Dissertation zur Erlangung des Doktorgrades (Dr. rer. nat.)
des Fachbereichs Mathematik/Informatik der Universität
Osnabrück

Combinatorial and algebraic properties of balanced simplicial complexes

vorgelegt von:
Lorenzo Venturello

Betreuer:
Prof. Dr. Martina Juhnke-Kubitzke

June 12, 2019

"*We often hear that mathematics consists mainly of 'proving theorems.' Is a writer's job mainly that of 'writing sentences'?*"

Gian-Carlo Rota

Contents

List of Figures

List of Tables

List of Symbols

$[n]$	$\{1, \ldots, n\}$
Δ_d	the d-simplex
$\langle F_1, \ldots, F_m \rangle$	the simplicial complex generated by $F_1, \ldots, F_m$
$\lvert\Delta\rvert$	the geometric realization of Δ
$V(\Delta)$	the vertex set of Δ
$\Delta * \Gamma$	the join of two simplicial complexes Δ and Γ
$\mathrm{lk}_\Delta(F)$	the link of a face F in Δ
$\mathrm{st}_\Delta(F)$	the star of a face F in Δ
$\mathrm{Skel}_j(\Delta)$	the j-th skeleton of Δ
$f(\Delta)$	the f-vector of Δ
$h(\Delta)$	the h-vector of Δ
$\partial\Delta$	the boundary complex of Δ
$\mathcal{C}_d$	the d-dimensional cross-polytope
$\mathbb{F}$	a field
$\mathbb{F}[\Delta]$	the Stanley-Reisner ring of Δ
I_Δ	the Stanley-Reisner ideal of Δ
$\mathbb{F}[x_1, \ldots, x_n]$	the polynomial ring in n variables with coefficients in $\mathbb{F}$
$\mathrm{Hilb}(R, t)$	the Hilbert series of a graded ring R
$H_i(\Delta; \mathbb{F})$	the i-th vector space of simplicial homology of Δ
$\widetilde{H}_i(\Delta; \mathbb{F})$	the i-th vector space of reduced simplicial homology of Δ
$\beta_i(\Delta; \mathbb{F})$	the i-th topological Betti number of Δ
$\beta^S_{i,i+j}(R)$	the graded Betti numbers of the graded S-module R
$\overset{\mathsf{PL}}{\cong}$	the PL-homeomorphism

Introduction

In the study of topological spaces and manifolds discretization plays often a major role. Among the many different structures and techniques employed in this field, simplicial complexes are definitely of crucial importance extending to geometry and combinatorics. Moreover, the recent development of computational methods, which allowed for experiments and applications, produced a wide variety of new research problems. In particular, the following question gathered a lot of attention: What conditions does the topology/geometry pose on the number of faces that a simplicial complex can have? In 1971 McMullen conjectured a characterization of these numbers for the case of boundary complexes of simplicial polytopes, a special class of simplicial complexes homeomorphic to spheres. Not even a decade later his conjecture became the celebrated g-theorem thanks to the work of Billera, Lee [BL81] and Stanley [Sta80]. Stanley's revolutionary contribution consisted in tackling the problem with commutative algebra tools such as the Cohen-Macaulay property, which enters the picture via a natural correspondence between simplicial complexes and squarefree monomial ideals. Interestingly, even though most of the combinatorial types of simplicial spheres cannot be realized convexly, the main conjecture in the area asserts that there is no face vector of a simplicial sphere which cannot be attained by a simplicial polytope. After almost 40 years from Stanley's paper, Adiprasito [Adi18] recently announced a proof confirming this conjecture on its highest level of generality. Anyway many other problems remain open, especially regarding simplicial complexes with additional combinatorial properties. In this thesis we focus on a family of complexes with a certain coloring property. The notion of vertex coloring of a graph dates back to the 19th century, with the work of mathematicians like Cayley, Heawood and Kempe, while one of the first references for a theory of coloring applied to higher dimensional objects is due to Fisk [Fis77a]. In 1979 Stanley [Sta79] defined *completely balanced* simplicial complexes as pure d-dimensional simplicial complexes endowed with a minimal vertex coloring, i.e., a coloring using $d+1$ colors. His main result makes ingenious use of combinatorial commutative algebra to study the face numbers of balanced Cohen-Macaulay complexes, but the definition can be naturally extended to more general, i.e., non-pure, simplicial complexes. From now on we shall use the term balanced simplicial complexes to refer to an arbitrary d-dimensional $(d+1)$-colorable simplicial complex, as it is customary in more modern literature.

There are many reasons to be interested in balanced complexes, since they often appear in different areas of combinatorics. As an example, order complexes of graded posets are balanced, and therefore so is the barycentric subdivision of any regular CW-complex. This allows us to interpret balanced complexes as a generalization of the former and use refined information on their numbers of faces via flag numbers. As we will see in Chapter 2, this idea yields interesting results under the viewpoint of face enumeration of balanced spheres and manifolds. An even broader family of balanced complexes is given by Coxeter complexes, whose importance in algebraic combinatorics cannot be overstated.

In combinatorial topology a minimal vertex coloring of a simplicial complex corresponds to a simplicial map to the simplex of the same dimension, with the property

that the dimension of faces is preserved under such a map. In this way we can characterize balanced simplicial complexes as those d-dimensional simplicial complexes which admit such a non-degenerate simplicial map to the d-simplex.
From a more algebraic viewpoint, the coloring map yields an $\mathbb{N}^d$-grading of the associated Stanley-Reisner ring which guarantees the existence of an $\mathbb{N}^d$-homogeneous system of parameters.
The goal of this thesis is to exhibit several results on balanced simplicial complexes under different perspectives, from pure combinatorics, to combinatorial topology and commutative algebra. The original contribution is contained in Chapters 3, 4, 5 and 6, whose content can be found in the preprints [JV17; JV18a; JV18b; Ven18]. During the preparation of this thesis, [JV17] has been accepted for publication in Algebraic Combinatorics, while an extended abstract of [Ven18] has been accepted for a software demonstration at FPSAC 2019 and will appear in the proceedings volume of the Séminaire Lotharingien de Combinatoire.
Each of these chapters is concluded with a paragraph briefly describing some related open problems.

Summary of the thesis

In Chapter 1 we begin with some basic definitions on simplicial complexes, their geometric realizations and some interesting decomposition properties which will appear in the following chapters. Next we introduce the Stanley-Reisner correspondence between simplicial complexes and squarefree monomial ideals. Since Stanley's celebrated proof of the g-theorem many results in face enumeration made use of standard commutative algebra tools, such as the study of the Hilbert series of graded algebras and the theory of Cohen-Macaulay rings. We give an overview of this topic and conclude the chapter with the classical lower and upper bound theorems for the face numbers of polytopes, spheres and manifolds.

Chapter 2 treats balanced simplicial complexes, and along the way we give examples and state elementary properties. In analogy with the first chapter we state known results on face numbers of balanced complexes. Even though a complete characterization of f-vectors of balanced polytopes remains elusive, a special set of linear forms in the Stanley-Reisner ring of balanced Cohen-Macaulay complexes led to interesting proofs of lower bound theorems for spheres and manifolds, which we sketch and comment.

In Chapter 3 we prove a result which answers a question raised in [Duv+16]. Stanley conjectured that every Cohen-Macaulay simplicial complex is *partitionable*, a property which in particular guarantees non-negativity of the h-vector. This is known to hold true for simplicial balls which are convexly realizable. In 2016, Duval, Goeckner, Klivans and Martin [Duv+16] disproved Stanley's conjecture by constructing an infinite sequence of 3-dimensional counterexamples, obtained by gluing together along a fixed subcomplex sufficiently many copies of a particular simplicial complex. Among the special cases left open was the question whether balanced Cohen-Macaulay complexes are partitionable. In this chapter we answer this question in the negative via a balanced modification of the original counterexamples, hence providing an infinite sequence of 3-dimensional balanced Cohen-Macaulay non-partitionable simplicial complexes. Moreover, as in [Duv+16], our counterexamples are even *constructible*.

The focus of Chapter 4 is on the balanced counterpart of a classical result in combinatorial topology. In 1978 Pachner [Pac78] proved that two combinatorial manifolds are PL-homeomorphic if and only if they can be transformed one into the other via a sequence of moves called *bistellar flips*. Clearly this result still holds for balanced combinatorial manifolds, but since bistellar flips do not preserve the minimal coloring, the intermediate steps will not be balanced. Izmestiev, Klee and Novik [IKN17] defined a coloring-preserving analog of bistellar flips called *cross-flips*, based on the boundary complex of the cross-polytope, and they proved that two balanced combinatorial manifolds are PL-homeomorphic if and only if they can be transformed one into the other via a sequence of cross-flips. In particular they proved that a subfamily of cross-flips, called *basic cross-flips*, suffices to relate any two PL-homeomorphic balanced combinatorial manifolds, and asked for a characterization and enumeration of those flips. We provide an explicit description, showing that there are $2^{d+1} - 1$ non-isomorphic basic cross-flips. Moreover, we show that some of them can be obtained as combination of others, from which we infer that roughly half of the basic cross-flips still suffice to connect any two PL-homeomorphic balanced combinatorial manifolds. A second theorem by Pachner [Pac91] states that two combinatorial manifolds *with boundary* are PL-homeomorphic if and only if they can be transformed one into the other via a sequence of shellings and inverse shellings. As in the aforementioned case without boundary, inverse shellings do not preserve balancedness in general. Our main result in Chapter 4 shows that there exists a sequence of shellings and inverse shellings preserving the coloring which connects any two balanced PL-homeomorphic balanced combinatorial manifolds with boundary, hence solving Problem 1 in [IKN17]. We also obtain the following intermediate result: if two PL-homeomorphic balanced combinatorial manifolds have a combinatorially isomorphic boundary, and this isomorphism preserves the coloring, then the two can be connected via a sequence of cross-flips. A similar result for the case of bistellar flips was proved in [Cas95].

Chapter 5 has a more algebraic flavor. Via the Hilbert function of the Stanley-Reisner ring of a simplicial complex it is possible to obtain bounds on the number of faces that a simplicial complex can have if, for instance, the homology is prescribed. A finer algebraic invariant which is even more closely related with simplicial homology is the graded minimal free resolution of a Stanley-Reisner ring, when regarded as a module over the polynomial ring. In Chapter 5 we study upper bounds for the graded Betti numbers of the Stanley-Reisner ring of a balanced complex in various cases. We start with arbitrary balanced simplicial complexes, for which the bounds follow from a direct inspection of Hochster's formula. Next we provide two strategies for the case of balanced Cohen-Macaulay complexes, based respectively on the lex and the lex-plus-squares ideals associated with a certain Hilbert function. Finally we obtain bounds for the linear strand of the resolution in the case of *normal pseudomanifolds* and we compute the graded Betti numbers of *cross-polytopal stacked spheres*, a family of polytopal spheres which plays a special role in the balanced lower bound theorem.

