \times X)$, induced by the projection, is a weak equivalence.*

Therefore by 1.4.7

$$KO_i^{[n]}(X) = \pi_i(KO^{[n]}(X)) \cong Hom_{H_\bullet(S)}(S^i \wedge X_+, KO^{[n]}) \tag{2.5.5}$$

In particular, this result holds over $Spec(\mathbb{Z}[\frac{1}{2}])$. Again by 2.5.1, we have

$$KSp_i^{[n]}(X) = \pi_i(KSp^{[n]}(X)) \cong Hom_{H_\bullet(S)}(S^i \wedge X_+, KSp^{[n]}) \tag{2.5.6}$$

Further, the open immersion $U \hookrightarrow X$ is a cofibration and hence the quotient map of pointed presheaves

$$U_+ \to X_+ \to X_+/U_+$$

is a homotopy cofibration sequence. By 1.2.5, we then have a fibration sequence of simplicial sets.

$$\mathbf{Hom}_{sPSh_\bullet(S)}(X_+/U_+, KO^{[n]}) \to \mathbf{Hom}_{sPSh_\bullet(S)}(X_+, KO^{[n]}) \to \mathbf{Hom}_{sPSh_\bullet(S)}(U_+, KO^{[n]})$$

We have an isomorphism of simplicial sets $\mathbf{Hom}_{sPSh_\bullet(S)}(X_+, F) \xrightarrow{\sim} F(X)$ for any simplicial presheaf F and hence,

$$\mathbf{Hom}_{sPSh_\bullet(S)}(X_+, KO^{[n]}) \xrightarrow{\sim} KO^{[n]}(X)$$

for any scheme $X \in Sm_S$. This coupled with localization implies that there is an isomorphism of homotopy fibers in Ho**sSet**,

$$\mathbf{Hom}_{sPSh_\bullet(S)}(X_+/U_+, KO^{[n]}) \xrightarrow{\sim} KO^{[n]}(X, U) \tag{2.5.7}$$

We can go further and show that there is a spectrum $\mathbf{KO} \in SH_\bullet(S)$ such that it represents orthogonal and symplectic K-theory. More precisely,

Theorem 2.5.6. *S be any regular Noetherian scheme of finite dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Let $KO_f^{[n]}$ be a choice of fibrant replacement of $KO^{[n]}$ for each n. There exist morphisms $S^{2,1} \wedge KO_f^{[n]} \to KO_f^{[n+1]}$ in $\mathbf{Spc}_\bullet(S)$ such that the resulting motivic spectrum $\mathbf{KO} = (KO_f^{[0]}, KO_f^{[1]}, \ldots, KO_f^{[n]}, \ldots) \in SH_\bullet(S)$ satisfies*

$$Hom_{SH(S)}(S^{2n-i,n} \wedge X_+, \mathbf{KO}) \cong KO_i^{[n]}(X) \tag{2.5.8}$$

for all schemes $X \in Sm_S$ and for all $i, n \in \mathbb{Z}$.

Here $S^{2,1} \cong S_s^1 \wedge \mathbb{G}_m \cong \mathbb{P}^1 \cong \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\}) \in H_\bullet(S)$. It follows that,

$$Hom_{SH(S)}(S^{2n+4-i,n+2} \wedge X_+, \mathbf{BO}) \cong KSp_i^{[n]}(X). \tag{2.5.9}$$

Panin and Walter show this using the structure of oriented cohomology theories. We will go into more detail about this in Chapter 4. What we need for representability are the isomorphisms,

$$\mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\}) \wedge KO^{[n]} \xrightarrow{\sim} KO^{[n+1]}$$

in $H_\bullet(S)$. These are obtained using the Thom space isomorphisms,

$$th(E, \phi) : KO^{[n]}(X) \xrightarrow{\sim} KO^{[n+m]}(E, E - X)$$

defined for each symplectic space (E, ϕ). In fact, Panin and Walter prove the existence of Thom isomorphisms for the broader class of SL^c-bundles [PW18, Th.5.1].

Chapter 3

Geometry of Grassmannians

3.1 Introduction

In this chapter, we discuss the geometry of the Grassmannian schemes $Gr(r,n)$ and their open subschemes $RGr(r,n)$ and $HGr(2r,2n)$, the real and quaternionic Grassmannians respectively. These schemes are closely related to K-theory. Specifically the infinite Grassmannians $\mathbb{Z}\times Gr$, $\mathbb{Z}\times RGr$ and $\mathbb{Z}\times HGr$ represent algebraic, orthogonal and symplectic K-theory respectively in the unstable motivic homotopy category $H_\bullet(S)$ over well behaved schemes.

Definition 3.1.1. Let E be a vector bundle over a scheme S. A *subbundle* of E, denoted by $V\subset E$, is the equivalence class of monomorphisms of $\mathcal{O}_S$-modules, $V\rightarrowtail E$, such that the V and E/V are vector bundles, identified *up to isomorphisms*

$$\begin{array}{ccc} V & \rightarrowtail & E \\ \downarrow\wr & \nearrow & \\ W & & \end{array}$$

Theorem 3.1.1. *Let E be a vector bundle over a scheme S and $r\in\mathbb{N}$. Let* $\mathrm{Gr}(r,E)(-): Sch_S^{op}\to \mathrm{Set}$ *be the presheaf which sends an S-scheme, $f: X\to S$ to the set of subbundles $\{V\rightarrowtail f^*E \mid \mathrm{rank}(V)=r\}$. There exists a smooth scheme $Gr(r,E)\in Sm_S$, such that there is an isomorphism of functors*

$$Hom_{Sch_S}(-,Gr(r,E))\xrightarrow{\sim}\mathrm{Gr}(r,E)(-)$$

To prove this, we will first deal with the case when S is an affine scheme $Spec(R)$.

Definition 3.1.2. For any $n\in\mathbb{N}$ and $I\subseteq\{1,\dots,n\}$, let $R[x]_I$ be the quotient of the polynomial ring $R[x_{i,j}]$, $1\le i\le n$ and $1\le j\le r=|I|$, by the ideal generated by polynomials $\{x_{i_k,j}-\delta_{k,j}\}_{i_k\in I,j\le r}$, where i_k denotes the k^{th} element of I and $\delta_{i,j}$ is the Kronecker delta function.

$R[x]_I$ is isomorphic to the polynomial ring with $r(n-r)$ variables given by the $x_{i,j}$ not set to 1 or 0.

Lemma 3.1.2. *Let $S = Spec(R)$ be an affine scheme. There exist natural transformations $Hom(-, Spec(R[x]_I)) \to Gr(r,n)(-)$ for all $I \subset \{1, \dots n\}$ of size r such that $\{Hom(-, Spec(R[x]_I))\}_I$ is an open cover of $Gr(r,n)$*

Proof of Lemma. For simplicity of notation, denote $Hom(-, X)$ by h_X and $Hom(-, Spec(R))$ by h_R. For any scheme X, we can identify morphisms $X \to Spec(R[x]_I)$, with morphisms of $\mathcal{O}_X$-modules $f : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n}$, with $p_I \circ f : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus r}$ an isomorphism, where $p_I : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{O}^{\oplus r}$ is the projection onto the coordinates corresponding to I. This is then an embedding of $\mathcal{O}_X^{\oplus r}$ as a subbundle of $\mathcal{O}^{\oplus n}$, uniquely determined by such an f, and hence we have a morphism $h_{R[x]_I} \to Gr(r,n)$ as a subfunctor. Note that, even when S is non-affine we can define this subfunctor as consisting of all the subbundles which are isomorphic to $\mathcal{O}^{\oplus r}$, by the projection p_I. We will denote this subfunctor by $Gr(r,n)_I$. To show these give an open cover, we need to show that for any morphism $h_T \to Gr(r,n)$, where $Spec(T)$ is an affine scheme, the pullbacks $h_T \times_{Gr(r,n)} h_{R[x]_I}$ are all representable, say by U_I, and $\{U_I \to Spec(T)\}$ is an open cover. Further, as a morphism $h_T \to Gr(r,n)$ is equivalent to a direct summand $P \rightarrowtail T^{\oplus n}$, and for every projective module P there is a distinguished open cover $\bigcup_i D(f_i) = Spec(R)$, such that P_{f_i} is a free module, it is enough to consider morphisms $h_T \to Gr(r,n)$ corresponding to free subbundles $i : T^{\oplus r} \rightarrowtail T^{\oplus n}$. In such a case, the pullback $h_T \times_{Gr(r,n)} h_{R[x]_I}$ is represented by the affine scheme $Spec(T[det(p_I \circ i)]^{-1})$. As i is a submodule embedding, these form an open cover. $\square$

Note that $Gr(r,n)_I$ is still an open subscheme of $Gr(r,n)$ in the non-affine case and $Gr(r,n)_I \cong \mathbb{A}^{r(n-r)}$.

Proof of theorem. From the above lemma, to show that $Gr(r,n)$ is representable it is enough that it is a Zariski sheaf. This follows easily as giving a subbundle embedding $\mathcal{V} \hookrightarrow \mathcal{O}_X^n$ is equivalent to giving such subbundles on an open cover $X = \cup_i U_i$ which agree on intersections. As every vector bundle is locally trivial we get that $Gr(r, \mathcal{F})$ is representable over any scheme. As $Gr(r,n)_I \cong \mathbb{A}^{n(n-k)}$ is a smooth scheme, $Gr(r,n)$ is smooth. $\square$

From the above isomorphism, we get a rank r vector subbundle

$$\mathcal{U}_{r,E} \to E_{|_{Gr(r,E)}}$$

called the canonical bundle, over the Grassmannian, corresponding to the identity morphism $1_{Gr(r,E)} \in Hom(Gr(r,E), Gr(r,E))$. The isomorphism in one direction is given by sending a map

$$f : X \to Gr(r,E) \in Hom_{Sch_S}(X, Gr(r,E))$$

to the pullback $f^*\mathcal{U}_{r,E} \rightarrowtail \mathcal{O}^{\oplus n}$. We will mostly use the canonical bundles over $Gr(r,n)$, which we denote by $\mathcal{U}_{r,n}$. When E is understood, we will suppress it in the subscript. Let $\mathcal{U}_r \rightarrowtail \mathcal{O}_{Gr(r,n)}^{\oplus n}$ be the canonical bundle over $Gr(r,n)$, then $\mathcal{U}_{r|Gr(r,n)_I} \cong \mathcal{O}^{\oplus r}$ as giving a morphism $X \to Gr(r,n)_I$ is equivalent to giving a morphism $X \to Gr(r,n)$ which factors through $Gr(r,n)_I$. By construction this implies that $f : X \to Gr(r,n)$ factors through $Gr(r,n)_I$ if and only if,

$$\begin{array}{ccc} f^*\mathcal{U}_r & \rightarrowtail & \mathcal{O}^{\oplus n} \\ & \searrow^{\sim} & \downarrow p_I \\ & & \mathcal{O}^{\oplus r} \end{array}$$

It will be useful to have a description of the transition functions of $\mathcal{U}_r$ on the intersections $Gr(r,n)_I \cap Gr(r,n)_J$.