In the last chapter we investigate from a computational viewpoint balanced 2- and 3-manifolds with few vertices. The *barycentric subdivision* of an arbitrary (non-balanced) triangulation of any topological space provides a balanced triangulation of the same space, with a large vertex set. Currently no other systematic procedure to convert arbitrary triangulations into balanced ones is known. We present an implementation of cross-flips with the purpose of reducing the number of vertices of a balanced triangulation of a manifold, starting from barycentric subdivisions. Especially in the 3-dimensional case an effective reduction strategy is non-trivial, and requires to include flips which increase the number of vertices in order to avoid local minima. After discussing the details of the program, we exhibit several small balanced triangulations, whose lists of facets are included in Appendix A. In particular, we found a unique vertex-minimal balanced triangulation of the real projective plane on 9 vertices and several vertex-minimal balanced triangulations of the real projective space on 16 vertices, one of which has a very strong symmetry. We found a triangulation of the Poincaré homology 3-sphere on 26 vertices and, via its iterated suspensions, balanced non-combinatorial d-spheres with $2d + 20$ vertices for every $d \geq 5$. Using obstructions from knot theory we exhibit a 3-sphere with 28 vertices and one with 22 vertices, which are the smallest balanced non-shellable and shellable but non-vertex-decomposable 3-spheres known. Regarding simplicial complexes which are not combinatorial manifolds, we present a balanced triangulation of the dunce hat on 11 vertices, and we prove that this is indeed vertex-minimal. Finally we run our program on some normal 3-pseudomanifolds which are not combinatorial manifolds, and list the smallest balanced f-vectors found for several homeomorphism types.

Acknowledgements

An exhaustive list of people who deserve to be mentioned here would double the length of the thesis.
I wish to express my deepest gratitude to Prof. Martina Juhnke-Kubitzke for her guidance, patience and invaluable help. In these three years she always found the time to answer my admittedly many questions and never made me miss her support. It is a great honor for me to be the first of what I am sure will become a long list of grateful PhD students.
My friend and office mate Jan-Marten Brunink deserves a special thanks for all the moments we enjoyed together in the office, traveling for conferences and at the stadium. He never gave up in his noble attempt to teach me German, even when the toughness of his goal became evident.
I want to thank Prof. Holger Brenner for hosting me in my first month in Osnabrück and Davide Alberelli for showing me around and helping me taking my first steps in the city and in the university. They made me feel home from the very beginning.
In the institute I could always sense a great and positive atmosphere which made these three years go by very quickly. With Carina Betken, Gilles Bonnet, Stephan Bussmann, Alex Grosdos, Jens Grygierek, Arun Kumar, Dinh Le Van, Dominik Nagel, Alex Nover, Richard Sieg, Jonathan Steinbuch, Sabrina Syed, Paula Verdugo, Konrad Voelkel and Markus Wageringel I had a great time, both inside and outside the department. I will miss especially the coffee breaks, the waffle breaks, the ice cream breaks and all the (rather many) other excuses to take a break.
In many occasions I benefited from the experience and knowledge of the professors in the department, and I am grateful to all of them.
I wish to thank all the secretaries, with a special mention for Maria Anna Gausmann, for their patience and help filling in the many unpronounceable German forms.
Attending a number of workshops in Germany and beyond, I had the chance to share new experiences and laughs with Christopher Borger, Giulia Codenotti, Rodica Dinu, Frank Röttger and Oscar Iglesias Valiño. Together with Giulia we managed to make the world a little more octahedral, a task that would not have been accomplished without her tenacity.
I owe special thanks to Davide Bolognini, Alessio D'Alì and Hailun Zheng for their help. I wish them the best luck for their careers.
I wish to express my gratitude to Aldo Conca for the hospitality and the help during the week I spent in Genova, and to Timo De Wolff, Eran Nevo, Isabella Novik, Paco Santos and Raman Sanyal for listening attentively to my talks and providing insightful comments.
In the non-math Osnabrück I had a great time with Matthew Adamson, Leonid Berov, Leonardo Pimpini and everyone in the Spielverein 16.
I was always looking forward to meeting my lifelong friends Davide, Emanuele, Francesco, Marco and Nicolò on my trips back to Italy. Their encouragement and support has been fundamental during these three years.
I want to thank Irene, Marco, Noemi and Rosanna for the holidays we shared and Simone and Stefano for hosting me anytime I felt like taking a break.
Finally, I am deeply grateful to my parents for their constant support.

Chapter 1

Background on simplicial complexes

1.1 Basic notions

An (abstract) *simplicial complex* Δ on a finite vertex set $V = V(\Delta)$ is a collection of subsets of $V(\Delta)$ that is closed under inclusion. Elements $F \in \Delta$ are called *faces* of Δ, and maximal faces with respect to inclusion are called *facets*. We sometimes use $\mathcal{F}(\Delta)$ to denote the set of facets of Δ. The *dimension* of a face F is $\dim(F) := |F| - 1$ and the *dimension* of Δ is $\dim(\Delta) := \max\{\dim(F) \ : \ F \in \Delta\}$. 0-dimensional and 1-dimensional faces are also called *vertices* and *edges* of Δ, respectively. If all facets of Δ are of the same dimension, then Δ is called *pure*. A basic but important combinatorial invariant associated to a d-dimensional simplicial complex Δ is its *f-vector* $f(\Delta) = (f_{-1}(\Delta) \ldots, f_d(\Delta))$, where $f_j(\Delta)$ denotes the number of j-dimensional faces of Δ. Often, it is more convenient to consider the so-called *h-vector* $h(\Delta) = (h_0(\Delta), \ldots, h_{d+1}(\Delta))$ of Δ, defined by the equality

$$\sum_{i=0}^{d+1} f_{i-1}(\Delta)(t-1)^{d+1-i} = \sum_{i=0}^{d+1} h_i(\Delta) t^{d+1-i}, \tag{1.1}$$

which yields

$$h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d+1-i}{d+1-j} f_{i-1}(\Delta) \ \text{ and } \ f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{d+1-i}{d+1-j} h_i(\Delta).$$

This last definition is motivated by several algebraic and combinatorial reasons that will appear in different chapters of this thesis. Note that the h-numbers can be a priori negative, and that since the f-numbers can be expressed as non-negative combination of the h-numbers inequalities on the latter imply inequality on the former. A *subcomplex* Γ of Δ is any simplicial complex $\Gamma \subseteq \Delta$. For any subset $W \subseteq V$ we define the subcomplex *induced* by W as

$$\Delta_W := \{F \in \Delta : F \subseteq W\}.$$

Hence a subcomplex $\Gamma \subseteq \Delta$ is induced if any $F \subseteq V(\Gamma)$ with $F \in \Delta$ is a face of Γ. Given a collection of subsets $F_1, \ldots, F_m$ of a set V, we denote by $\langle F_1, \ldots, F_m \rangle$ the smallest simplicial complex that contains $F_1, \ldots, F_m$, i.e.,

$$\langle F_1, \ldots, F_m \rangle := \{F \subseteq V \ : \ F \subseteq F_i \text{ for some } 1 \le i \le m\}.$$

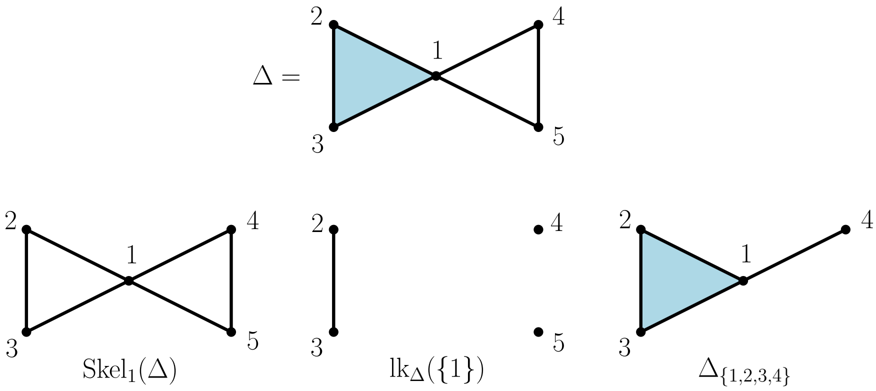

FIGURE 1.1: The 1-skeleton, a vertex link and an induced subcomplex of Δ.

To every face $F \in \Delta$ we associate two simplicial complexes, namely the *link* $\mathrm{lk}_\Delta(F)$ and the *star* $\mathrm{st}_\Delta(F)$ of F in Δ, providing a local description of Δ around F:

$$\mathrm{lk}_\Delta(F) := \{G \in \Delta \ : \ F \cup G \in \Delta, F \cap G = \emptyset\},$$

$$\mathrm{st}_\Delta(F) := \{G \in \Delta \ : \ F \cup G \in \Delta\}.$$

Note that if Δ is pure so are $\mathrm{lk}_\Delta(F)$ and $\mathrm{st}_\Delta(F)$, and $\dim(\mathrm{lk}_\Delta(F)) = \dim(\Delta) - \dim(F) - 1$. The *deletion* of a face $F \in \Delta$ describes the simplicial complex Δ outside of F:

$$\Delta \setminus F := \{G \in \Delta \ : \ F \nsubseteq G\}.$$

Similarly, the *deletion* $\Delta \setminus \Gamma$ of a subcomplex Γ from a simplicial complex Δ is defined as

$$\Delta \setminus \Gamma := \langle \mathcal{F}(\Delta) \setminus \mathcal{F}(\Gamma) \rangle,$$

The *join* of the simplicial complexes Δ and Γ on disjoint vertex sets is

$$\Delta * \Gamma := \{F \cup G \ : \ F \in \Delta, G \in \Gamma\}.$$

As an example $\mathrm{st}_\Delta(F) = \mathrm{lk}_\Delta(F) * \langle F \rangle$, and in general $\dim(\Delta * \Gamma) = \dim(\Delta) + \dim(\Gamma) + 1$. If F consists of a single vertex then we omit brackets and simply write $\Delta \setminus v$, $\mathrm{lk}_\Delta(v)$ and $\mathrm{st}_\Delta(v)$ for the deletion, link and star of $F = \{v\}$ in Δ. Borrowing terminology from classical topology we define the *cone* and the *suspension* over Δ to be $\langle \{v\} \rangle * \Delta$ and $\langle \{v\}, \{w\} \rangle * \Delta$ respectively, where v and w are new vertices not in $V(\Delta)$. Another subcomplex associated to Δ is its *j-skeleton*

$$\mathrm{Skel}_j(\Delta) := \{F \in \Delta \ : \ \dim(F) \leq j\},$$

consisting of all faces of dimension at most j (for $0 \leq j \leq \dim(\Delta)$). Finally we say that two simplicial complexes Δ and Γ are *combinatorially isomorphic*, and we write $\Delta \cong \Gamma$, if there exists a face-preserving bijection between their vertex sets, i.e., one can be transformed into the other by relabeling the vertices.

1.2 Spheres, manifolds and pseudomanifolds

There exists a functorial operation which associates to each abstract simplicial complex Δ a topological space $|\Delta|$ called the *geometric realization* of Δ. Explicitly one can construct $|\Delta|$ in the following way: consider the vectors in the standard basis

$\{e_v : v \in V(\Delta)\}$ of the Euclidean space $\mathbb{R}^{f_0(\Delta)}$ and for $F \in \Delta$ let X_F be the convex hull of the vectors $\{e_v : v \in F\}$. Then $|\Delta| = \bigcup_{F\in\Delta} X_F$. The number $f_0(\Delta)$ is in general far from being the smallest dimension in which Δ can be embedded: indeed a fundamental results in combinatorial topology states that every d-dimensional simplicial complex can be realized in $\mathbb{R}^{2d+1}$, see [Mat03, Theorem 1.6.1]. Anyway, up to homeomorphism, the geometric realization of an abstract simplicial complex is unique. We will often visualize abstract simplicial complexes via their geometric realization, as in Figure 1.1, which depicts some of the definitions in the first paragraph. In what follows we denote by Δ_d the *d-simplex*, that is any d-dimensional simplicial complex isomorphic to $\{F : F \subseteq [d+1]\}$ and with $\partial\Delta_d$ its boundary, which is $\partial\Delta_d \cong \langle [d+1] \setminus \{i\} : i \in [d+1]\rangle$. Note that $|\Delta_d|$ is a d-ball and $|\partial\Delta_d| \cong S^{d-1}$. We define now four fundamental classes of simplicial complexes.

Definition 1.1. A pure d-dimensional simplicial complex Δ is:

- an *$\mathbb{F}$-homology d-sphere* if $\widetilde{H}_i(\mathrm{lk}_\Delta(F);\mathbb{F}) \cong \widetilde{H}_i(S^{d-\dim(F)-1};\mathbb{F})$ for every face $F \in \Delta$ and for a fixed field $\mathbb{F}$;
- a *simplicial d-sphere* if $|\Delta| \cong S^d$;
- a *combinatorial (or PL) d-sphere* if $\Delta \overset{\mathsf{PL}}{\cong} \partial\Delta_{d+1}$, or equivalently if Δ and $\partial\Delta_{d+1}$ have a common subdivision;
- a *polytopal d-sphere* if $\Delta \cong \partial P$, where P is a *simplicial $(d+1)$-polytope*, i.e., a polytope in which all faces are simplices.

While it is not hard to see that the four classes in Definition 1.1 are listed in decreasing level of generality, the reverse containments deserve some discussion. All four definitions coincide for $d = 2$, and every simplicial 3-sphere is combinatorial. In 1965 Grünbaum [Grü03, Section 11.5] showed that there exist simplicial 3-spheres that are not polytopal, while Poincaré described the first instance of a (triangulable) 3-manifold that has the homology of S^3 but a non-trivial fundamental group. Surprisingly it is an open problem whether every simplicial 4-sphere is combinatorial, but for $d \geq 5$ the following result by Edwards and Cannon provides a negative answer.

Theorem 1.2. [Edw75; Can79] *The double suspension of any homology d-sphere is homeomorphic to S^{d+2}.*

On the level of simplicial complexes this implies that taking twice the suspension of any triangulation Γ of the Poincaré homology 3-sphere (or any other homology 3-sphere) yields a simplicial 5-sphere $\Delta = \langle\{v_1\},\{v_2\}\rangle * \langle\{w_1\},\{w_2\}\rangle * \Gamma$, with the property that $\mathrm{lk}_\Delta(\{v_1, w_1\}) = \Gamma$. Since the class of combinatorial spheres is closed w.r.t. taking links we have that Δ is not a combinatorial 5-sphere. Finally we mention a result of Adiprasito and Izmestiev [AI15] showing that iterated *barycentric subdivision* (see Chapter 2 for a definition) transforms every combinatorial sphere into a polytopal one.
These different families of spheres are the building blocks for the corresponding families of manifolds.

Definition 1.3. A pure d-dimensional simplicial complex Δ is:

- an *$\mathbb{F}$-homology d-manifold* if $\widetilde{H}_i(\mathrm{lk}_\Delta(F);\mathbb{F}) \cong \widetilde{H}_i(S^{d-\dim(F)-1};\mathbb{F})$ for every non-empty face $F \in \Delta$ and for a fixed field $\mathbb{F}$;

- a *simplicial d-manifold* if $|\Delta| \cong M$ for some triangulable compact d-manifold M without boundary;
- a *combinatorial (or PL) d-manifold* if $\mathrm{lk}_\Delta(v)$ is a combinatorial $(d-1)$-sphere for every vertex v;
- a *locally polytopal d-manifold* if $\mathrm{lk}_\Delta(v)$ is a polytopal $(d-1)$-sphere for every vertex v.

We turn now our attention to the case of manifolds with non-empty boundary, which will be the main focus of Chapter 4.

Definition 1.4. A pure d-dimensional simplicial complex Δ is: '

- a *combinatorial d-ball* if $\Delta \overset{\mathrm{PL}}{\cong} \Delta_d$;
- a *combinatorial d-manifold with boundary* if for any vertex $v \in \Delta$ the link $\mathrm{lk}_\Delta(v)$ is either a combinatorial $(d-1)$-sphere or a combinatorial $(d-1)$-ball.

If Δ is a combinatorial d-manifold with boundary, then its *boundary complex* $\partial\Delta$ is defined as

$$\partial\Delta := \{F \in \Delta \ : \ \mathrm{lk}_\Delta(F) \text{ is a combinatorial ball}\} \cup \{\emptyset\}.$$

Equivalently, the boundary complex of Δ, which is itself a combinatorial $(d-1)$-manifold without boundary, is the simplicial complex whose facets are the $(d-1)$-faces of Δ contained in exactly one facet of Δ. We will take this as the definition of the boundary complex of any simplicial complex Δ. So, in particular $\partial\Delta = \emptyset$, if Δ is a manifold without boundary. If $F \in \Delta$ is a face of a simplicial complex Δ, we write ∂F for its boundary complex, i.e., $\partial F = \{G \subsetneq F\}$. We define the *interior* $\overset{\circ}{\Delta}$ of a combinatorial d-manifold with boundary Δ to be the set $\overset{\circ}{\Delta} = \Delta \setminus \partial\Delta$. Observe that if $\partial\Delta \neq \emptyset$, then $\overset{\circ}{\Delta}$ is not a simplicial complex.

Remark 1.5. In the rest of this thesis, in particular in Chapter 4, we will use the word (homology/simplicial/combinatorial) manifold in the sense of Definition 1.3, i.e., to indicate triangulations without boundary.

A simplicial complex is *strongly connected* if its dual graph (see Definition 6.5) is connected.

Definition 1.6. A pure d-dimensional strongly connected simplicial complex is a *pseudomanifold* if every $(d-1)$-dimensional face is contained in exactly two facets. A d-dimensional pseudomanifold Δ is a *normal d-pseudomanifold* if the link of each face of dimension at most $d-2$ is connected.

The class of normal d-pseudomanifolds coincides with the one of simplicial (combinatorial, homology) d-manifolds for $d=2$, but in higher dimension it properly contains homology manifolds. For an elementary example of a normal pseudomanifold that is not an homology manifold we can consider the suspension $\Sigma = \langle\{v\},\{w\}\rangle * \Delta$ of a combinatorial $(d-1)$-manifold Δ that is not a sphere. Since for every face F containing only vertices of Δ, suspension and link commute, $\mathrm{lk}_\Sigma(F)$ is a combinatorial $(d-|F|)$-sphere, whereas if $v \in V(F)$, then $\mathrm{lk}_\Sigma(F) = \mathrm{lk}_\Sigma(v) \cap \mathrm{lk}_\Sigma(F \setminus v) = \Delta \cap \mathrm{lk}_\Delta(F \setminus v) = \mathrm{lk}_\Delta(F \setminus v)$. If $F \neq \{v\}$, then $\mathrm{lk}_\Delta(F \setminus v)$ is a combinatorial $(d-|F|)$-sphere, while if $F = \{v\}$, then $\mathrm{lk}_\Delta(\{\emptyset\}) = \Delta$ is a combinatorial $(d-1)$-manifold, which implies that Σ is a normal pseudomanifold, but not an homology manifold. In general we have the following proposition, through which one can make use of induction to prove statements to normal pseudomanifolds of arbitrary dimension.

Lemma 1.7. *A pure simplicial complex Δ is a normal pseudomanifold if and only if $\mathrm{lk}_\Delta(F)$ is a normal pseudomanifold for every $F \in \Delta$.*

Proof. The claim follows from $\mathrm{lk}_{\mathrm{lk}_\Delta(F)}(G) = \mathrm{lk}_\Delta(F \cup G)$. □

Remark 1.8. Geometric realizations of simplicial complexes unfortunately do not cover all homeomorphism types of topological manifolds. Indeed in dimension $d \geq 4$ there are compact manifolds which are neither homeomorphic to a combinatorial manifold nor to a simplicial one. This last statement was recently established in any dimension by Manulescu [Man16].

1.3 Decompositions of simplicial complexes

It is often convenient to study classes of simplicial complexes defined and constructed recursively from a single simplex. This allows to use the induction machinery in proofs and often to obtain information on the face numbers and on the topology. In this section we focus on three families, namely vertex decomposable, shellable and constructible simplicial complexes, which will appear again in Chapter 6. We point out that in the literature there is an endless list of further properties that a simplicial complex can enjoy. Moreover, in Chapter 4 we will introduce partitionability and discuss how it relates to the properties discussed in this section.

Proposition 1.9. [HH11, Proposition 8.2.8] *Let Δ be a pure d-dimensional simplicial complex. The following conditions are equivalent:*

i. There exists an ordering $F_1, \dots, F_m$ of the facets of Δ such that the complex $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i\rangle$ is pure and $(d-1)$-dimensional for every $i = 1, \dots, m$.

ii. For every $i = 1, \dots, m$ the set $\{F \subseteq F_i : F \notin \langle F_1, \dots, F_{i-1}\rangle\}$ has a unique minimal element denoted by R_i.