Theorem 3.1.3. *Let $S = Spec(R)$ be affine. Under the isomorphism $Gr(r,n)_I \cong Spec(R[x]_I)$ we have $Gr(r,n)_I \cap Gr(r,n)_J \cong Spec(R[x]_I)[(detM_J)^{-1}]$ where M_J is the $r \times r$-matrix with the ik^{th} entry is x_{ij_k}, and the transition functions $R[x]_I[(detM_J)^{-1}] \xrightarrow{\sim} R[x]_J[(detM_I)^{-1}]$ are given by sending x_{ij} to the ij^{th}-entry of $[x_{ij}]M_I^{-1}$.*

Proof. For every subbundle in $Gr(r,n)_I(X)$, we can associate an $n \times r$ matrix A, and this identification is unique if we set the $r \times r$ minor, with the rows corresponding to I, to the identity matrix. The identification $Gr(r,n)_{IJ} = Gr(r,n)_I \cap Gr(r,n)_J \cong Spec(R[x]_I)[(detM_J)^{-1}]$ comes from the fact that the intersection corresponds to subbundles $i : \mathcal{V} \rightarrowtail \mathcal{O}^{\oplus n}$ such that $p_I \circ i$ and $p_J \circ i$ are isomorphisms and therefore the $r \times r$ minors A_I and A_J of the associated $n \times r$ matrix A are invertible. For a subbundle $\mathcal{V} \in Gr(r,n)_{IJ}$, the representative matrices A_1 and A_2 differ by multiplication by the inverse of the J minor, $A_2 = A_1 A_{1J}^{-1}$. From this, we get the description of the transition function. □

We can use this description of the intersections to get the transition functions for the canonical bundles $\mathcal{U}_r$. We have $\mathcal{U}_r \cong \mathcal{O}^{\oplus r} \xrightarrow{i_I} \mathcal{O}^{\oplus n}$. Using the matrix identification in the above proof, it is easy to see that the transition functions $\mathcal{O}^{\oplus r}_{Gr(r,n)_{IJ}} \xrightarrow{\sim} \mathcal{O}^{\oplus r}_{Gr(r,n)_{IJ}}$ are given by the matrix M_J. We have enough details to show the following.

Theorem 3.1.4. *The tangent bundle of the morphism $Gr(r,E) \to S$ is given by, $T_S Gr(r,E) \cong Hom(\mathcal{U}_r, E/\mathcal{U}_r)$.*

Proof. This result is given in [Ful98, Sec.14.6]. Let $p : Gr(r,E) \to S$ be the structure map. Applying the second fundamental form construction to the exact sequence $\mathcal{U}_r \rightarrowtail p^*E \twoheadrightarrow p^*E/\mathcal{U}_r$, we get a morphism $\mathcal{U}_r \to \Omega^1_{Gr(r,E)/S} \otimes p^*E/\mathcal{U}_r$. Dualizing we get a morphism $T_S Gr(r,E) \to \mathcal{U}_r^\vee \otimes p^*E/\mathcal{U}_r$. It is easy to check on local coordinates that this is an isomorphism. □

3.2 Relations between Grassmannians

The description of the Grassmannian schemes as a presheaf gives us a clean way to give morphisms between them. Let F and E be a vector bundles over a scheme S of rank n and m respectively. There exist morphisms of smooth schemes $Gr(r,F) \to Gr(r, F \oplus E)$ and $Gr(r,F) \to Gr(r+m, F \oplus E)$, given by morphisms of functors

$$V \rightarrowtail F \mapsto V \rightarrowtail F \oplus E \quad \text{and} \quad V \rightarrowtail F \mapsto V \oplus E \rightarrowtail F \oplus E$$

respectively. Let i be $i : Gr(r, F) \to Gr(r, F \oplus E)$ and j be $j : Gr(r, F) \to Gr(r + m, F \oplus E)$ as above. These maps are characterized by the subbundles $i^*\mathcal{U}_{r,F\oplus E} \cong \mathcal{U}_{r,F}$ and $j^*\mathcal{U}_{r+m,F\oplus E} \cong \mathcal{U}_{r,F} \oplus E_{|Gr(r,F)}$ respectively. These maps between Grassmannians play an important role in the definition of the algebraic K-theory spectrum, so we will look at them more closely.

Theorem 3.2.1. *The map $i : Gr(r, F) \to Gr(r, F \oplus E)$ is a closed embedding with normal bundle $\mathcal{U}_r^\vee \otimes E$. There is an open embedding $f : \mathcal{U}_r^\vee \otimes E \to Gr(r, F \oplus E)$, such that i factors as $i = f \circ z$, where z is the zero section.*

Proof. For simplicity we will first deal with the case of $E \cong \mathcal{O}_S$. For any open subscheme $U \hookrightarrow S$, $F_{|U} \cong \mathcal{O}_U^{\oplus n}$ implies $F \oplus \mathcal{O}_{|U} \cong \mathcal{O}_U^{\oplus n+1}$. Therefore, it is enough to show that $Gr(r, n) \hookrightarrow Gr(r, n + 1)$ is a closed embedding, as being a closed embedding is local on the base. The schemes $Gr(r, n)_I$ can actually be described as vector bundles $Hom(\mathcal{O}^{\oplus r}, \mathcal{O}^{\oplus n}/\mathcal{O}_I^{\oplus r})$, where $\mathcal{O}_I^r$ denotes the embedding $\mathcal{O}^{\oplus r} \to \mathcal{O}^{\oplus n}$ given by elements of I. The above embedding then arises by gluing monomorphisms $i_I : Hom(\mathcal{O}^{\oplus r}, \mathcal{O}^{\oplus n}/\mathcal{O}_I^{\oplus r}) \hookrightarrow Hom(\mathcal{O}^{\oplus r}, \mathcal{O}^{\oplus n+1}/\mathcal{O}_I^{\oplus r})$. As monomorphisms between vector bundles induce closed embedding of corresponding schemes, $Gr(r, n) \hookrightarrow Gr(r, n + 1)$ is a closed embedding. In fact, the embedding is smooth and therefore the normal bundle is the cokernel of the map,

$$T_S Gr(r, F) \to i^* T_S Gr(r, F \oplus \mathcal{O}). \tag{3.2.1}$$

We know $T_S Gr(r, F) \cong Hom(\mathcal{U}_r, F/\mathcal{U}_r)$ and $i^*(\mathcal{U}_r \hookrightarrow F \oplus \mathcal{O}_{Gr(r,F\oplus\mathcal{O})}) \cong \mathcal{U}_r \hookrightarrow (F \oplus \mathcal{O}_{Gr(r,F)})$. Putting this together, the normal bundle is the cokernel of the map

$$Hom(\mathcal{U}_r, F/\mathcal{U}_r) \to Hom(\mathcal{U}_r, F \oplus \mathcal{O}/\mathcal{U}_r) \tag{3.2.2}$$

Therefore, the normal bundle is $Hom(\mathcal{U}_r, \mathcal{O}) = \mathcal{U}_r^\vee$. To construct the open embedding, consider the open subscheme of $Gr(r, F \oplus \mathcal{O})$ given by the functor

$$U(X \xrightarrow{f} S) = \{E \rightarrowtail f^*F \oplus \mathcal{O} \mid E \to f^*F \text{ is a monomorphism}\}$$

Note that this gives an open subscheme as the condition is locally given by the determinant of some $r \times r$ minor being invertible. It is easy to see that the image of $Gr(r, F)$ lands in U. We have a map $U \to Gr(r, F)$ which, as a morphism of functors, sends

$$E \rightarrowtail f^*F \oplus \mathcal{O} \mapsto E \rightarrowtail f^*F$$

Sections of this map correspond to maps $\mathcal{U}_r \to \mathcal{O}$ and hence we are done with the case $E \cong \mathcal{O}_S$. The general case follows along the same lines, 3.2.1 just becomes,

$$Hom(\mathcal{U}_r, F/\mathcal{U}_r) \to Hom(\mathcal{U}_r, F \oplus E/\mathcal{U}_r). \tag{3.2.3}$$

□

Similarly, we can prove a slightly weaker result about j,

Theorem 3.2.2. *The tangent bundle of the map j is $(F/\mathcal{U})_r^\vee$. There exists an open embedding $f : (F/\mathcal{U}_r)^\vee \hookrightarrow Gr(r + 1, F \oplus E)$, such that j factors as $j = f \circ z$, where z is the zero section.*

The proof follows along similar lines as the previous theorem. Hence j is an *immersion.* By the maps $Gr(r,F) \to Gr(r, F \oplus \mathcal{O}_S)$ and $Gr(r,F) \to Gr(r+1, F \oplus \mathcal{O}_S)$ we get a diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & Gr(r,n) & \longrightarrow & Gr(r,n+1) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & Gr(r+1,n+1) & \longrightarrow & Gr(r+1,n+2) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \end{array}$$

Taking the colimit in the category of presheaves, along the horizontal directions we get ind-schemes $Gr(r,\infty)$ and taking the colimit of the whole diagram we get the infinite Grassmannian $Gr(\infty,\infty)$ which we will denote plainly by Gr. We do not need all the $Gr(r,n)$ to define Gr, it is enough to have a cofinal subset of them. In particular we have $colim_n Gr(n,2n) \cong Gr$. Note that this colimit does not exist in the category of finite-dimensional schemes over S. We know that every projective module P, over a commutative ring R is a direct summand of a free module $P \oplus Q \cong R^{\oplus n}$.

Theorem 3.2.3. *Let $Spec(R) \to S$ be an affine S-scheme. For any projective module P of rank r, there exists a map $Spec(R) \to Gr(r,n)$ such that the corresponding submodule is $P \rightarrowtail R^{\oplus n}$. This implies, every P corresponds to some element in $Gr(Spec(R))$.*

It is important to note that there might be several distinct embeddings $P \rightarrowtail R^{\oplus n}$, so the above map is not unique, even as an element of $Gr(Spec(R))$.

3.3 The real and quaternionic Grassmannians

To define the motivic spectrum **BO**, we need open subschemes of $Gr(r,n)$ related to the structure of bilinear forms. Let ϕ be a bilinear form on a vector bundle F over S.

Definition 3.3.1. Define $Gr(r,F,\phi)$ to be the subfunctor of $Gr(r,F)$, given by

$$Gr(r,F,\phi)(X \xrightarrow{f} S) = \{E \subset f^*F \mid f^*\phi_{|E} \text{ is nondegenerate}\}.$$

When ϕ is a symmetric or a symplectic form (2.2.1), we denote $Gr(r,F,\phi)$ by $RGr(r,F,\phi)$ and $HGr(r,F,\phi)$ respectively.

$RGr(r,F,\phi)$ and $HGr(r,F,\phi)$ are called the real and quaternionic Grassmannians respectively. As there are no non-degenerate symplectic forms of odd rank, $HGr(r,F,\phi)$ is empty for r odd and ϕ a symplectic form.

Theorem 3.3.1. *$Gr(r,F,\phi)$ is representable and an open subscheme of $Gr(r,F)$. In particular, $HGr(r,F,\phi)$ and $RGr(r,F,\phi)$ are open subschemes of $Gr(r,F)$.*

Proof. As before, it is enough to show that if $\phi : \mathcal{O}_S^{\oplus n} \xrightarrow{\sim} (\mathcal{O}_S^{\oplus n})^\vee$ is a bilinear form, then $Gr(r, n, \phi) = Gr(r, \mathcal{O}_S^{\oplus n}, \phi)$ is an open subscheme of $Gr(r, n)$. On an affine open subscheme, $U = Spec(R)$, any bilinear form ϕ is given by a matrix M_ϕ, which is determined by our choice of basis, i,e $\phi(x, y) = x^T M_\phi y$ for all $x, y \in R^n$. ϕ is then non-degenerate if and only if M_ϕ is invertible. Restricting $Gr(r, n)_I$ to $Spec(R)$, we have $Gr(r, n)_I \cong Spec(R[x_{ij}]_I)$ with $j \in \{1, \ldots, n\} - I$ and $i \in I$. To show the functor $Gr(r, n, \phi)$ is representable, it is enough to show that there exist subfunctors $Gr(r, n, \phi)_I \hookrightarrow Gr(r, n, \phi)$ such that $Gr(r, n, \phi)_I$ are representable and form an open cover of $Gr(r, n, \phi)$.