Definition 1.10. A pure d-dimensional simplicial complex satisfying the equivalent properties in Proposition 1.9 is called *shellable.* An ordering as in Condition *i.* of Proposition 1.9 is called *shelling order* while the sets R_i are the *restriction faces* of the ordering $F_1, \dots, F_m$.

For an extensive treatment of shellability we refer to [Sta96, Section III.2] or [Zie95, Section 8.1]. Notable examples of shellable complexes are given by polytopal spheres due to the work of Bruggesser and Mani [BM71]. Shellability offers a simple combinatorial interpretation of the h-numbers, via the identity

$$h_i(\Delta) = |\{1 \leq j \leq m : |R_j| = i\}|, \tag{1.2}$$

which also shows that the h-vector of any shellable simplicial complex is non-negative. In the next section we shall see that these numbers are subject to stronger inequalities. Moreover it is well known that shellability poses restrictions on the topology.

Proposition 1.11. [BW97, Corollary 13.3] *Let Δ be a pure d-dimensional simplicial complex. Then Δ is homotopy equivalent to a wedge of $h_{d+1}(\Delta)$ many d-spheres.*

In particular for a d-dimensional shellable complex Δ we have $\widetilde{H}_i(\Delta;\mathbb{F}) = 0$ if $i \leq d$ and $\dim_{\mathbb{F}} \widetilde{H}_d(\Delta;\mathbb{F}) = h_{d+1}(\Delta)$.

Definition 1.12. A pure d-dimensional simplicial complex Δ is *vertex decomposable* if $\Delta \cong \Delta_d$, or there exists a vertex v such that $\mathrm{lk}_\Delta(v)$ and $\Delta \backslash v$ are vertex decomposable.

This property, in a slightly more general form, was introduced by Provan and Billera [PB80], where they proved that vertex decomposable complexes satisfy a form of the Hirsch Conjecture (a statement on the combinatorial diameter of simplicial complexes), which raised the question whether all polytopal spheres are vertex decomposable. A negative answer was provided already in dimension 4 (see [KK87]), way before the Hirsch Conjecture was disproved by Santos [San12]. Nevertheless vertex decomposability relates to shellability.

Proposition 1.13. [PB80, Corollary 2.9] *Every vertex decomposable simplicial complex is shellable.*

Definition 1.14. A pure d-dimensional simplicial complex Δ is *constructible* if $\Delta \cong \Delta_d$ or $\Delta = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are constructible d-dimensional complexes and $\Gamma_1 \cap \Gamma_2$ is constructible and $(d-1)$-dimensional.

Comparing the above definition with Definition 1.10 it follows by induction that every shellable simplicial complex is constructible. To summarize we have the following hierarchy:

$$\text{vertex decomposable} \quad \Rightarrow \quad \text{shellable} \quad \Rightarrow \quad \text{constructible}$$

Examples of constructible 3-balls that are not shellable due to Rudin [Rud58] and Ziegler [Zie98] show that the right implication is strict, and Hachimori [Hac00a, Section 5.3] gave a 2-dimensional example of a shellable but not vertex decomposable simplicial complex. Finally, all three families in this section are closed under taking links and joins (see e.g., [Jon08, Theorem 3.30]).

Proposition 1.15. *The classes of vertex decomposable, shellable and constructible simplicial complexes are closed under taking links and joins.*

Even though the three properties treated in this section are typically defined for pure simplicial complexes there exist non-pure versions for which we defer to [BW96; BW97].

1.4 Stanley-Reisner correspondence and Cohen-Macaulay rings

This section is devoted to the study of a certain algebraic structure associated to a simplicial complex. In order to proceed with the study of this graded $\mathbb{F}$-algebra we recall some basic concepts in commutative algebra. For more details we refer to [BH93; Sta96]. Let $S := \mathbb{F}[x_1, \dots, x_n]$ denote the polynomial ring in n variables over an arbitrary field $\mathbb{F}$ and let $\mathfrak{m} = (x_1, \dots, x_n)$ be its maximal homogeneous ideal. For an $\mathbb{N}$-graded S-algebra (or S-module) R we use R_i to denote its graded component of degree i. Unless otherwise specified we will employ the standard $\mathbb{N}$-grading of S. The *Hilbert function* of R is the function from $\mathbb{N}$ to $\mathbb{N}$ that maps i to $\dim_{\mathbb{F}}(R_i)$ and the corresponding univariate generating function $\mathrm{Hilb}(R,t) := \sum_{i\geq 0} \dim_{\mathbb{F}}(R_i)t^i$ is called the *Hilbert series* of R. There exists a bijective correspondence between simplicial complexes on $[n]$ and squarefree monomial ideals of S.

Definition 1.16. Given a simplicial complex Δ with $V(\Delta) = [n]$ its *Stanley-Reisner ideal* is the squarefree monomial ideal $I_\Delta \subseteq S$ defined by

$$I_\Delta := (x_F \ : \ F \notin \Delta) \subseteq S,$$

where $x_F = \prod_{i \in F} x_i$. The quotient $\mathbb{F}[\Delta] := S/I_\Delta$ is called the *Stanley-Reisner ring* of Δ.

For the map in the other direction we associate to any squarefree monomial ideal I the simplicial complex whose faces correspond to squarefree monomials not contained in I. As an $\mathbb{F}$-vector space the Stanley-Reisner ring naturally decomposes as

$$\mathbb{F}[\Delta] = \bigoplus_{F \in \Delta} x_F \mathbb{F}[x_i : i \in F].$$

This yields an easy computation of its Hilbert series:

$$\begin{aligned}
\mathrm{Hilb}(\mathbb{F}[\Delta], t) &= \sum_{F \in \Delta} t^{|F|} \frac{1}{(1-t)^{|F|}} \\
&= \sum_{i=0}^{\dim(\Delta)+1} f_{i-1}(\Delta) \frac{t^i}{(1-t)^i} \\
&= \frac{\sum_{i=0}^{\dim(\Delta)+1} h_i(\Delta) t^i}{(1-t)^{\dim(\Delta)+1}}.
\end{aligned} \tag{1.3}$$

Moreover since $h_0(\Delta) + \cdots + h_{\dim(\Delta)+1}(\Delta) = f_{\dim(\Delta)}(\Delta) > 0$ the numerator of (1.3) does not vanish when $t = 1$, which implies via the Hilbert-Serre theorem (see e.g. [BH93, Corollary 4.1.8]) that $\dim(\mathbb{F}[\Delta]) = \dim(\Delta) + 1$, where $\dim(\mathbb{F}[\Delta])$ is the Krull dimension of the ring $\mathbb{F}[\Delta]$. For this reason in the rest of this chapter, as well as in Chapter 5, we will use $d - 1$ as the standard dimension of a simplicial complex.
This correspondence is extremely useful to study how algebraic invariants of Stanley-Reisner rings reflect combinatorial and topological properties of the corresponding simplicial complex, and vice versa. A special instance for this is provided by Hochster's formula (see [BH93, Theorem 5.5.1]), which allows to express the Hilbert function of the Artinian graded $\mathbb{F}$-algebras $\mathrm{Tor}_i(\mathbb{F}[\Delta], \mathbb{F})$ in terms of the simplicial homology of induced subcomplexes of Δ.
Some of the standard operations on simplicial complexes introduced in the previous sections have a simple meaning in commutative algebra.

Lemma 1.17. *Let Δ and Γ be simplicial complexes. Then*

- $\mathbb{F}[\Delta * \Gamma] \cong \mathbb{F}[\Delta] \otimes_{\mathbb{F}} \mathbb{F}[\Gamma]$;
- *For $S \subseteq V(\Delta)$, $\mathbb{F}[\Delta_S] \cong \mathbb{F}[\Delta]/(x_i : i \in V(\Delta) \setminus S)$;*
- *If $V(\Delta) = V(\Gamma)$, then $I_{\Delta \cap \Gamma} = I_\Delta + I_\Gamma$ and $I_{\Delta \cup \Gamma} = I_\Delta \cap I_\Gamma$.*

The first statement of Lemma 1.17 implies that

$$\begin{aligned}
\sum_{i=0}^{\dim(\Delta*\Gamma)+1} h_i(\Delta * \Gamma) t^i =& \mathrm{Hilb}(\mathbb{F}[\Delta * \Gamma], t)(1-t)^{\dim(\Delta*\Gamma)+1} \\
=& \mathrm{Hilb}(\mathbb{F}[\Delta], t)(1-t)^{\dim(\Delta)+1} \mathrm{Hilb}(\mathbb{F}[\Gamma], t)(1-t)^{\dim(\Gamma)+1} \\
=& \left(\sum_{i=0}^{\dim(\Delta)+1} h_i(\Delta) t^i \right) \left(\sum_{i=0}^{\dim(\Gamma)+1} h_i(\Gamma) t^i \right).
\end{aligned}$$

Comparing the coefficients of the two polynomials we obtain:

Corollary 1.18. *For $i = 0, \dots, \dim(\Delta) + \dim(\Gamma) + 1$ it holds that*

$$h_i(\Delta * \Gamma) = \sum_{j=0}^{i} h_j(\Delta) h_{i-j}(\Gamma).$$

A straightforward application of (1.1) yields the same formula for the f-vector of the join of two simplicial complexes. We remark the analogy with Künneth formula for the simplicial homology.

Theorem 1.19 (Künneth formula)**.** [Mun84a, Section 58] *Let Δ and Γ be simplicial complexes. Then*

$$\widetilde{H}_i(\Delta * \Gamma; \mathbb{F}) \cong \bigoplus_{j=0}^{i} \widetilde{H}_j(\Delta; \mathbb{F}) \otimes_{\mathbb{F}} \widetilde{H}_{i-j}(\Gamma; \mathbb{F}).$$

We consider now two sequences of linear forms which play an important role in commutative algebra.

Definition 1.20. Let $I \subseteq S$ be a homogeneous ideal and let $R = S/I$ be of Krull dimension d. Let $\Theta = \{\theta_1, \dots, \theta_d\} \subseteq S_1$. Then:

- Θ is a *linear system of parameters* (*l.s.o.p.*) for R if $\dim(R/(\theta_1, \dots, \theta_i)R) = \dim(R) - i$, for all $1 \le i \le d$.
- Θ is a *regular sequence* for R if θ_i is not a zero divisor of $R/(\theta_1, \dots, \theta_{i-1})R$, for all $1 \le i \le d$.

Due to the Noether normalization lemma [BH93, Theorem 1.5.17], an l.s.o.p. for $R = S/I$ always exists, under the assumption that $\mathbb{F}$ is an infinite field. Observe that from the definition above it follows that $R/\Theta R$ is an *Artinian* algebra, that is an $\mathbb{F}$-algebra of Krull dimension zero, which is equivalent to say that $R/\Theta R$ is a finite dimensional $\mathbb{F}$-vector space. Moreover, if Θ is a regular sequence, then Θ is an l.s.o.p., but the converse is far from being true in general. The class of rings for which the converse holds is of particular interest.

Definition 1.21. A graded $\mathbb{F}$-algebra R is *Cohen-Macaulay* over $\mathbb{F}$ if one (equivalently every) l.s.o.p. is a regular sequence for R.

Theorem 1.22. [BH93, Proposition 2.2.11] *A graded $\mathbb{F}$-algebra R is Cohen-Macaulay over $\mathbb{F}$ if it is free as an $\mathbb{F}[\Theta]$-module for one (equivalently every) l.s.o.p. Θ.*

A simplicial complex Δ is called *Cohen-Macaulay* over $\mathbb{F}$ if $\mathbb{F}[\Delta]$ is a Cohen-Macaulay ring. The theory of Cohen-Macaulay rings plays a key role in combinatorial commutative algebra and its importance cannot be overstated (see e.g., [BH93; Sta96]).

Proposition 1.23. *Every Cohen-Macaulay simplicial complex is pure.*

Proof. A Cohen-Macaulay ring R is *unmixed* (see [HH11, Appendix A]), i.e., $\dim(R) = \dim(R/\mathfrak{p})$ for every associated prime ideal $\mathfrak{p} \subseteq R$. Let Δ be a Cohen-Macaulay simplicial complex on $[n]$. Since the minimal primes of a squarefree monomial ideal I_Δ are exactly $\mathrm{Min}(I_\Delta) = \{\mathfrak{p}_{[n]\setminus F} : F \in \mathcal{F}(\Delta)\}$, where $\mathfrak{p}_{[n]\setminus F} = (x_i : i \in [n] \setminus F)$, we have that all facets F have the same dimension. □

Proposition 1.24. [Sta96, III, Theorem 2.5] *Let Δ be a shellable $(d-1)$-dimensional simplicial complex. Let $F_1, \ldots, F_m$ be a shelling order for Δ and R_i the corresponding restriction faces. Then for any l.s.o.p. Θ it holds that*

$$\mathbb{F}[\Delta] = \bigoplus_{i=1}^{m} x_{R_i}\mathbb{F}[\Theta].$$

In particular, Δ is Cohen-Macaulay over any field $\mathbb{F}$.

For an example which shows that the converse of Proposition 1.24 does not hold (not even in dimension 2) we can consider any triangulation of the dunce hat (see Section 6.3.2). Another crucial property of Cohen-Macaulay complexes is the following.

Lemma 1.25. [Sta96, II Corollary 3.2] *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex and let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be an l.s.o.p. for $\mathbb{F}[\Delta]$. Then*

$$h_i(\Delta) = \dim_{\mathbb{F}} \left(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]\right)_i.$$

Proof. Since Δ is Cohen-Macaulay we have that Θ is a regular sequence, and hence we have that for every $1 \leq i \leq d$ the sequence

$$0 \longrightarrow \mathbb{F}[\Delta]/\Theta_{i-1}\mathbb{F}[\Delta](-1) \xrightarrow{\cdot\theta_i} \mathbb{F}[\Delta]/\Theta_{i-1}\mathbb{F}[\Delta] \longrightarrow \mathbb{F}[\Delta]/\Theta_i\mathbb{F}[\Delta] \longrightarrow 0$$

is exact, where $\Theta_i = \{\theta_1, \ldots, \theta_i\}$ for $1 \leq i \leq d$ and $\Theta_0 = \{0\}$. Moreover by the additivity of Hilbert series on short exact sequences we obtain

$$(1-t)\mathrm{Hilb}(\mathbb{F}[\Delta]/\Theta_{i-1}\mathbb{F}[\Delta], t) = \mathrm{Hilb}(\mathbb{F}[\Delta]/\Theta_i\mathbb{F}[\Delta], t)$$

which together with (1.3) yields $\mathrm{Hilb}(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta], t) = \sum_{i=0}^{d} h_i(\Delta)t^i$. □

Lemma 1.25 implies that if Δ is a Cohen-Macaulay complex then the numbers $h_i(\Delta)$ are non-negative and that they essentially count the number of degree i monomials in any Artinian reduction of $\mathbb{F}[\Delta]$. If $\dim(\Delta) = d-1$ and $|V(\Delta)| = n$, such an Artinian algebra can be written as the quotient of a polynomial ring with $n-d$ variables, which provides an upper bound for the h-numbers.

Corollary 1.26. [BH93, Theorem 5.1.10] *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex with $|V(\Delta)| = n$. Then $0 \leq h_i(\Delta) \leq \binom{n-d+i-1}{i}$.*

We now define another important class of Cohen-Macaulay algebras, properly containing complete intersections. Recall that the *socle* of an S-module R is the $\mathbb{F}$-vector space $\mathrm{Soc}(R) := 0 :_R \mathfrak{m} = \{r \in R : r\mathfrak{m} = (0)\}$.

Definition 1.27. Let R be a graded Cohen-Macaulay S-module and $\mathbb{F}$-algebra and let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be an l.s.o.p. for R. Then R is a *Gorenstein ring* if $\dim_{\mathbb{F}}(\mathrm{Soc}(R/\Theta R)) = 1$.

Observe that, since by definition the set $\{\theta_i, \ldots, \theta_d\}$ is a regular sequence for $R/(\theta_1, \ldots, \theta_{i-1})R$, we have that R is Gorenstein if and only if the Artinian algebra $R/\Theta R$ is Gorenstein. The following proposition establishes a certain symmetry in the Hilbert series of graded Gorenstein algebras.

Proposition 1.28. [Sta96, Theorem 12.6] *Let R be a standard graded Gorenstein $\mathbb{F}$-algebra with $\dim(R) = d$ and let $\mathrm{Hilb}(R, t) = \dfrac{\sum_i h_i t^i}{(1-t)^d}$. Then*

$$\mathrm{Hilb}(R, 1/t) = (-1)^d t^{s-d} \mathrm{Hilb}(R, t),$$

where $s := \max\{j : h_j \neq 0\}$. In particular, we have $h_i = h_{s-i}$ for every $i = 0, \ldots, s$.

The number s as in Proposition 1.28 is called *socle degree* of R. If we let Δ be a *Gorenstein simplicial complex* over $\mathbb{F}$ (i.e., $\mathbb{F}[\Delta]$ is Gorenstein), then Proposition 1.28 and Lemma 1.25 show a certain symmetry of $h(\Delta)$.

Remark 1.29. Given a standard graded Cohen-Macaulay $\mathbb{F}$-algebra R of dimension d, with $\mathrm{Hilb}(R, t) = (\sum_i h_i t^i)/(1-t)^d$ there exists a simplicial complex Δ with $h_i(\Delta) = h_i$ for every $i = 0, \ldots, d$. The corresponding statement for Gorenstein algebras is not true, see [Sta96, III Section 6].

Remark 1.30. There is an important subfamily of Gorenstein algebras, namely *complete intersections*. A finitely generated $\mathbb{F}$-algebra $R = S/I$ is a complete intersection if I is generated by a regular sequence. Since this class does not have an interesting topological counterpart it will not appear in the rest of this manuscript, but we mention it for the sake of completeness.

1.5 Face numbers of spheres and manifolds

One of the main reasons behind the study of Cohen-Macaulay complexes is that they are meaningful from a topological point of view. Indeed, the next two results exhibit criteria to decide whether a simplicial complex is Cohen-Macaulay or Gorenstein based on simplicial homology. The second condition of Theorem 1.31 is due to Reisner, and therefore known as *Reisner's criterion*, while Theorem 1.32 is a result of Stanley.

Theorem 1.31. [Rei76, Theorem 1][Mun84b, Corollary 3.5] *Let Δ be a simplicial complex. The following are equivalent:*

- *Δ is Cohen-Macaulay over $\mathbb{F}$;*
- *$\widetilde{H}_i(\mathrm{lk}_\Delta(F); \mathbb{F}) = 0$ for every $F \in \Delta$ and for every $i < \dim(\mathrm{lk}_\Delta(F))$;*
- *$\widetilde{H}_i(\Delta; \mathbb{F}) \cong H_i(|\Delta|, |\Delta| \setminus p; \mathbb{F}) = 0$ for every point $p \in |\Delta|$ and for every $i < \dim(\Delta)$.*

Theorem 1.32. [Sta77, Theorem 7] *Let Δ be a simplicial complex and let $\Gamma = \Delta_C$ with $C := \{v \in V(\Delta) : \mathrm{st}_\Delta(v) \neq \Delta\}$. The following are equivalent:*

- *Δ is Gorenstein over $\mathbb{F}$;*
- *For every $F \in \Delta$*

$$\widetilde{H}_i(\mathrm{lk}_\Gamma(F); \mathbb{F}) = \begin{cases} 0 & \text{if } i < \dim(\mathrm{lk}_\Gamma(F)) \\ \mathbb{F} & \text{if } i = \dim(\mathrm{lk}_\Gamma(F)) \end{cases};$$

- *For every point $p \in |\Gamma|$*

$$\widetilde{H}_i(\Gamma; \mathbb{F}) \cong H_i(|\Gamma|, |\Gamma| \setminus p; \mathbb{F}) = \begin{cases} 0 & \text{if } i < \dim(\Gamma) \\ \mathbb{F} & \text{if } i = \dim(\Gamma) \end{cases}.$$

In particular, once the field $\mathbb{F}$ is fixed, both properties only depend on the topology of $|\Delta|$, and not on the combinatorics of its triangulation. Since homology spheres are closed under taking links we immediately obtain the following.

Corollary 1.33. *If Δ is an $\mathbb{F}$-homology $(d-1)$-sphere, then Δ is Gorenstein over $\mathbb{F}$ with socle degree d. In particular, $h_i(\Delta) = h_{d-i}(\Delta)$ for every $i = 0, \dots, d$.*

Remark 1.34. We really need to fix a field to make both properties topological, the standard example being Δ any triangulation of the projective plane $\mathbb{R}\mathbf{P}^2$. Indeed, we know that $\widetilde{H}_1(\mathrm{lk}_\Delta(\emptyset);\mathbb{Z}) = \widetilde{H}_1(\Delta;\mathbb{Z}) \cong \mathbb{Z}_2$, which implies that $\widetilde{H}_1(\mathrm{lk}_\Delta(\emptyset);\mathbb{F}) = 0$ if and only if $\mathrm{char}(\mathbb{F}) \neq 2$. Hence Δ is Cohen-Macaulay over any field of characteristic other than 2. For a simplicial complex that is even Gorenstein over all fields of characteristic other than 2 we can consider any triangulation of $\mathbb{R}\mathbf{P}^3$.