Let $Gr(r, n, \phi)_I$ be the open subscheme of $Spec(R[x_{ij}]_I)$ given by

$$Spec(R[x_{ij}]_I[det(x_I^T M_\phi x_I)^{-1}])$$

where x_I is the matrix whose ij-th coordinate is x_{ij} in $R[x_{ij}]_I$. For any R-algebra A, the A-points of $Gr(r, n, \phi)_I$ are given by a collection of $r(n - r)$ elements of A such that the matrix A is invertible. This gives an embedding of $Gr(r, n, \phi)_I$ as a subfunctor. □

We denote $RGr(r, H_+^{\oplus n})$ and $HGr(2r, H_-^{\oplus n})$ by $RGr(r, 2n)$ and $HGr(2r, 2n)$ respectively. We have for each $HGr(r, F, \phi)$, a canonical symplectic subbundle $(U_{r,\phi}, \phi)$ and for each $RGr(r, F, \psi)$, a canonical symmetric subbundle $(V_{r.\psi}, \psi)$ corresponding to the identity elements. On $HGr(2r, 2n)$ and $RGr(r, n)$ we denote the canonical bundles by $U_{2r,2n}$ and $V_{r,n}$ respectively.

Definition 3.3.2 (Quaternionic projective space)**.** For any $n \geq 0$, we denote $HGr(2, 2n + 2)$ by HP^n and call it the *Quaternionic projective space.*

Remark 3.3.1. In [PW10] they define $RGr(r, 2n)$ not using symmetric spaces but quadratic spaces q_{2n}. These are equivalent over schemes where 2 is invertible, but give different subschemes of $Gr(r, n)$ otherwise.

Similar to the case of $Gr(r, F)$, we have morphisms (we suppress the form for simplicity)

$$RGr(r, F) \to RGr(r, F \oplus H_+) \quad \text{and} \quad RGr(r, F) \to RGr(r + 2, F \oplus H_+)$$

$$HGr(2r, F) \to HGr(2r, F \oplus H_-) \quad \text{and} \quad HGr(2r, F) \to HGr(2r + 2, F \oplus H_-).$$

One advantage when dealing with $RGr(r, F)$ and $HGr(r, F)$ is that for any symplectic (or symmetric subbundle) $\mathcal{U} \subset E$, there is a decomposition $E \cong \mathcal{U} \oplus \mathcal{U}^\perp$. Hence the canonical bundles have orthogonal complements.

Theorem 3.3.2. *The morphisms*

$$RGr(r, F) \to RGr(r, F \oplus H_+) \quad \textit{and} \quad HGr(2r, F) \to HGr(2r, F \oplus H_-)$$

are closed embeddings with normal bundles, $V_{r,n}^\vee \otimes \mathcal{O}^{\oplus 2}$ *and* $\mathrm{U}_{2r,2n}^\vee \otimes \mathcal{O}^{\oplus 2}$ *respectively. The tangent bundle of the morphisms*

$$RGr(r, F) \to RGr(r + 1, F \oplus H_+) \quad \textit{and} \quad HGr(2r, F) \to HGr(2r + 2, F \oplus H_-)$$

are $F/V_r^\vee \otimes \mathcal{O}^{\oplus 2}$ *and* $F/U_{2r}^\vee \otimes \mathcal{O}^{\oplus 2}$ *respectively.*

Proof sketch. These statements can be derived easily by extending 3.2.1 and 3.2.2 to $Gr(r,F) \to Gr(r, F \oplus \mathcal{O}^{\oplus 2})$ and $Gr(r,F) \to Gr(r+1, F \oplus \mathcal{O}^{\oplus 2})$ and using the fact that tangent bundles are well behaved with respect to restrictions to open subschemes. □

Remark 3.3.2. Note that as the canonical bundles $U_{2r,F}$ and $V_{r,E}$ come equipped with non-degenerate forms, we have $U_{2r,F} \xrightarrow{\sim} U_{2r,F}^{\vee}$ and $V_{r,E} \xrightarrow{\sim} V_{r,E}^{\vee}$.

We can also construct $RGr(r,\infty)$, $HGr(2r,\infty)$, RGr and HGr along the lines of the ordinary Grassmannians. We then have, $RGr = \operatorname{colim}_n RGr(n,2n)$ and $HGr = \operatorname{colim}_n HGr(2n,4n)$. As all these structure maps add a bundle of even rank, they do not change the parity of the initial subbundle. This means that RGr splits as $RGr = colim_n RGr(2n,4n) \coprod RGr(2n+1, 4n+2)$. We will now explore in more detail the geometry of $HGr(2r,2n)$. All of the following theorems are given in chapters 3 and 4 in [PW10].

Theorem 3.3.3. *Let (E,ϕ) be a symplectic space and $F := E \oplus H_-$.*

(a) The normal bundle $N = U_{r,\phi}^{\vee} \otimes \mathcal{O}^{\oplus 2}$ of $HGr(2r,E)$ in $HGr(2r,F)$ has a canonical embedding as an open subscheme of $Gr(2r,F)$ containing $HGr(2r,E)$.

(b) The closed subschemes $N^+ = HGr(2r,F) \cap Gr(2r, E \oplus \mathcal{O} \oplus 0)$ and $N^- = HGr(2r,F) \cap Gr(2r, E \oplus 0 \oplus \mathcal{O})$ of $HGr(2r,F)$ satisfy $N^+, N^- \subset N$ and $N^+ \cap N^- = HGr(2r,E)$.

(c) By the maps $N^+, N^- \to N \to HGr(2r,E)$, both N^+ and N^- are vector subbundles of N over $HGr(2r,E)$.

(d) There are natural vector bundle isomorphisms $N^+ \cong U_{2r,E} \cong N^-$.

(e) Let $\pi_+, \pi_- : N^+ \to HGr(2r,E)$ be the structure maps from above. Then $\pi_+^*(U_{2r,E}, \phi_{U_E})$ is isometric to $(U_F, (\phi_{U_F}, h_-))_{|N^+}$ where h_- is the form on H_- and similarly for π_-.

Proof. Consider the subfunctor $\mathcal{N}$ of $Gr(2r,F)$ given by all subbundles $V \rightarrowtail E \oplus H_-$ such that $V \to E$ is a monomorphism and ϕ_E is non-degenerate. Both these conditions can be described locally as certain matrices being invertible and therefore is representable by an open subscheme. Giving an element of $\mathcal{N}(T)$ is equivalent to giving a symplectic subbundle $V \rightarrowtail E$ and a morphism $V \to \mathcal{O}^{\oplus 2}$. This is equivalent to giving a morphism $f : T \to HGr(2r,E)$ and a pair of sections $f^*U_{2r,E} \to \mathcal{O}^{\oplus 2}$. Hence $\mathcal{N}$ is isomorphic to $N \cong U_{2r,\phi}^{\vee} \otimes \mathcal{O}^{\oplus 2} \cong Hom(U_{2r,\phi}, \mathcal{O}^{\oplus 2})$. Hence we have showed (a).

Giving a morphism $t : T \to N^+$ is equivalent to giving a symplectic subbundle $V \rightarrowtail t^*E \oplus \mathrm{H}_-$ such that the restriction $V \to \mathcal{O}_T \oplus 0$ is zero. The symplectic form on F is (ϕ, h_-). We write the morphism $V \rightarrowtail t^*E \oplus \mathrm{H}_-$ as $(i,f,0) : V \rightarrowtail t^*E \oplus H_-$, where $i : V \to t^*E$, $(f,0) : V \rightarrowtail \mathcal{O} \oplus \mathcal{O}$. As V is a symplectic subbundle, $(i,f,0)^{\vee}(\phi,h_-)(i,f,0) = (i^{\vee}\phi i, (f,0)^{\vee}h_-(f,0))$ is non-degenerate. $(f,0)^{\vee}h_-(f,0) = 0$ as h_- restricted to $\mathcal{O} \oplus 0$ is the zero form. From this we get that $i^{\vee}\phi i : V \to t^*E \to t^*E^{\vee} \to V^{\vee}$ is an isomorphism. A similar result holds for N^- and hence we have $N^+, N^- \subset N$. In fact, by the above description, N^+ and N^- are the subbundles $U_{2r,E}^{\vee} \otimes (\mathcal{O} \oplus 0)$ and $U_{2r,E}^{\vee} \otimes (0 \oplus \mathcal{O})$

of $U_{2r,E}^{\vee} \otimes \mathcal{O}^{\oplus 2}$ respectively and hence we have (b) and (c). We get (d) as $U_{2r,E}$ is a symplectic space and hence $U_{2r,E} \cong U_{2r,E}^{\vee}$.

For (e), we know that giving a morphism $T \to \pi_+^*(U_{2r,E}, \phi_{U_E})$ or $T \to (U_{2r,F}, (\phi_{U_F}, h_-)_{|N^+})$ is equivalent to giving a morphism $f : T \to N^+$ and a section of the corresponding subbundle V, hence $\pi_+^*(U_{2r,E}) \cong (U_{2r,F})$. The forms are equivalent by the description of N^+ above. □

Let $Y = HP^n - N^+$, where $N^+ = HP^n \cap Gr(2, \mathcal{O}^{2n} \oplus \mathcal{O} \oplus 0)$ is as above. Y is an open subscheme of HP^n and giving a morphism $T \to Y$ is equivalent to giving a morphism $f : T \to S$ and a rank 2 symplectic subbundle $V \rightarrowtail H_-^{\oplus n} \oplus H_-$ such that the morphism $\lambda : V \to \mathcal{O}$ induced by the last projection $p_{2n+2} : \mathcal{O}^{\oplus 2n+1} \oplus \mathcal{O} \twoheadrightarrow 0 \oplus \mathcal{O}$, is an epimorphism.

Theorem 3.3.4. *Y is the quotient of $\mathbb{A}^{4n+1}$ under the free group action of $\mathbb{G}_a = \mathbb{A}^1$ given by,*

$$\mathbb{G}_a \times \mathbb{A}^{2n} \times \mathbb{A}^{2n} \times \mathbb{A}^1 \to \mathbb{A}^{2n} \times \mathbb{A}^{2n} \times \mathbb{A}^1$$

$$t \cdot (a, b, r) = (a, b + ta, r + t(1 - \phi(a, b))), \tag{3.3.1}$$

where $\phi_n : \mathbb{A}^{2n} \times \mathbb{A}^{2n} \to \mathbb{A}^1$ is the standard symplectic form $H_-^{\oplus n}$.