If Δ is an $\mathbb{F}$-homology $(d-1)$-sphere, then Theorem 1.31 together with Corollary 1.26 state that $h_i(\Delta) \leq \binom{n-d+i-1}{i}$, and a natural question is whether this bound is tight. Indeed for every d and $n \geq d+1$ there exists a simplicial d-polytope on n vertices whose h-vector (i.e., the h-vector of its boundary) attains this inequality for every $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Definition 1.35. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be the *moment map* $\gamma(t) = (t, t^2, \dots, t^d)$. We define the *cyclic polytope* $\mathcal{C}(n,d)$ to be the convex hull of any n distinct points on the curve $\gamma(\mathbb{R})$.

It is well known that the combinatorial type of $\mathcal{C}(n,d)$ only depends on n and d. The face structure of cyclic polytopes is well understood and, together with Corollary 1.26 and Theorem 1.31, it plays an important role in what is called the upper bound theorem for spheres. We call a simplicial complex Δ on n vertices *k-neighborly* if $\mathrm{Skel}_k(\Delta) \cong \mathrm{Skel}_k(\Delta_{n-1})$, that is if all $(k+1)$-subsets of $V(\Delta)$ are faces of Δ. Note that if Δ is k-neighborly then it is j-neighborly for every $j \leq k$.

Proposition 1.36. [Grü03, Section 4.7] *$\partial\mathcal{C}(n,d)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly and hence $h_i(\partial\mathcal{C}(n,d)) = \binom{n-d+i-1}{i}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.*

In other words Proposition 1.36 guarantees the existence of (polytopal) $(d-1)$-spheres on n vertices for any d and $n \geq d+1$ which attain simultaneously the bound in Corollary 1.26 for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$. Interestingly we remark that while every $\lfloor \frac{d}{2} \rfloor$-neighborly d-polytope is simplicial when d is even, the same does not hold in odd dimension, pyramids over $\frac{d}{2}$-neighborly d-polytopes being counterexamples. Moreover, even though cyclic d-polytopes are important members of this family, there exist $\lfloor \frac{d}{2} \rfloor$-neighborly d-polytopes that are not combinatorially equivalent to a cyclic polytope (see [Grü03; Zie95]).

For a $(d-1)$-dimensional simplicial complex Δ define $g_0(\Delta) := h_0(\Delta)$ and $g_i(\Delta) := h_i(\Delta) - h_{i-1}(\Delta)$ if $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$. The vector $g(\Delta) := (g_0(\Delta), \dots, g_{\lfloor \frac{d}{2} \rfloor}(\Delta))$ is called the *g-vector* of Δ. The celebrated *g-theorem* gives a complete characterization of g-vectors, and hence also of h and f-vectors, of simplicial polytopes.

Theorem 1.37. *(g-theorem)*[Sta80; BL81] *Let $g = (g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$ be a sequence of non-negative integers. Then $g = g(\partial P)$ for some simplicial d-polytope P if and only if there exists an Artinian standard graded $\mathbb{F}$-algebra A such that $g_i = \dim_{\mathbb{F}}(A_i)$ for every $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$.*

The first proof of the necessity statement was due to Stanley [Sta80] via an ingenious connection with algebraic geometry. Indeed for any rational simplicial d-polytope P (any simplicial polytope can be realized with rational coordinates) one can consider the complex projective toric variety associated to its normal fan. This variety is not smooth in general, but it has only mild singularities, namely finite quotient ones, which allow to define a singular cohomology ring. Stanley showed that this algebra is isomorphic to a certain Artinian reduction A of $\mathbb{R}[\partial P]$ and made use of the *Hard Lefschetz theorem* for toric varieties to conclude the existence of a linear form ω such that the multiplication map $\cdot\omega : A_i \to A_{i+1}$ is injective for every $i \leq \lfloor \frac{d}{2} \rfloor$. Finally

$$\dim_{\mathbb{R}}(A/\omega A)_i = \dim_{\mathbb{R}} A_i - \dim_{\mathbb{R}} A_{i-1} = h_i(\partial P) - h_{i-1}(\partial P) = g_i(\partial P).$$

Here the use of $\mathbb{R}$ in place of an arbitrary field follows [Sta80] but it can be replaced with any infinite field. The sufficiency of the g-theorem relies on a result of Billera and Lee [BL81] who explicitly constructed a simplicial polytope P with the prescribed g-vector. The g-theorem hence completely characterizes the face numbers of polytopal spheres and both Stanley's and McMullen's proof rely on geometric tools which make them unsuitable for larger classes of spheres. Moreover, Kalai [Kal88] showed that for $d \geq 5$ the number of combinatorial types of simplicial spheres on n vertices is asymptotically much larger than the number of combinatorial types of polytopal spheres on n vertices (i.e., their ratio tends to infinity as n grows), and it seems therefore plausible the existence of f-vectors of simplicial spheres that are not f-vectors of any polytopal sphere. The unsuccessful struggle to find such a counterexample pushed researchers to conjecture that the statement of Theorem 1.37 holds if we replace polytopal spheres with larger classes of spheres. Since Stanley's proof of the g-theorem in 1980 this problem, known as the *g-conjecture*, has become perhaps the most challenging question in geometric and topological combinatorics.

Remark 1.38. During the preparation of this manuscript Adiprasito [Adi18] uploaded on the arXiv a breakthrough preprint in which he proves a generic Lefschetz theorem for generic Artinian reductions of Stanley-Reisner rings of rational homology spheres, hence establishing the g-conjecture in its full generality.

The last part of this chapter is devoted to the inequality $h_2(\Delta) \geq h_1(\Delta)$ in different levels of generality. For the treatment of the equality case we start with a definition.

Definition 1.39 (Connected sum)**.** Let Δ and Γ be pure simplicial complexes of the same dimension on disjoint vertex sets, let F and G be two facets of Δ and Γ respectively and let $\varphi : F \longrightarrow G$ be a bijection. Then the *connected sum* $\Delta \#_\varphi \Gamma$ is the simplicial complex obtained from $\Delta \setminus F$ and $\Gamma \setminus G$ by identifying v and $\varphi(v)$, for every $v \in F$.

Lemma 1.40. *Let Δ and Γ be pure $(d-1)$-dimensional simplicial complexes such that $\Delta\#\Gamma$ is of dimension $(d-1)$. Then*

$$f_{i-1}(\Delta\#\Gamma) = \begin{cases} f_{i-1}(\Delta) + f_{i-1}(\Gamma) - \binom{d}{i} & \text{if } i = 1, \dots, d-1 \\ f_{i-1}(\Delta) + f_{i-1}(\Gamma) - 2 & \text{if } i = d \end{cases},$$

$$h_i(\Delta\#\Gamma) = \begin{cases} h_i(\Delta) + h_i(\Gamma) & \text{if } i = 1, \dots, d-1 \\ h_i(\Delta) + h_i(\Gamma) - 1 & \text{if } i = d \end{cases}.$$

Definition 1.41. Any simplicial $(d-1)$-sphere that can be obtained as the connected sum of k copies of $\partial\Delta_d$ for some $k \geq 1$ is called a *stacked $(d-1)$-sphere* on $d+k$ vertices.

Note that for $k \geq 4$ there are several combinatorial types of such spheres, and we denote by $\mathcal{ST}(n,d)$ the set of stacked $(d-1)$-spheres on n vertices. Nevertheless their f-vector only depends on n and d and it can be easily computed via Lemma 1.40:

$$\begin{aligned} f_{i-1}(\mathcal{ST}(n,d)) &= kf_{i-1}(\partial\Delta_d) - (k-1)f_{i-1}(\Delta_{d-1}) \\ &= k\binom{d}{i} - (k-1)\binom{d-1}{i} = \binom{d}{i}\frac{ki+d-i}{d}, \end{aligned}$$

with $k = n-d$, for every $i = 1, \dots, d-1$ and $f_{d-1}(\mathcal{ST}(n,d)) = kd - 2k + 2$. Moreover we have that

$$h_i(\mathcal{ST}(n,d)) = k,$$

for every $i = 1, \dots, d-1$.
There is an equivalent reformulation of Definition 1.41.

Lemma 1.42. [Kal87] *A simplicial $(d-1)$-sphere is stacked if and only if it can be realized as the boundary of a simplicial d-ball having only i-faces with $i \geq d-1$ in the interior.*

This familiy of spheres plays a role in the *Lower Bound Theorem (LBT).*

Theorem 1.43 (LBT)**.** [Bar73; Kal87; Fog88; Tay95] *Let Δ be a normal $(d-1)$-pseudomanifold on n vertices with $d \geq 3$. Then*

- $h_2(\Delta) \geq h_1(\Delta)$ *or, equivalently,* $f_i(\Delta) \geq f_i(\mathcal{ST}(n,d))$ *for every* $i \geq 0$*;*
- *If* $d \geq 4$ *and* $h_2(\Delta) = h_1(\Delta)$*, then* $\Delta \in \mathcal{ST}(n,d)$*.*

The inequality in Theorem 1.43 was first proved for the case of simplicial polytopes by Barnette [Bar73], who addressing to an unpublished result by Walkup observed that the statement still holds for the more general setting of simplicial manifolds. Kalai [Kal87] gave another proof of the inequality and established the equality case via the theory of rigidity of frameworks, while Fogelsanger [Fog88] (by means of commutative algebra) and Tay [Tay95] (purely combinatorially) extended Theorem 1.43 to all normal pseudomanifolds.

Remark 1.44. It might seem surprising that an inequality involving the numbers $h_2(\Delta)$ and $h_1(\Delta)$ (which directly translates into an inequality for $f_1(\Delta)$) is equivalent to a bound for $f_i(\Delta)$ for every i. This is due to an argument often referred to as *MPW-reduction* (McMullen-Perles-Walkup) [Kal87, Section 5], which consists of inductively reducing inequalities for $f_i(\Delta)$ to one for $f_1(\Delta)$ using the identity

$$f_i(\Delta) = \frac{1}{i+1}\sum_{v \in V(\Delta)} f_{i-1}(\mathrm{lk}_\Delta(v))$$

which follows from a double counting argument. Therefore it applies to any class of simplicial complexes closed under taking links of faces.

Remark 1.45. As a consequence of the g-theorem, the h-vector of a polytopal $(d-1)$-sphere ∂P is unimodal, i.e., $h_i(\partial P) \leq h_{i+1}(\partial P)$ for every $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$. This result goes under the name of *Generalized Lower Bound Theorem (GLBT)*. Using commutative algebra tools Murai and Nevo [MN13] characterized the equality case, which corresponds to a higher notion of stackedness.

Via Lemma 1.42 we can define a natural generalization of stacked spheres in the world of manifolds. Indeed an homology $(d-1)$-manifold is *stacked* if it can be realized as the boundary of a homology manifold with boundary whose interior contains only i-faces with $i \geq d-1$.

Definition 1.46 (Handle addition)**.** Let Δ be a pure $(d-1)$-dimensional simplicial complex, with $F, G \in \mathcal{F}(\Delta)$. Assume moreover that there exists a bijection $\varphi : F \to G$ such that $\mathrm{lk}_\Delta(v) \cap \mathrm{lk}_\Delta(\varphi(v)) = \emptyset$ for every $v \in F$. The simplicial complex obtained by identifying H and $\varphi(H)$ for every $H \subseteq F$ and then removing the facet $F = G$ is called the *handle addition* on Δ.

Definition 1.47 (Walkup class)**.** The set of simplicial $(d-1)$-manifolds which can be obtained from $\partial\Delta_d$ by successively applying connected sum and handle addition is called the *Walkup class* and denoted by $\mathcal{H}_d$.

If Δ is a simplicial $(d-1)$-manifold belonging to the Walkup class, its topology is determined by $\beta_1(\Delta; \mathbb{F})$ and by the orientability: indeed $|\Delta|$ is either homeomorphic to S^{d-1}, to $(S^{d-1} \times S^1)^{\beta_1(\Delta;\mathbb{F})}$ or to the non-orientable bundle $(S^{d-1} \rtimes S^1)^{\beta_1(\Delta;\mathbb{F})}$ (see e.g., [Kal87]). Moreover, it is not hard to see that every stacked homology manifold is combinatorial.

Proposition 1.48. [DM17, Corollary 4.6] *Let Δ be a homology $(d-1)$-manifold, with $d \geq 3$. Then Δ is stacked if and only if it belongs to the Walkup class.*

For a stacked $(d-1)$-dimensional manifold Δ, the number $g_2(\Delta)$ attains the minimum among all normal $(d-1)$-pseudomanifolds on $f_0(\Delta)$ vertices. This characterizes the case of equality in the following theorem, known as the *Lower Bound Theorem for Manifolds.*

Theorem 1.49 (LBT for manifolds)**.** [NS09b; Mur15] *Let Δ be a normal $(d-1)$-pseudomanifold with $d \geq 3$. Then $h_2(\Delta) - h_1(\Delta) \geq \binom{d+1}{2}(\widetilde{\beta}_1(\Delta; \mathbb{F}) - \widetilde{\beta}_0(\Delta; \mathbb{F}))$, and equality holds if and only if Δ is a stacked manifold.*

Theorem 1.49 is due to Novik and Swartz (for orientable homology manifolds) and to Murai (for normal pseudomanifolds). In particular Murai's result is proved via upper bounds on the graded Betti numbers of the Stanley-Reisner ring of a normal pseudomanifold (see Chapter 5).

Remark 1.50. In this section we provided a very general overview on face enumeration of spheres and manifolds, which in particular does not treat the case of manifolds with non-empty boundary. There is of course a strong connection between the techniques used to study manifolds without and with boundary, but overall fewer results are known in the latter case. However, we refer the interested reader to the two remarkable surveys [KN16a; Swa14], in which details and further references can be found.

Chapter 2

The class of balanced complexes and manifolds

2.1 Definition and first properties

We begin this chapter by defining the main object of study of this thesis.

Definition 2.1. A $(d-1)$-dimensional simplicial complex Δ is *balanced* if there is a partition of its vertex set $V(\Delta) = \dot{\bigcup}_{i=1}^{d} V_i$ such that for every $F \in \Delta$, $|F \cap V_i| \leq 1$ for every $i = 1, \ldots, d$.

We often refer to the sets V_i as *color classes*. Another way to phrase this definition is to observe that Δ is balanced if and only if its 1-skeleton is *d-colorable*, in the classical graph theoretic sense, that is there exists a *coloring map* $\kappa : V(\Delta) \longrightarrow [d]$ which is injective when restricted to the edges, i.e., $\kappa(i) \neq \kappa(j)$ for every $\{i, j\} \in \Delta$. Note that, without extra assumptions on its structure, a balanced simplicial complex does not uniquely determine the size of the color classes, not even if it is pure, as shown by the middle and right complex in Figure 2.2. However, in this thesis we will always assume the vertex partition to be part of the data defining Δ.

Example 2.2. Let Q be a graded finite poset. We can associate to Q a simplicial complex $\mathcal{O}(Q) := \{\{q_{i_1}, \ldots, q_{i_k}\} \subseteq Q : q_{i_1} < \cdots < q_{i_k}\}$ called the *order complex* of Q which satisfies $\dim(\mathcal{O}(Q)) = \operatorname{rank}(Q)$. If we partition the vertices of $\mathcal{O}(Q)$ (i.e., the elements of Q) as $V(\mathcal{O}(Q)) = \dot{\bigcup}_{i=1}^{d} V_i$, where $V_i := \{q \in Q : \operatorname{rank}(q) = i\}$, we obtain that the order complex of any graded poset is balanced. In particular, if Q is the face poset of a simplicial complex Δ ranked by size, then $\mathcal{O}(Q \setminus \{\emptyset\})$ is called the *barycentric subdivision* of Δ, and we denote it by $\operatorname{Bd}(\Delta)$. Since $|\operatorname{Bd}(\Delta)| \cong |\Delta|$ any triangulable topological space has a balanced triangulation.

Example 2.3. Let $\mathcal{C}_d := \operatorname{conv}\{\pm e_1, \ldots, \pm e_d\}$, with $\{e_i\}_{i=1}^{d}$ the standard basis of $\mathbb{R}^d$. The simplicial polytope $\mathcal{C}_d$ is called the d-dimensional *cross-polytope* and its boundary (more generally any simplicial complex combinatorially equivalent to it) is a balanced $(d-1)$-dimensional simplicial sphere, which can be described combinatorially as the join of d pairs of points

$$\partial\mathcal{C}_d \cong \{e_1, -e_1\} * \cdots * \{e_d, -e_d\}.$$

A coloring map κ is given by $\kappa(e_i) = \kappa(-e_i) = i$.

Lemma 2.4. *Let Δ be a balanced $(d-1)$-dimensional simplicial complex. Then:*

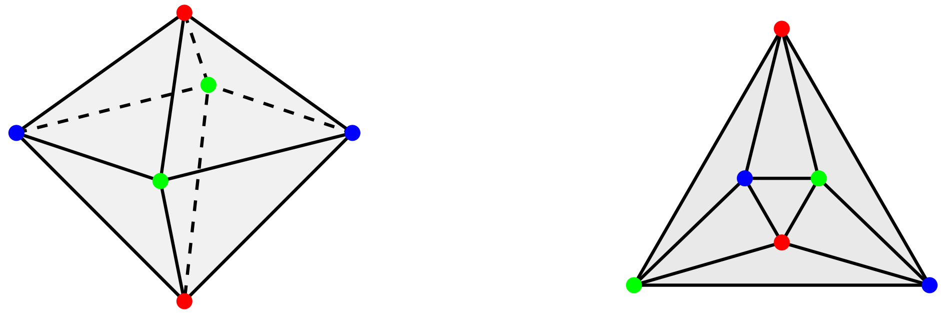

FIGURE 2.1: The simplicial complexes $\partial\mathcal{C}_3$ and $\partial\mathcal{C}_3 \setminus F$, where F is a facet.

i. $\Delta * \Gamma$ *is a balanced simplicial complex for any balanced complex* Γ*. In particular the cone and the suspension of* Δ *are balanced.*

ii. Any $(d-1)$*-dimensional subcomplex* $\Gamma \subseteq \Delta$ *is balanced.*

If Δ *is pure, then:*

iii. $\mathrm{lk}_\Delta(F)$ *and* $\mathrm{st}_\Delta(F)$ *are balanced pure simplicial complexes.*

Proof. i. We consider the partitions $V(\Delta) = \dot{\bigcup}_{i=1}^{\dim(\Delta)+1} A_i$ and $V(\Gamma) = \dot{\bigcup}_{i=1}^{\dim(\Gamma)+1} B_i$ of $V(\Delta)$ and $V(\Gamma)$ as in Definition 2.1. We define $V(\Delta * \Gamma) = \dot{\bigcup}_{i=1}^{\dim(\Delta)+\dim(\Gamma)+2} V_i$, with $V_i = A_i$ for $i = 1, \ldots, \dim(\Delta)+1$ and $V_i = B_i$ for $i = \dim(\Delta)+2, \ldots, \dim(\Delta)+\dim(\Gamma)+2$. Since $\dim(\Delta)+\dim(\Gamma)+2 = \dim(\Delta*\Gamma)+1$ and taking the join only adds edges of the form $\{v, w\}$ with $v \in V(\Delta)$ and $w \in V(\Gamma)$ the condition in Definition 2.1 is still satisfied.
ii. Let κ be a coloring map for Δ. Clearly any full dimensional subcomplex $\Gamma \subseteq \Delta$ is balanced, with a coloring map given by the restriction $\kappa|_{V(\Gamma)}$.
iii. If Δ is pure, then every facet contains one vertex per color class, which in turns implies that the pure $(d-|F|-1)$-dimensional simplicial complex $\mathrm{lk}_\Delta(F)$ can be properly colored using elements in $[d] \setminus [\kappa(F)]$. To show that $\mathrm{st}_\Delta(F)$ is balanced it suffices to observe that $\mathrm{st}_\Delta(F) \cong F * \mathrm{lk}_\Delta(F)$ and then to use the first statement of this lemma.

□

Remark 2.5. It is not hard to see that the skeleton $\mathrm{Skel}_j(\Delta)$ of a balanced $(d-1)$-dimensional simplicial complex Δ is not balanced for $0 < j < d-1$. Moreover induced subcomplexes and subcomplexes which are not full dimensional of Δ need not to be balanced, as the boundary of the right complex in Figure 2.1 shows.

In more topological terms a $(d-1)$-dimensional simplicial complex is balanced if it admits a *non-degenerate simplicial map* to the $(d-1)$-simplex. The class of pure balanced simplicial complexes agrees with the class of so-called *completely balanced* complexes, originally introduced by Stanley in [Sta79]. However, a balanced simplicial complex in the sense of Definition 2.1 does not need to be pure.