This is [PW10, Th.3.4]. It is proved there in greater generality, here we give a shorter proof for our desired special case,

Proof. Let Z be the closed subscheme of $\mathbb{A}^{2n+2} \times \mathbb{A}^{2n+2}$ given by the equations

$$Z = \{(e, f) \in \mathbb{A}^{2n+2} \times \mathbb{A}^{2n+2} | p_{2n+2}(f) = 1, p_{2n+2}(e) = 0, \phi_{n+1}(e, f) = 1\} \tag{3.3.2}$$

By this definition, every pair $(e, f) \in Z$ defines a symplectic subspace $\langle e, f \rangle \rightarrowtail H_-^{\oplus n+1}$, where $\langle e, f \rangle \cong H_-$ is the subspace generated by e and f. This gives a morphism $\pi : Z \to HP^n$, sending $(e, f) \mapsto \langle e, f \rangle$. As $p_{2n+2}(f) = 1$, π factors through the subspace Y. Let $e = (a_1, a_2, \ldots, a_{2n}, a_{2n+1}, a_{2n+2})$ and $f = (b_1, b_2, \ldots, b_{2n}, b_{2n+1}, b_{2n+2})$. The conditions on Z give us $a_{2n+2} = 0$, $b_{2n+2} = 1$ and $b_{2n+1} + \sum_{j=1}^{n} a_{2j-1}b_{2j} - a_{2j}b_{2j-1} = 1$. This gives us an isomorphism $\mathbb{A}^{2n} \times \mathbb{A}^{2n+1} \xrightarrow{\sim} Z$, defined as

$$((a_1, \ldots, a_{2n}), (b_1, \ldots, b_{2n+1})) \mapsto$$
$$((a_1, \ldots, a_{2n}, 1 - \phi_{2n}((a_1, \ldots, a_{2n}), (b_1, \ldots, b_{2n})), 0), (b_1, \ldots, b_{2n+1}, 1)) \tag{3.3.3}$$

There is a $\mathbb{G}_a$-action on Z given by $t \cdot (e, f) = (e, f + te)$. This action agrees with 3.3.1 under the above isomorphism and $\langle e, f \rangle = \langle e, f + te \rangle$, hence $Z \to Y$ is a $\mathbb{G}_a$-bundle. We will show that $Z \to Y$ is a $\mathbb{G}_a$-torsor. The restriction of the canonical bundle to Y satisfies, $U_{2,2n+2|Y} \to \mathcal{O}_Y$ is an epimorphism. Let $Y = \bigcup_\alpha Y_\alpha$ be an open cover of Y such that, there exist splittings $s_\alpha : \mathcal{O}_{Y_\alpha} \to U_{2,2n+2|Y_\alpha}$. Let us fix such a splitting s_α for each α and let $Z_\alpha = Z \times_Y Y_\alpha$. Y has an open cover by the subschemes $Y_I = Y \cap Gr(2, 2n+2)_I$, such that every symplectic subbundle in Y_I has, as its underlying vector bundle, the trivial bundle $\mathcal{O}^{\oplus 2}$. Further, as every non-degenerate symplectic form on $\mathcal{O}^{\oplus 2}$ is isometric to H_-, every element of Y_I corresponds to a symplectic subspace isometric to H_-. Let $Y_{\alpha,I} = Y_I \cap Y_\alpha$ and $Z_{\alpha,I} = Z \times_Y Y_{\alpha,I}$. Suppose we have a symplectic subbundle $(V, \psi) \rightarrowtail H_{-T}^{\oplus n+1} \in Y_{\alpha,I}(T \to S)$, then $(V, \psi) \xrightarrow{\sim} H_-$ and there exists a splitting $s_{\alpha,V} : \mathcal{O}_T \to V$ of $p_{2n+2|V} : V \twoheadrightarrow \mathcal{O}_T$ by pulling back $s_\alpha : \mathcal{O}_Y \to U_{\alpha,I}$ over Y_α. Let $f_V = s_{\alpha,V}(1)$ and $e_V \in V$ be the unique element such that $\phi(e_V, -) = p_{2n+2|V}(-)$. This gives maps $\eta_{\alpha,I} : Y_{\alpha,I}(T \to S) \times \mathbb{G}_a(T \to S) \to Z_{\alpha,I}(T \to S)$, given by $(V \rightarrowtail H_{-T}^{\oplus n+1}, t) \mapsto (e_V, f_V + te_V)$. Suppose we have two elements of $Z_{\alpha,I}(T \to S)$, (e, f_1) and (e, f_2). Writing $f_2 = ae + bf_1$, we get $1 = \phi(e, f_2) = b$ and hence for a fixed e, the $\mathbb{G}_a$ action is transitive. Given a pair (e, f), the subspace $\langle e, f\rangle$ satisfies $\phi(e, -) = p_{2n+2|\langle e,f\rangle}(-)$ and hence $\eta_{\alpha,I}$ is surjective. For any two subspaces V and W, $(e_V, f_V + t_1 e_V) = (e_W, f_W + t_2 e_W)$ implies $V = \langle e_V, f_V\rangle = \langle e_W, f_W\rangle = W$ as subspaces and $f_V = s_{\alpha,V}(1) = s_{\alpha,W}(1) = f_W$. This finally gives us $t_1 = t_2$ and hence each $\eta_{\alpha,I}$ is injective. Therefore we have shown that $\eta_{\alpha,I}$ are bijections and hence $Z \to Y$ is a $\mathbb{G}_a$-torsor. □

3.4 Grassmannians and K-theory

Let S be a scheme. The algebraic K-theory presheaf over S is isomorphic to $\mathbb{Z} \times Gr = \coprod_{i\in\mathbb{Z}} Gr$ in $H_\bullet(S)$ (cf.[MV99]),

Theorem 3.4.1. *There exists an equivalence $\mathbb{Z} \times Gr \xrightarrow{\sim} K$ in $H_\bullet(S)$ and hence for any $X \in Sm_S$ and for all $n \in \mathbb{N}$,*

$$Hom_{H_\bullet(S)}(S^n \wedge X_+, \mathbb{Z} \times Gr) \cong K_n(X)$$

The main goal of this section is to prove an analogous result for symplectic K-theory.

Theorem 3.4.2. *The presheaves $\mathbb{Z} \times HGr$ and KSp are isomorphic as objects in $H_\bullet(S)$.*

The proof is very similar to the algebraic K-theory situation. We will prove this result in stages.

Lemma 3.4.3. *BSp_∞ is a unital magma in $H_\bullet(S)$.*

Proof. By 1.5.4 the homotopy colimit of the diagram $BSp_{2n} \to BSp_{2n+2}$ is $\mathbb{A}^1$-equivalent to the colimit BSp_∞. Similarly $BSp_\infty \times BSp_\infty \cong \operatorname{hocolim}_n BSp_n \times BSp_n$. For all $n \in \mathbb{N}$, we let $p_n : BSp_{2n} \times BSp_{2n} \to BSp_{4n}$ be induced by the group homomorphisms $Sp_{2n} \times Sp_{2n} \to Sp_{4n}$, given by $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. We will construct maps $\sigma_n : BSp_{4n} \to BSp_{4n}$ such that each σ_n is $\mathbb{A}^1$-homotopic to the identity and the diagram

$$\begin{array}{ccccc} BSp_{2n} \times BSp_{2n} & \xrightarrow{p_n} & BSp_{4n} & \longrightarrow & BSp_{4n+4} \\ \downarrow & & & & \downarrow{\scriptstyle \sigma_{n+1}} \\ BSp_{2n+2} \times BSp_{2n+2} & & \xrightarrow{p_{n+1}} & & BSp_{4n+4} \end{array} \tag{3.4.1}$$

commutes. For simplicity, let us look at 4×4 matrices. Consider the signed permutation matrix $Q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, then Q is orthogonal. Given any block matrix $\begin{pmatrix} A_{2\times 2} & 0 \\ 0 & B_{2\times 2} \end{pmatrix}$, the inner automorphism $M \mapsto QMQ^t$ permutes the two blocks and hence $Q \in Sp_4$. There is a morphism $Q_t : \mathbb{A}^1 \to Sp_4$ such that $Q_t(0) = I_4$ and $Q_t(1) = Q$. To construct this morphism, first note that Q has a decomposition

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We define Q_t to be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where we denote $\mathbb{A}^1_X \cong Spec(R[t])$ for any affine scheme $X = Spec(R)$. All three of these matrices in the product are symplectic, $Q_t(0) = I$ and $Q_t(1) = Q$. In 3.4.1 the image of the top row is contained in the subgroup of block matrices of the form $\begin{pmatrix} A_{2n\times 2n} & 0 & 0 \\ 0 & B_{2n\times 2n} & 0 \\ 0 & 0 & \Omega_4 \end{pmatrix}$, where $\Omega_{2n} = \oplus_{i=1}^n \Omega$ and the bottom half of the diagram has image of the form $\begin{pmatrix} A_{2n\times 2n} & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & B_{2n\times 2n} & 0 \\ 0 & 0 & 0 & \Omega \end{pmatrix}$.
Therefore $\sigma_n : BSp_{4n} \to BSp_{4n}$ needs to be an automorphism which permutes these blocks. The desired permutations can be decomposed into products of permutations of the form $(2i+1, 2i+3)(2i+2, 2i+4)$. Like the 4×4 case, this permutation can be replaced with the corresponding signed permutation, which will be of the form $I_{2i} \oplus Q \oplus I_{2j}$. Let σ_n be the automorphism corresponding to this permutation. It follows that $\sigma_n : BSp_{4n} \to BSp_{4n}$ is $\mathbb{A}^1$ homotopic to the identity and hence is an $\mathbb{A}^1$-weak equivalence. σ_n induces an isomorphism of homotopy colimits $BSp_\infty \xrightarrow{\sim} BSp_\infty$ in $H_\bullet(S)$. Hence we have an induced map of homotopy colimits $BSp_\infty \times BSp_\infty \to BSp_\infty$ in $H_\bullet(S)$. We can similarly use permutations to show $\mathrm{colim}_n\, I_{2n}$ is a unit of this operation. □

The following is essentially [MV99, Th.4.1] but tweaked to correct a mistake. The mistake was initially pointed out in [ST15].

Definition 3.4.1. A graded monoid object in $sSh_\bullet(T)$, where T is any site, consists of a simplicial sheaf of monoids $(M, +)$, morphisms $\alpha : \mathbb{N} \to M$ and $f : M \to \mathbb{N}$ in $Mon(sSh(T))$ such that $f\alpha = id$. Let $M_n = f^{-1}(n)$ be the n^{th} degree component. We define $M_\infty = \mathrm{colim}_n\, M_n$ to be the filtered colimit where the maps are addition of $\alpha(1)$, $\alpha(1)+ : M_n \to M_{n+1}$.

Lemma 3.4.4. *Let $(M, +, \alpha, f)$ be a graded monoid object in $sSh_\bullet(S)_{Nis}$, with $M_\infty = colim_n f^{-1}(n)$. Assume the following two conditions hold.*

1. *the map $a\underline{\pi}_0^{\mathbb{A}^1}(f) : a\underline{\pi}_0^{\mathbb{A}^1}(M) \to \mathbb{N}$ is an isomorphism of constant sheaves;*

2. *$(M,+)$ is commutative in $H_\bullet(S)$ under the induced monoidal structure;*

3. *The diagram*

$$\begin{array}{ccc} M_n \times M_n & \xrightarrow{+} & M_{2n} \\ \downarrow & & \downarrow{\scriptstyle \alpha(1)+} \\ M_{n+1}\times M_{n+1} & \longrightarrow & M_{2n+2} \end{array}$$

commutes in $H_\bullet(S)$

Then the canonical morphism $M_\infty \times \mathbb{Z} \to R\Omega^1_s BM$ is an $\mathbb{A}^1$-weak equivalence.

Here $a\underline{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ is just the sheaf of Homs in the $\mathbb{A}^1$-homotopy category,

$$U \mapsto Hom_{H_\bullet(S)}(U, \mathcal{X})$$

Proof. By 1.6.2 we can replace M with a termwise free simplicial monoid and hence assume $M^+ \cong R\Omega^1_s BM$. The $Ex_{\mathbb{A}^1}$ functor might not preserve the monoid structure, but by [MV99, 1.7], there is a $\mathbb{A}^1$-fibrant replacement functor $M \to Ex^{Mon}_{\mathbb{A}^1}(M)$ taking monoid objects to monoid objects. As $a\underline{\pi}_0^{\mathbb{A}^1}(f) : a\underline{\pi}_0^{\mathbb{A}^1}(M) \to \mathbb{N}$ is an isomorphism, $Ex^{Mon}_{\mathbb{A}^1}(M)$ is also graded as

$$a\underline{\pi}_0^{\mathbb{A}^1}(Ex^{Mon}_{\mathbb{A}^1}(M))(U) \cong \pi_0(Ex^{Mon}_{\mathbb{A}^1}(M))(U).$$

The morphism $M \to Ex^{Mon}_{\mathbb{A}^1}(M)$ induces an $\mathbb{A}^1$-weak equivalence of each graded component as they are disjoint. By 1.6.3 and 1.5.6 we get $\mathbb{A}^1$-weak equivalences $R\Omega^1_s BM \xrightarrow{\sim} R\Omega^1_s BEx^{Mon}_{\mathbb{A}^1}M$ and $M_\infty \xrightarrow{\sim} Ex^{Mon}_{\mathbb{A}^1}(M)_\infty$. Therefore, we can replace M and M_∞ with $Ex^{Mon}_{\mathbb{A}^1}(M)$ and $(Ex^{Mon}_{\mathbb{A}^1}(M))_\infty$ respectively and reduce the question to the situation where M is $\mathbb{A}^1$-fibrant. Then by 1.4.2 (2), (3) are equivalent to there being simplicial homotopies between the relevant morphisms. We can thus reduce to the case where $(M,+)$ is commutative in the simplicial homotopy category $H^s(Sm_S)$ and the diagram in (3) commutes up to simplicial homotopy. Now we need to show that $M_\infty \times \mathbb{Z} \to M^+$ is a weak equivalence of simplicial sheaves. As the Nisnevich site has enough points we use the stalk functors to reduce to the case where all objects are Kan complexes. The first two conditions then imply that the map $M_\infty \times \mathbb{Z} \to M^+$, where M^+ is the group completion of M, is a homology isomorphism. (3) implies that M_∞ is an H-space and therefore $\pi_1(M_\infty)$ is abelian and acts trivially on all higher homotopy groups. The map is then a weak equivalence by Whitehead's theorem. □

Applying this to $M = \coprod_n BSp_{2n}$ with $\alpha(n) = I_{2n}$, we get an $\mathbb{A}^1$-weak equivalence $\mathbb{Z}\times BSp_\infty \to R\Omega^1_s B(\coprod_n BSp_{2n})$. As $R\Omega^1_s B(\coprod_n BSp_{2n}) \xrightarrow{\sim} \Omega^1_s B(Symp) = KSp$ by 2.4.2, we get an isomorphism $\mathbb{Z}\times BSp_\infty \xrightarrow{\sim} KSp$ in $H_\bullet(S)$.

Definition 3.4.2. An acceptable gadget over an $X \in Sm_S$ is a sequence of smooth quasi-projective X-schemes $(U_i)_{i\in\mathbb{N}}$ and closed embeddings $f_i : U_i \to U_{i+1}$ of X-schemes such that for any henselian regular local ring B over S and any commutative square

$$\begin{array}{ccc} \partial\Delta^n{}_B & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ \Delta^n{}_B & \longrightarrow & X \end{array}$$

of morphisms of S-schemes, there exists a $j \geq i$ and a map $\Delta^n{}_B \to U_j$ making the following diagram commute.

$$\begin{array}{ccccc} \partial\Delta^n{}_B & \longrightarrow & U_i & \longrightarrow & U_j \\ \downarrow & & \nearrow & & \downarrow \\ \Delta^n{}_B & & \longrightarrow & & X \end{array}$$

Theorem 3.4.5. *Given any acceptable gadget (U_i, f_i) over an S-scheme X, the canonical morphism $colim_i U_i = U_\infty \to X$ is an $\mathbb{A}^1$-weak equivalence. In fact $Sing_*^{\mathbb{A}^1}(U_\infty) \to X$ is a Nisnevich weak equivalence.*

Proof. First let us consider the case when the acceptable gadget is over S. To prove that $Sing_*^{\mathbb{A}^1}(U_\infty) \to S$ is a Nisnevich weak equivalence and hence in order to prove that $U_\infty \to S$ is an $\mathbb{A}^1$-weak equivalence, it is enough to show that the induced map $Sing_*^{\mathbb{A}^1}(U_\infty) \to S$ is an $\mathbb{A}^1$-weak equivalence. Let $x : \text{Spec K} \to X$ be a point of X in the Nisnevich site $(Sm_S)_{Nish}$. As U_∞ is an ind-scheme $U_\infty(\mathcal{O}^h_{X,x})$ is equal to evaluating U_∞ at $\mathcal{O}^h_{S,s}$ as a presheaf on all S-schemes. Similarly $Sing_*^{\mathbb{A}^1}(U_\infty)(\mathcal{O}^h_{S,s})$ is also equal to evaluating $Sing_*^{\mathbb{A}^1}(U_\infty)$ at $\mathcal{O}^h_{S,s}$ as a simplicial presheaf. It is enough to show $Sing_*^{\mathbb{A}^1} U_\infty(\mathcal{O}^h_{X,x})$ is a contractible simplicial set for all points x. But as each $\mathcal{O}^h_{X,x}$ is a henselian local ring, the definition of acceptable gadget implies that $Sing_*^{\mathbb{A}^1} U_\infty(\mathcal{O}^h_{X,x}) \to *$ is an acyclic fibration. When the gadget is over a scheme $X \in Sm_S$, then $U_\infty \to X$ is an isomorphism in $H_\bullet(X)$ and the map $Lp_\#$1.4.9, where $p : X \to S$ is the structure map, preserves $\mathbb{A}^1$-weak equivalences and hence $U_\infty \to X$ is an isomorphism in $H_\bullet(S)$. □

For all $r, n \geq 0$, let $HU(2r, 2n) \to HGr(2r, 2n)$ be the principal Sp_{2r}-bundle associated to the canonical rank $2r$ symplectic subbundle $U_{2r,2n} \to HGr(2r, 2n)$ that is, $HU(2r, 2n)$ is locally isomorphic to Sp_{2r} with the same transition functions as $\mathrm{U}_{r,n}$. As is the case with the symplectic bundles, there are closed embeddings $f_i : HU(2r, 2n) \to HU(2r + 2, 2n + 2)$ induced by $H_-^{\oplus 2n} \hookrightarrow H_-^{\oplus 2n+2}$. Panin and Walter show in [PW18] that $(HU(2r, 2n), f_r)$ form an acceptable gadget, and in fact given any rank $2r$ symplectic bundle (E, ϕ) they show the corresponding principal Sp_{2r} bundles $HU(E, \phi; 2n) \to X \times HGr(2r, 2n)$ give an acceptable gadget $(HU(E, \phi; n), f_n)_{n\geq}$ over $X \times HGr(2r, 2n)$. Giving a map $Y \to HU(E, \phi; 2n)$ is equivalent to giving maps $f : Y \to X$ and $g : Y \to HGr(2r, 2n)$ and an isometry $i : f^*(E, \phi) \cong g^* U_{2r,2n}$. As $U_{2r,2n}$ is a subbundle of $H_-^{\oplus n}$, this is equivalent to giving $f : Y \to X$ and sections $(u_i)_{i\leq 2n} : \mathcal{O}_Y^{\oplus 2n} \to f^*E^\vee$ such that

$$f^*\phi = \sum_{i=1}^{n} u_{2i-1} \wedge u_{2i}$$

this is just the condition for the pullback to be a subbundle of $H_-^{\oplus n}$.

Lemma 3.4.6. *For any symplectic bundle (E, ϕ) of rank $2r$ over a scheme X, the pairs $(HU(E, \phi; 2n), f_n)_{n\geq 0}$ form an acceptable gadget over X.*

The proof in [PW18, Prop.8.5] extends to the case when 2 is not invertible.

Proof of 3.4.2. Applying 3.4.5 to $HU(E,\infty) = \mathrm{colim}_n(HU(E,\phi;2n), f_n)$, we get that $HU(E,\infty) \to X$ is an $\mathbb{A}^1$-weak equivalence. When $X = HGr(2r,2n)$ this implies

$$(HU(2r,2n) \times HU(2r,\infty))/Sp_{2r} \to HGr(2r,2n)$$

is an $\mathbb{A}^1$-weak equivalence. Taking the (ho)colimit we get that $(HU(2r,\infty) \times HU(2r,\infty))/Sp_{2r} \to HGr(2r,\infty)$ is an $\mathbb{A}^1$-weak equivalence. Inductively this gives us an $\mathbb{A}^1$-weak equivalence $E(HU(2r,\infty))/Sp_{2r} \to HGr(2r,\infty)$ where $E(HU(2r,\infty))_n = HU(2r,\infty)^{\times n+1}$ is sectionwise contractible. The quotients are taken as Nisnevich sheaves. Therefore $E(HU(2r,\infty))/Sp_{2r}$ has a contractible Sp_{2r}-torsor in the Nisnevich topology and hence we have $E(HU(2r,\infty))/Sp_{2r} \cong BSp_{2r}$ in $H^s(S)$. Therefore $BSp_{2r} \cong HGr(2r,\infty)$ in $H(S)$. Taking homotopy colimits we get

$$\mathbb{Z} \times HGr \cong \mathbb{Z} \times BSp_\infty \cong KSp$$

in $H_\bullet(S)$. □

We have a stronger result analogous to 3.4.1 in the case when $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$.

Theorem 3.4.7. *Let S be a regular noetherian scheme of finite dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For any $X \in Sm_S$ and for all $n \in \mathbb{N}$,*

$$Hom_{H_\bullet(S)}(S^n \wedge X_+, \mathbb{Z} \times HGr) \cong KSp_n(X).$$

The statement follows from 2.5.5 and 3.4.2.

Chapter 4

The Motivic spectrum BO

4.1 Introduction

In this chapter we come to the main goal of this thesis, extending Panin and Walter's construction of the motivic spectrum **BO** to $Spec(\mathbb{Z})$. In [PW18], they construct several spectra to represent hermitian K-theory. The ones which most straightforwardly represent hermitian K-theory are constructed using Schlichting's hermitian K-theory presheaves $KO^{[n]} : Sm_S^{op} \to \mathbf{sSet}$, seen as motivic spaces. However, these spaces are large and therefore don't have good geometric properties. Another model of **BO** they construct has $\mathbf{BO}_{4n} = \mathbb{Z} \times RGr$ and $\mathbf{BO}_{4n+2} = \mathbb{Z} \times HGr$, where HGr and RGr are the infinite quaternionic and real Grassmannian respectively. As HGr and RGr are ind-schemes, this construction is much more geometric and the structure maps $T^{\wedge 2} \wedge (\mathbb{Z} \times HGr) \to RGr$ and $T^{\wedge 2} \wedge (\mathbb{Z} \times RGr) \to HGr$ can be constructed using only the properties of the schemes $HGr(2r, 2n)$ and $RGr(r, n)$.

The two main ideas we need are, the fact that that $HP^1 \cong T^{\wedge 2}$ in $H_\bullet(S)$ and the existence of maps $HP^1 \times HGr \to RGr$ and $HP^1 \times RGr \to HGr$ arising from tensoring of vector bundles. This construction will give us a motivic spectra and thus it represents *some* cohomology theory. The relationship to hermitian K-theory arises along similar lines of the relationship between algebraic K-theory and the infinite Grassmannian Ind-scheme Gr. In [MV99], they were able to show that over a nice scheme S, we have an isomorphism $K \cong \mathbb{Z} \times Gr$ in $H_\bullet(S)$, where K is Quillen's algebraic K-theory presheaf. As Quillen's K-theory presheaf is a homotopy sheaf on the Nisnevich site and is $\mathbb{A}^1$-invariant, this gives us that $\mathbb{Z} \times Gr$ represents algebraic K-theory in the unstable homotopy category $H_\bullet(S)$. Along these lines, Panin and Walter in [PW18] and Schlichting and Tripathi in [ST15] respectively showed that $KSp \cong \mathbb{Z} \times HGr$ and $KO \cong \mathbb{Z} \times RGr$ in $H_\bullet(S)$ for a nice scheme where 2 is invertible. In chapter 3 we showed that the condition of 2 being invertible is unnecessary for the equivalence $KSp \cong \mathbb{Z} \times HGr$ (cf.3.4.2). However KSp is not yet known to satisfy Nisnevich excision or $\mathbb{A}^1$-invariance so we do not know yet what $\mathbf{BO}^{p,q}(\Sigma^\infty(X, x))$ looks like for $(X, x) \in Sm_{S*}$.

4.2 Hermitian K-theory spectrum

Recall that for any scheme S, we have simplicial presheaves $KO^{[n]} : Sm_S^{op} \to \mathbf{sSet}$, the hermitian K-theory with respect to the n shifted dualities (2.5.2). When S is a regular Noetherian scheme of finite dimension, with $\frac{1}{2} \in \Gamma(\mathcal{O}_S, S)$, there is a motivic spectrum $\mathbf{KO} = (KO^{[0]}, KO^{[1]}, KO^{[2]}, \ldots)$. We will go into some detail about the structure maps of $\mathbf{BO}$ here. The morphism $T \wedge KO^{[n]} \to KO^{[n+1]}$ is induced by a functor

$$Ch^b(Vect(X), \vee^n, \eta, q) \to Ch^b(Vect(\mathbb{A}^1 \times X)^{q_{(\mathbb{A}^1 \setminus 0) \times X}}, \vee^n, \eta, q)$$

Definition 4.2.1 (Koszul complex)**.** Let $p : E \to X$ be a vector bundle of rank n. The pullback $p^*E = E \times_X E \to E$ is a vector bundle over E (as a scheme) and has a section $s : E \to E \times_X E$ given by a diagonal map. The *Koszul complex* $\kappa(E)$ is a chain complex of vector bundles over E given by,

$$\kappa(E) : (0 \to \Lambda^n p^*E^\vee \to \Lambda^{n-1} p^*E^\vee \to \ldots \to \Lambda^2 p^*E^\vee \to p^*E^\vee \to \mathcal{O}_E \to 0)$$

with grading $\kappa(E)_i = \Lambda^{n-i}E^\vee$ and differentials $d : \Lambda^{k+1}p^*E^\vee \to \Lambda^k p^*E^\vee$ given by

$$d(x_0 \wedge x_1 \wedge \ldots \wedge x_k) = \sum_{i=0}^{n} (-1)^i s^*(x_i) x_0 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_k$$

where s^* is the dual of the section $s : \mathcal{O}_E \to p^*E$.

$\kappa(E)$ restricted to $E - X$ is exact. The canonical isomorphism $\Lambda^r p^*E \xrightarrow{\sim} (\Lambda^{n-r}p^*E)^\vee \otimes \Lambda^n p^*E$ induces an isomorphism of chain complexes $\theta(E) : \kappa(E) \xrightarrow{\sim} \kappa(E)^\vee \otimes \Lambda^n p^*E[n]$. Given an isomorphism $\lambda : detE = \Lambda^n E \xrightarrow{\sim} \mathcal{O}_X$, this gives us a non-degenerate symmetric form in $Ch^b(Vect(E), \vee^n, \eta, q)$,

$$\kappa(E, \lambda) : \kappa(E) \xrightarrow{\sim} \kappa(E)^\vee \otimes \Lambda^n p^*E[n] \xrightarrow{\sim} \kappa(E)^\vee[n]$$

where we choose the sign of the isomorphisms $\Lambda^r p^*E \xrightarrow{\sim} (\Lambda^{n-r}p^*E)^\vee \otimes \Lambda^n p^*E$ which are compatible with $*^n$. Given a pair (E, λ), where $p : E \to X$ is a rank n vector bundle and $\lambda : detE = \Lambda^n E \xrightarrow{\sim} \mathcal{O}_X$ an isomorphism between the determinant bundle and the trivial bundle, the Thom class $th(E, \lambda) = [(\kappa(E, \lambda)]$ is the corresponding element in $KO_0^{[n]}(E, E - X)$. From this we see that for every SL_n-bundle (E, λ), where $\lambda : \Lambda^n E \xrightarrow{\sim} \mathcal{O}_X$, we have a functor

$$Ch^b(Vect(X), \vee^m, \eta, q) \to Ch^b(Vect(E)^{q_{E-X}}, \vee^{m+n}, \eta, q)$$

which on objects is given by tensoring with $\kappa(E)$

$$C. \mapsto p^*C. \otimes \kappa(E).$$

This induces a map of orthogonal K-theory spaces $KO^{[m]}(X) \to KO^{[m+n]}(E, E - X)$ which we denote by $\otimes th(E, \lambda)$.

Theorem 4.2.1. *Let $X \in Sm_S$ with S a regular Noetherian scheme of finite dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For any pair (E, λ) described above, the map $\otimes th(E, \lambda) : KO^{[m]}(X) \to KO^{[m+n]}(E, E - X)$ is a weak equivalence of spaces.*

This is a corollary of [PW18, Th.5.1]. Note that this map is well defined over any scheme. We need the regularity and the invertibility of 2 for this map to be an isomorphism. For any scheme X, let $E = \mathcal{O}_X$ be the trivial bundle with $\lambda = id$. $\kappa(E, \lambda)$ is then,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{A}^1_X} & \xrightarrow{t} & \mathcal{O}_{\mathbb{A}^1_X} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle -1} & & \big\downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{A}^1_X} & \xrightarrow{-t} & \mathcal{O}_{\mathbb{A}^1_X} & \longrightarrow & 0 \end{array}$$

where t is the variable in $\mathbb{A}^1_{Spec(R)} \cong Spec(R[t])$. $\kappa(\mathcal{O}, id)$ induces an isomorphism

$$\otimes th(\mathcal{O}, id) : KO^{[n]}(X) \xrightarrow{\sim} KO^{[n+1]}(\mathbb{A}^1 \times X, (\mathbb{A}^1 \setminus \{0\}) \times X)$$

which is functorial on X. Therefore there is a levelwise weak equivalence of simplicial presheaves $KO^{[n]}(-) \xrightarrow{\sim} KO^{[n+1]}(\mathbb{A}^1 \times -, (\mathbb{A}^1 \setminus \{0\}) \times -)$. If $KO^{[n]}_f$ is the fibrant replacement of $KO^{[n]}$, we have a zigzag of $\mathbb{A}^1$-weak equivalences

$$\mathbf{Hom}_{sPSh_\bullet(S)}(-\wedge T, KO^{[n]}_f) \xleftarrow{\sim} \mathbf{Hom}_{sPSh_\bullet(S)}(-\wedge T, KO^{[n]}) \xrightarrow{\sim} KO^{[n]}(\mathbb{A}^1 \times -, (\mathbb{A}^1 \setminus \{0\}) \times -).$$

$\mathbf{Hom}_{sPSh_\bullet(S)}(- \wedge T, KO^{[n]}_f)$ is fibrant and hence we can lift this to get a weak equivalence of simplicial presheaves

$$KO^{[n]}_f(-) \xrightarrow{\sim} \mathbf{Hom}_{sPSh_\bullet(S)}(- \wedge T, KO^{[n]}_f).$$

The structure map is the adjoint of this map.

4.3 The geometric HP^1 spectrum BO

As before, the stable homotopy category $SH(S)$ is the stabilization of $H_\bullet(S)$ with respect to the functor $(X, x_0) \mapsto T \wedge (X, x_0)$ where $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$. There is a Quillen equivalence between $\mathbf{Spt}(S)_T$ and $\mathbf{Spt}(S)_{T^{\wedge 2}}$ given by the adjoint pair $(X_n) \mapsto (X_{2n})$ and $(X_n) \mapsto (X_0, T \wedge X_0, X_1, T \wedge X_1, \ldots)$, inducing an isomorphism $SH(S)_T \cong SH(S)_{P^{\wedge 2}}$. To construct our desired spectrum, we need a different model of the stable homotopy category which utilizes the quaternionic projective space $HP^1 = HGr(2, 4)$.

Theorem 4.3.1. *Let $x_0 : pt \to HP^1$ be the distinguished point corresponding to the subbundle $[H \oplus 0]$. There is an isomorphism $\eta : (HP^1, x_0) \cong T^{\wedge 2}$ in $H_\bullet(S)$.*

This is proved in [PW18]. We elaborate some of the arguments below.

Proof. Applying 3.3.3 to HP^1, we have an open subscheme $\mathbb{A}^4 \cong N \hookrightarrow Gr(2, 4)$, such that $N = N^+ \oplus N^-$, where $N^+, N^- \cong \mathbb{A}^2$ are closed subschemes of HP^1. By 3.3.4 $HP^1 - N^+$ is the quotient of $\mathbb{A}^5 = \mathbb{A}^2 \times \mathbb{A}^2 \times \mathbb{A}^1$ by the $\mathbb{G}_a$-action,

$$t \cdot (a, b, r) = (a, b + ta, r + t(1 - \phi(a, b)))$$

The inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{A}^5$ given by $t \mapsto (0, 0, 0, 0, t)$ is then $\mathbb{G}_a$-invariant and an $\mathbb{A}^1$-equivalence. Therefore we have an induced $\mathbb{A}^1$-equivalence of the quotients

$x_0 : pt \to HP^1 - N^+$ given by the subspace $0 \oplus 0 \oplus H_-$. The commutative square,

$$\begin{array}{ccc} pt & \longrightarrow & HP^1 \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle =} \\ HP^1 - N^+ & \hookrightarrow & HP^1 \end{array}$$

gives us an equivalence of pointed spaces $(HP^1, x_0) \xrightarrow{\sim} HP^1/(HP^1 - N^+)$ We have a similar square induced by the equivalence $N^- \xrightarrow{\sim} N$,

$$\begin{array}{ccc} N^- - 0 & \longrightarrow & N^- \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ N - N^+ & \hookrightarrow & N \end{array}$$

The left hand side is an equivalence as $N - N^+ = \mathbb{A}^4 - \mathbb{A}^2$ is a rank 2 vector bundle over $N^- - 0$. Hence we have $N^-/(N^- - 0) \cong N/(N - N^+)$. We then have a zigzag of equivalences

$$\begin{array}{ccccc} T^2 \cong \mathbb{A}^2/(\mathbb{A}^2 - 0) \cong N^-/N^- - 0 & \longrightarrow & HP^1/(HP^1 - N^+) & \xleftarrow[\sim]{} & (HP^1, x_0) \\ \downarrow{\scriptstyle\sim} & \searrow & \uparrow{\scriptstyle \text{excision}} & & \\ N/N - N^+ & \xleftarrow[\text{excision}]{} & N \cap HP^1/((N \cap HP^1) - N^+) & & \end{array}$$

Using the 2 out of 3 property twice, this gives us an equivalence $T^{\wedge 2} \xrightarrow{\sim} HP^1/(HP^1 - N^+)$ and hence $T^{\wedge 2} \cong (HP^1, x_0)$ in $H_\bullet(S)$. $\square$

Therefore we have $SH(S) \cong SH(S)_{T^{\wedge 2}} \cong SH(S)_{HP^1}$. We now have the desired model of $SH(S)$. In the category $SH(S)_{HP^1}$, we have an alternate description of the structure maps of **KO**. For any rank 2 symplectic bundle (E, ϕ) over a scheme X, $\phi : E \otimes E \to \Lambda^2 E \to \mathcal{O}_X$ is an isomorphism. Let $(U_{2,4}\phi_{2,4}) \to HP^1 \times X$ be the canonical symplectic bundle. The canonical morphism $U_{2,4} \rightarrowtail H_-^{\oplus 2}{}_{HP^1}$, restricts to a set of four maps $U_{2,4} \rightarrowtail \mathcal{O}_{HP^1}$. The pair which factors through $U_{2,4} \rightarrowtail H_-$ differ upto isomorphism only by a sign. Therefore denote these pairs by $(x_0, -x_0)$ and $(x_\infty, -x_\infty)$ respectively (this notation in consistent with the fact the these are isomorphisms when pulled back along points x_0 and x_∞). Consider the symmetric form

$$\begin{array}{ccccc} \mathcal{O}_{HP^1} & \longrightarrow & U_{2,4} & \xrightarrow{x_0} & \mathcal{O}_{HP^1} \\ \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle \phi_{2,4}} & & \downarrow{\scriptstyle 1} \\ \mathcal{O}_{HP^1} & \xrightarrow{(-x_\infty)^\vee} & U_{2,4}^\vee & \longrightarrow & \mathcal{O}_{HP^1} \end{array}$$

in $Ch^b(Vect(HP^1 \times X), -\eta, q)$, indexed from degrees 0 to 2. By construction this form is equal to $[U_{2,4}] - [H_-]$ in $KSp(HP^1 \times X)$ under

$$KO^{[2]}(HP^1 \times X) \xrightarrow{\sim} KSp^{[0]}(HP^1 \times X) \xrightarrow{\sim} KSp(HP^1 \times X)$$

and is the pullback of $\kappa(U_{2,4}, \phi)$ along the zero section $z : HP^1 \to U_{2,4}$. We will call this element of $KO_0^{[2]}(HP^1 \times X)$ the *Borel class* $-b_1(U_{2,4})$. The Borel class will give us the desired structure map. For simplicity, given two complexes $C.$ and $D.$ over two distinct schemes X and Y, we denote by $C. \boxtimes D.$ the complex $p_1^*C. \otimes p_2^*D.$ over $X \times Y$ where p_i are the projections. We will similarly use the $\boxtimes$ notation when taking tensor products of symplectic bundles $p_1^*U \otimes p_2^*V = U \boxtimes V$.

Theorem 4.3.2. *Let S be a regular Noetherian scheme of finite dimension with $\frac{1}{2} \in \Gamma(\mathcal{O}_S, S)$. The structure morphisms of* **KO** *are represented by maps*

$$KO^{[n]}(-) \to KO^{[n+2]}(- \times HP^1) \quad C. \mapsto C. \boxtimes (-b_1(\mathrm{U}_{2,4}))$$

Proof. It follows from taking the image of the Borel class under the zigzag of weak equivalences between HP^1 and $T^{\wedge 2}$. □

Let $\mathbb{Z} \times HGr := \mathrm{colim}_n [-n, n] \times HGr(2n, 4n)$, where by $[-n, n] \times X$ we mean disjoint union of $2n$ copies of X. Note that this is equivalent to $\coprod_n HGr$. Let

$$[-n, n]' = \{i \in \mathbb{Z} | -n \le i \le n \, and \, i \equiv n \bmod 2\}$$

For real Grassmannians, $RGr(2n, 4n)$ and $RGr(2n-1, 4n-2)$ behave differently. We define,

$$\mathbb{Z} \times RGr := \operatorname*{colim}_n [-2n, 2n]' \times RGr(2n, 4n) \cup [-2n+1, 2n-1]' \times RGr(2n-1, 4n-2)$$

Note that, in both these definitions, the terms in the colimit are schemes.

Lemma 4.3.3. *For all $n \ge 0$, there exist morphisms of pointed schemes*

$$f_{2n} : ([-n, n] \times HGr(2n, 4n)) \times HP^1 \to RGr(16n, 32n)$$

such that $f_{2n|(0, H_-^{\oplus n} \oplus 0) \times HP^1}$ is constant, $f_{2n|[-n,n] \times HGr(2n,4n) \times (H_- \oplus 0)}$ is $\mathbb{A}^1$-homotopic to a constant morphism and all the morphisms and homotopies are compatible with inclusions $HGr(2n, 4n) \to HGr(2(n+1), 4(n+1))$ and $RGr(16n, 32n) \to RGr(16(n+1), 32(n+1))$.

Proof. We have decompositions $U_{2n} \oplus U_{2n}^{\perp} \cong H_-^{\oplus 2n}$ for all n. As discussed earlier in 2.2.6, the tensor product of symplectic spaces are symmetric. This means we have inclusions of symmetric spaces over $HGr(2n, 4n) \times HP^1$,

$$U_{2n} \boxtimes U_2 \rightarrowtail H_-^{\oplus 2n} \boxtimes U_2 \quad \text{and} \quad U_{2n}^{\perp} \boxtimes H_- \rightarrowtail H_-^{\oplus 2n} \boxtimes H_-,$$

where we suppress the base schemes for simplicity. Further, for each $i \in [-n, n]$ we have

$$H_-^{\oplus n-i} \boxtimes U_2^{\perp} \rightarrowtail H_-^{\oplus 2n} \boxtimes U_2^{\perp} \quad \text{and} \quad H_+^{\oplus 2n+2i} \rightarrowtail H_+^{\oplus 4n},$$

putting these together, we get

$$(U_{2n} \boxtimes U_2) \oplus (U_{2n}^{\perp} \boxtimes H_-) \oplus (H_-^{\oplus n-i} \boxtimes U_2^{\perp}) \oplus H_+^{\oplus 2n+2i}, \tag{4.3.1}$$

which is a rank $16n$ symmetric subspace of the rank $32n$ symmetric space

$$(H_-^{\oplus 2n} \boxtimes U_2) \oplus (H_-^{\oplus 2n} \boxtimes H_-) \oplus (H_-^{\oplus 2n} \boxtimes U_2^{\perp}) \oplus H_+^{\oplus 4n}, \tag{4.3.2}$$

which is isometric to the hyperbolic space $H_+^{\oplus 16n}$. Hence we have $2n+1$ subspaces of $H_+^{\oplus 16n}$ each of which is rank $16n$. Hence we have a morphism

$$f_{2n} : ([-n,n] \times HGr(2n,4n)) \times HP^1 \to RGr(16n,32n). \tag{4.3.3}$$

restricting to $(0, H_-^{\oplus n} \oplus 0) \times HP^1$ we have $U_{2n} \cong U_{2n}^{\perp} \cong H^{\oplus n}$. So the subspace becomes

$$(H_-^{\oplus n} \boxtimes U_2) \oplus (H_-^{\oplus n} \boxtimes H_-) \oplus (H_-^{\oplus n} \boxtimes U_2^{\perp}) \oplus H_+^{\oplus 2n},$$

We choose the isometry to $H_+^{\oplus 16n}$ such that the subspace above goes to $H_+^{\oplus 8n} \oplus 0 \rightarrowtail H_+^{\oplus 16n}$. Hence the restriction corresponds to the point of $RGr(16n,32n)$ given by the subspace $H_+^{\oplus 8n} \oplus 0 \rightarrowtail H_+^{\oplus 16n}$. The restriction to $([-n,n] \times HGr(2n,4n) \times (H_- \oplus 0)$ gives us $U_2 \cong U_2^{\perp} \cong H_-$. The subspace then becomes

$$(U_{2n} \boxtimes H_-) \oplus (U_{2n}^{\perp} \boxtimes H_-) \oplus (H_-^{\oplus n-i} \boxtimes H_-) \oplus H_+^{\oplus 2n+2i}$$

For each i, this is isometric to $H_+^{\oplus 8n}$ but to different summands of $H_+^{\oplus 16n}$.

As given in 2.2.6, there are signed permutations which permute 2×2 diagonal blocks of the matrix Ω_{2n}, and hence induce isomorphisms of these subbundles, which are $\mathbb{A}^1$-homotopic to identity. The composition

$$HGr(2n,4n) \times HP^1 \to HGr(2n+2,4n+4) \times HP^1 \to RGr(16+16,32n+32),$$

for a fixed i, is given by the subspace

$$(U_{2n} \oplus H_- \boxtimes U_2) \oplus (U_{2n}^{\perp} \oplus H_- \boxtimes H_-) \oplus (H_-^{\oplus n+1-i} \boxtimes U_2^{\perp}) \oplus H_+^{\oplus 2n+2+2i},$$

collecting the terms representing the subspace of f_{2n} we get

$$(U_{2n}\boxtimes U_2)\oplus(U_{2n}^{\perp}\boxtimes H_-)\oplus(H_-^{\oplus n-i}\boxtimes U_2^{\perp})\oplus H_+^{\oplus 2n+2i}\oplus H_-\boxtimes(U_2\oplus U_2^{\perp})\oplus H_-\boxtimes H_-\oplus H_+^{\oplus 2}.$$

The last three terms add up to $H_+^{\oplus 8}$. It is easy to verify that the composition

$$HGr(2n,4n) \times HP^1 \to RGr(16n,32n) \to RGr(16n+16,32n+32),$$

gives the same subspace. Hence the maps are compatible with the inclusions $HGr(2n,4n) \to HGr(2n+2,4n+4)$ and $RGr(16n,32n) \to RGr(16n+16,32n+32)$. □

From the above lemma, we get a morphism of pointed spaces $f : (HP^1)^+ \wedge \mathbb{Z} \times HGr \to RGr$ (cf.1.8.5), where $\mathbb{Z} \times HGr$ is pointed by $(0,[H_-])$ and RGr is pointed by $[H_+]$. The next lemma can be proved analogously,

Lemma 4.3.4. *For all $n \geq 0$, there exist morphisms of pointed schemes*

$$g_n : ([-n,n]' \times RGr(n,2n)) \times HP^1 \to HGr(8n,16n)$$

such that, $g_{n|(0,H_+^{\oplus n}\oplus 0)\times HP^1}$ is constant when n is even, $g_{n|[-n,n]\times RGr(2n,4n)\times(H_+\oplus 0)}$ is $\mathbb{A}^1$-homotopic to a constant morphism and all the morphisms and homotopies are compatible with inclusions $RGr(n,2n) \to RGr(n+1,2(n+1))$ and $HGr(8n,16n) \to HGr(8(n+1),16(n+1))$.

Proof. The proof is the same except for a change in indices. g_n is defined on the i^{th} component, where $i \in [-n, n]'$, by the subspace

$$(V_n \boxtimes V_1) \oplus (V_n^{\perp} \boxtimes H_+) \oplus (H_+^{\oplus \frac{n-i}{2}}) \oplus (H^{\oplus n+1} \boxtimes V_1^{\perp}) \tag{4.3.4}$$

□

From the above lemma, we get a morphism $g : (HP^1)^+ \wedge \mathbb{Z} \times RGr \to HGr$. Putting these together we get the main theorem by 1.8.5.

Theorem 4.3.5. *For any scheme S, there exists an HP^1-spectrum $\mathbf{BO}_S$ such that $\mathbf{BO}_{2i+1} = \mathbb{Z} \times HGr$, $\mathbf{BO}_{2i} = \mathbb{Z} \times RGr$ and structure maps given by f and g respectively.*

This proof is essentially given in [PW18] but has the extra assumption of $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Here we see that we do not need it. Let $f : S_1 \to S_2$ be a morphism schemes. We have $f^*(HGr(2r, 2n)_{S_2}) \cong HGr(2r, 2n)_{S_1}$ and $f^*(\mathbb{Z} \times HGr_{S_2}) \cong \mathbb{Z} \times HGr_{S_1}$. This pullback isomorphism is also compatible with the structure maps of **BO** as the tensor product is preserved under pullback. By 1.8.9 we then have the following.

Theorem 4.3.6. *For any morphism $f : S_1 \to S_2$, there is an isomorphism $Lf^*\mathbf{BO}_{S_2} \xrightarrow{\sim} \mathbf{BO}_{S_1}$ in $SH(S_1)$. In particular, for any scheme S, $\mathbf{BO}_S$ is isomorphic to the pullback of $\mathbf{BO}_{\mathbb{Z}} = \mathbf{BO}_{Spec(\mathbb{Z})}$ by the structure map $S \to Spec(\mathbb{Z})$.*

Proof. By 1.8.9 we have that $Lf^*\Sigma^\infty HGr(2r, 2n)_{S_2+} \cong \Sigma^\infty HGr(2r, 2n)_{S_1+}$. We have by modifying 1.8.1, $\mathbf{BO} = \mathrm{hocolim}_n \Sigma^{-4n,-2n}\Sigma^\infty \mathbb{Z} \times HGr$. Again $\Sigma^\infty \mathbb{Z} \times HGr_+$ is a homotopy colimit of $\Sigma^\infty\{i\} \times HGr(2n, 4n)$, as suspension preserves homotopy colimits. The result then follows by 1.5.2. □

Remark 4.3.1. In such cases **BO** is called an *absolute spectrum* (cf.[Dég18, Def.1.1.1]).

4.4 Properties of BO

Having constructed **BO** we will look at some of its properties. The first thing to note is that **BO** over $Spec(\mathbb{Z}[\frac{1}{2}])$ gives us back the spectrum in [PW18]. We then have the representability result for hermitian K-theory.

Theorem 4.4.1. *The HP^1-spectrum **BO** is isomorphic to $\mathbf{KO}_{HP^1}$ over $Spec(\mathbb{Z}[\frac{1}{2}])$. Hence **BO** represents hermitian K-theory over regular Noetherian schemes with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$.*

Proof. We have isomorphisms $\tau_{4n+2} : \mathbb{Z} \times HGr \xrightarrow{\sim} KSp \xrightarrow{\sim} KO^{[4n+2]}$ in $H_\bullet(S)$ for all $n \geq 0$. We need to show that the diagram

$$\begin{array}{ccc} HP^1 \wedge HP^1 \wedge (\mathbb{Z} \times HGr) & \longrightarrow & \mathbb{Z} \times HGr \\ \downarrow{\scriptstyle 1\wedge 1 \wedge \tau_{4n-2}} & & \downarrow{\scriptstyle \tau 4n+2} \\ HP^1 \wedge HP^1 \wedge KO^{[4n-2]} & \longrightarrow & KO^{[4n+2]} \end{array}$$

commutes in $H_\bullet(S)$. To see this we use the fact that $\mathbb{Z} \times HGr$ is an ind-scheme and hence the restriction $\{i\} \times HGr(2r, 2n) \hookrightarrow \mathbb{Z} \times HGr \xrightarrow{\sim} KSp$ is classified by an element in $KSp_0(HGr(2r, 2n))$. The map $BSp_{2n} \to BSymp$ in $H_\bullet(S)$ is given at the level of schemes by sending principal Sp_{2n}-bundles to the associated symplectic bundle. Consequently the map $HGr(2r, 2n) \to BSp_{2n} \to BSymp$ corresponds to the tautological symplectic bundle $\mathrm{U}_{r,n} \to HGr(2r, 2n)$. From this and the definition of the map $\mathbb{Z} \times M_\infty \to R\Omega^1 BM$ it follows that the isomorphism $\tau : \mathbb{Z} \times HGr \to KSp$ satisfies

$$\begin{aligned}\tau_{|\{i\}\times HGr(2r,2n)} &= [U_{2r,2n}] + (i - r)[H_-] \\ &\quad \in KSp_0(HGr(2r, 2n)) \to Hom_{H_\bullet(S)}(HGr(2r, 2n), KSp).\end{aligned}$$

The last map is an isomorphism over regular Noetherian schemes S with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. We denote by τ_{4k+2} the isomorphism

$$\mathbb{Z} \times HGr \xrightarrow{\tau} KSp \xrightarrow{\sim} KO^{[4k+2]}$$

where $KSp \xrightarrow{\sim} KO^{[4k+2]}$ is the isomorphism induced by $X \mapsto X[2k + 1]$. We then have

$$\tau_{4k+2|\{i\}\times HGr(2n,4n)} = ([U_{2n,4n}] + (i - 2n[H_-]))[2k + 1]$$

Similarly the map $HP^1 \wedge HP^1 \wedge (\mathbb{Z} \times HGr) \to \mathbb{Z} \times HGr \to KSp$ restricted to $HP^1 \times HP^1 \times \{i\} \times HGr(2n, 4n)$ is given by $([U_{2,4}] - [H_-]) \boxtimes ([U_{2,4}] - [H_-]) \boxtimes ([U_{2n,4n}] + (i - 2n)[H_-])$ (to see this note that $[U^\perp_{2n,4n}] = 2n[H] - [U_{2n,4n}]$). But tensoring twice with $([U_{2,4}] - [H_-])$ is exactly the structure map of **KO** (4.3.2) and hence the diagram commutes when restricted to the finite Grassmannians. As $\mathbb{Z} \times HGr$ is the colimit of $\{i\} \times HGr(2n, 4n)$ we have a map

$$Hom_{H_\bullet(S)}(\operatorname{colim}_n HGr(2n, 4n), X) \to \lim_n Hom_{H_\bullet(S)}(HGr(2n, 4n), X)$$

for any $X \in \mathbf{Spc}_\bullet(S)$. This is an isomorphism if

$$Hom_{H_\bullet(S)}(S^1_s \wedge HGr(2n + 2, 4n + 4), X) \to Hom_{H_\bullet(S)}(S^1_s \wedge HGr(2n, 4n), X)$$

is a surjection. To see this take a fibrant replacement of X to get the set of simplicial homotopy classes. Surjectivity then implies that we can lift a collection of homotopy classes, uniquely up to homotopy, to the colimit. This holds for $X = KO^{[k]}$ as then we have

$$Hom_{H_\bullet(S)}(S^1_s \wedge HGr(2n, 4n), KO^{[k]}) \cong KO^{[k]}_1(HGr(2n, 4n))$$

and the maps $KO^{[k]}_i(HGr(2n + 2, 4n + 4)) \to KO^{[k]}_i(HGr(2n, 4n))$ are surjections by [PW10, Th.11.4] applied to $KO^{[*]}_*$ which is a cohomology theory with a -1-commutative ring structure [PW18, Th.1.4]. ☐

Remark 4.4.1. Note that we needed $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ to get the isomorphism $KSp \xrightarrow{\sim} KO^{[2k+1]}$ as only then do we have the identification between skew-symmetric and alternating forms.

We can also extend the cellularity result in [RSØ18] to arbitrary schemes.

Theorem 4.4.2. *Let S be any scheme. The motivic spectrum $\mathbf{BO}_S$ is cellular.*

Proof. The proof is essentially given in [RSØ18]. By 4.3.6 and 1.8.10, it is enough prove this over $Spec(\mathbb{Z})$. As before we have $\mathbf{BO} = \text{hocolim}_n\, \Sigma^{-4n,-2n}\Sigma^\infty \mathbb{Z} \times HGr$. Therefore by 1.8.11 and 1.8.8(3) it is enough to show that $\Sigma^\infty HGr(2n, 4n)_+$ is cellular for each n. □

Lemma 4.4.3. *Let $m \geq 0$, the suspension spectrum of the Thom space of $U_{2r,2n}^{\oplus m}$ on $HGr(2r, 2n)$ is a finite cellular spectrum. In particular when $m = 0$ we get $\Sigma^\infty HG(2r, 2n)_+$.*

Proof. The proof is by induction on r and n. As $HGr(0, 2n) \cong pt$ the statement holds for $r = 0$. Let $0 \leq r \leq n$ and Y be the open subscheme $HGr(2r, 2n) \setminus N^+$ (where N^+ is a summand of the normal bundle of $HGr(2r, 2n-2) \to HGr(2r, 2n)$ (by 3.3.3). By [Spi10, Lem.3.5] the cofiber of the map

$$Th(U_{2r,2n|Y}^{\oplus m}) \to Th(U_{2r,2n}^{\oplus m}),$$

is isomorphic to $Th(U_{2r,2n|N^+}^{\oplus m} \oplus \mathcal{N})$ where $\mathcal{N}$ is the normal bundle of the closed embedding $N^+ \to HGr(2r, 2n)$. We have $U_{2r,2n|N^+} \cong \pi_+^* U_{2r,2n-2}$ by 3.3.3(3) and in fact the proof shows us that $\pi_+^* U_{2r,2n-2} \cong \mathcal{N}$. We therefore have a cofiber sequence

$$Th(U_{2r,2n|Y}^{\oplus m}) \to Th(U_{2r,2n}^{\oplus m}) \to Th(\pi_+^* U_{2r,2n-2}^{\oplus m+1}).$$

π_+ is the structure map of a vector bundle and hence an $\mathbb{A}^1$-weak equivalence. We then have $Th(\pi_+^* U_{2r,2n-2}^{\oplus m+1}) \cong Th(U_{2r,2n-2}^{\oplus m+1})$ by 1.7.3. By induction on n we have reduced to showing that $\Sigma^\infty Th(U_{2r,2n|Y}^{\oplus m})$ is cellular. By [PW10, Th.5.1] we have

$$Y \leftarrow Y_1 \leftarrow Y_2 \to HG(2r-2, 2n-2)$$

where every map is an affine bundle and there is an isomorphism of symplectic bundles

$$U_{2r,2n|Y_2} \cong \mathcal{O}_{|Y_2}^{\oplus 2} \oplus U_{2r-2,2n|Y_2}.$$

Furthermore, the map $Y_2 \to HG(2r-2, 2n-2)$ has a section by the proof of [PW10, Th.5.2] and hence every scheme in the sequence has a point. By 1.7.1, 1.7.2 and 1.7.3 we then have isomorphisms

$$Th(U_{2r,2n|Y}) \cong Th(U_{2r,2n|Y_2}) \cong Th(\mathcal{O}_{|Y_2}^{2m} \oplus U_{2r-2,2n|Y_2}^{\oplus m}) \cong S^{2m,m} \wedge Th(U_{2r-2,2n}^{\oplus m}).$$

Therefore we are done by induction. □

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