Proposition 2.6. [Sta79, Corollary 4.2] *Let* Δ *be a Cohen-Macaulay* $(d-1)$*-dimensional balanced simplicial complex. Moreover let* $\theta'_i := \sum_{\substack{v \in V(\Delta) \\ \kappa(v)=i}} x_v$ *for* $i = 1, \ldots, d$*. Then:*

i. The set $\Theta' = \{\theta'_1, \ldots, \theta'_d\}$ *is an l.s.o.p. for* $\mathbb{F}[\Delta]$*.*

ii. $x_v^2 = 0$ *in* $\mathbb{F}[\Delta]/\Theta'\mathbb{F}[\Delta]$ *for every* $v \in V(\Delta)$*.*

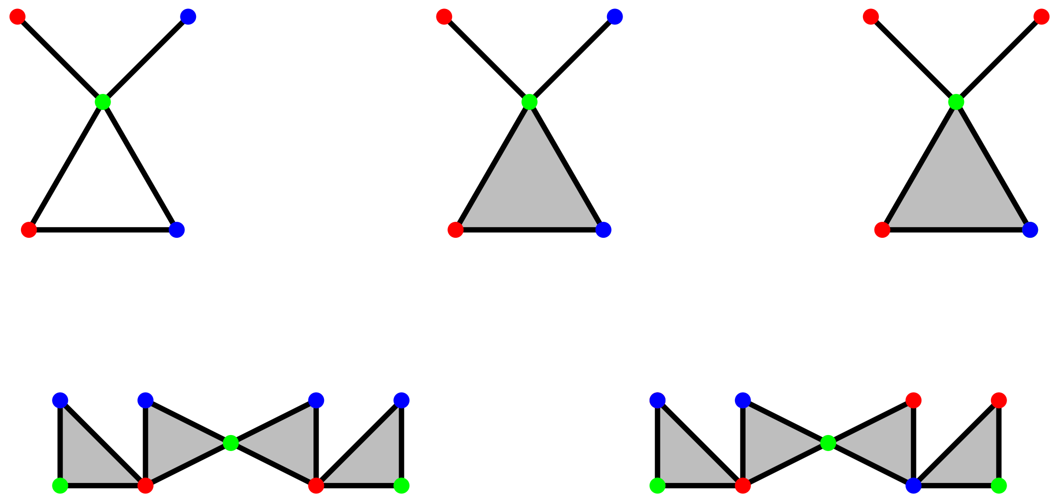

FIGURE 2.2: First row: a simplicial complex that is not balanced and two balanced complexes with different partitions in color classes. Second row: a pure balanced complex with two different partitions.

Proof. i. A result by Kind and Kleinschmidt [KK79] states that a sequence of linear forms $\{\theta_1, \ldots, \theta_d\}$ is an l.s.o.p. for $\mathbb{F}[\Delta]$ if and only if the restrictions $\{\theta_1|_F, \ldots, \theta_d|_F\}$ span a $|F|$-dimensional $\mathbb{F}$-vector space for every facet F. Here by $\theta_i|_F$ we mean the linear form obtained by considering only the terms of $\theta_i|_F$ corresponding to vertices in F. If Δ is balanced, then $\theta'_i|_F$ consists only of the variable x_v where $v \in F$ and $\kappa(v) = i$, and hence the forms $\{\theta'_1|_F, \ldots, \theta'_d|_F\}$ are linearly independent for every $F \in \Delta$.
ii. If we look at the monomial in x_v^2 in $\mathbb{F}[\Delta]/\Theta'\mathbb{F}[\Delta]$ we have that

$$x_v^2 = x_v \cdot \left(- \sum_{w:\kappa(w)=\kappa(v), w \neq v} x_w \right) = 0 \in \mathbb{F}[\Delta]/\Theta'\mathbb{F}[\Delta],$$

where the last equality follows from the fact that all monomials $x_v x_w$ with $\kappa(v) = \kappa(w)$ belong to I_Δ. □

Remark 2.7. The Stanley-Reisner ring of a $(d-1)$-dimensional balanced complex can be naturally equipped with a $\mathbb{N}^d$-grading by setting $\deg(x_v) := e_{\kappa(v)}$, where $\{e_i : i \in [d]\}$ are the standard unit vectors of $\mathbb{R}^d$. The linear forms θ'_i defined in Proposition 2.6 are still homogeneous with respect to this refined grading, which represents another peculiar feature of these algebras, since in general not every $\mathbb{N}^d$-graded $\mathbb{F}$-algebra has a homogeneous system of parameters if $d > 1$.

Clearly a balanced simplicial complex cannot have *too many* edges, since all monochromatic edges are forbidden. This idea will be made more precise and used intensively in Chapter 5. The h-vectors of balanced simplicial complexes which are Cohen-Macaulay are well understood.

Theorem 2.8. [BFS87; Sta79] *Let $h = (h_0, \ldots, h_d) \subseteq \mathbb{Z}^{d+1}_{\geq 0}$. The following are equivalent:*

i. $h = h(\Delta)$*, with Δ a balanced Cohen-Macaulay $(d-1)$-dimensional simplicial complex;*

ii. $h = h(\Delta)$*, with Δ a balanced shellable $(d-1)$-dimensional simplicial complex;*

iii. $h = f(\Delta)$*, with* Δ *a simplicial complex whose* 1*-skeleton is* d*-colorable.*

Sketch of proof. *ii* $\Rightarrow$ *i* is trivial since every shellable complex is Cohen-Macaulay. The implication *i* $\Rightarrow$ *ii* was first proved by Björner, Frankl and Stanley [BFS87] with the technique of *compression*. It is also a consequence of [Mur08, Proposition 4.2] where the author proves that the *colored algebraic shifting* of a pure balanced complex Δ, which is balanced and has the same h-vector as Δ, is shellable. Finally it follows from Proposition 2.6 that if Δ is Cohen-Macaulay and balanced there is an Artinian reduction of $\mathbb{F}[\Delta]$ in which $x_v^2 = 0$ for every $v \in V(\Delta)$, which implies that its monomial $\mathbb{F}$-basis is given by squarefree monomials and hence it forms a simplicial complex Γ (where inclusion is replaced by divisibility). Since all the monomials $x_i x_j$ with $\kappa(i) = \kappa(j)$ are in I_Δ the coloring map κ yields a proper d-coloring of the 1-skeleton of Γ, from which *i* $\Rightarrow$ *iii* follows. For *iii* $\Rightarrow$ *i* we again refer to [BFS87]. $\square$

Observe that *i* $\Rightarrow$ *iii* in Theorem 2.8 does not guarantee that the h-vector of a balanced Cohen-Macaulay complex is the f-vector of a balanced simplicial complex, since in general $\max\{i : h_i \neq 0\} < d$. As an example we can consider the balanced 2-ball Δ in Figure 2.1: its h-vector $h(\partial\mathcal{C}_3 \setminus F) = (1, 3, 3, 0)$ is the f-vector of a unique simplicial complex, namely $\partial\Delta_2$ which is 1-dimensional and whose 1-skeleton is not 2-colorable. Using the partition in color classes we can define a natural refinement of the f- and h-numbers.

Definition 2.9. Let Δ be a balanced $(d-1)$-dimensional simplicial complex. The *flag f-vector* $(f_S(\Delta))_{S\subseteq[d]}$ and the *flag h-vector* $(h_S(\Delta))_{S\subseteq[d]}$ of Δ are given by

$$f_S(\Delta) := |\{F \in \Delta : \kappa(F) = S\}|\,, \text{ and}$$
$$h_S(\Delta) := \sum_{T\subseteq S} (-1)^{|S|-|T|} f_T(\Delta),$$

for any $S \subseteq [d]$.

Flag numbers were originally introduced in the theory of posets, and therefore are of particular interest in the case $\Delta \cong \mathcal{O}(P)$ for some graded poset P, where they count chains of elements of P of certain ranks.

Definition 2.10. For a balanced $(d-1)$-dimensional simplicial complex Δ the subcomplexes $\Delta_S := \Delta_{\{v:\kappa(v)\in S\}}$, with $S \subseteq [d]$, are called *rank-selected subcomplexes.*

It is straightforward to see that a rank-selected subcomplex Δ_S is a balanced $(|S|-1)$-dimensional complex, even though as we pointed out in Remark 2.5 this is not the case for arbitrary induced subcomplexes.

Lemma 2.11. [Sta96, III Theorem 4.5] *Let* Δ *be a balanced* $(d-1)$*-dimensional Cohen-Macaulay simplicial complex. Then* Δ_S *is Cohen-Macaulay for every* $S \subseteq [d]$*.*

Proof. Using the characterization of Cohen-Macaulayness given in Theorem 1.22 we can write $\mathbb{F}[\Delta] = \bigoplus_{m\in B} m\mathbb{F}[\Theta']$, for a certain set of monomials B and the l.s.o.p. $\Theta' = \{\theta'_1, \dots, \theta'_d\}$ defined in Theorem 1.22. Since any rank-selected subcomplex Δ_S is induced, its Stanley-Reisner ideal can be obtain from the one of Δ by modding out all variables corresponding to vertices not in $V(\Delta_S)$. More precisely, we have

$$\mathbb{F}[\Delta_S] = \mathbb{F}[\Delta]/(x_i : \kappa(i) \notin S) = \bigoplus_{m\in B_S} m\mathbb{F}[\theta'_i : i \in S],$$

where B_S is the set of monomials in B not contained in the ideal $(x_i : \kappa(i) \notin S)$. Note that by the Kind-Kleinschmidt criterion [Sta96, III Lemma 2.4] the set $\{\theta'_i : i \in S\}$ is an l.s.o.p. for $\mathbb{F}[\Delta_S]$. Since we showed that $\mathbb{F}[\Delta_S]$ is a free $\mathbb{F}[\theta'_i : i \in S]$-module, the claim follows from Theorem 1.22. $\square$

Remark 2.12. There is a natural way of writing the f- and h-vector of Δ_S in terms of flag f and h-numbers of Δ, namely

$$f_{i-1}(\Delta_S) = \sum_{\substack{T\subseteq S\\|T|=i}} f_T(\Delta) \qquad \text{and} \qquad h_i(\Delta_S) = \sum_{\substack{T\subseteq S\\|T|=i}} h_T(\Delta).$$

2.2 Balanced spheres and manifolds

In this section we investigate which restrictions balancedness imposes on the face numbers of a sphere or manifold. Most of the results in this section can be found in [BK11; GKN11; JKM17; JK+18; KN16b]. Since the h-vector of a balanced homology $(d-1)$-sphere satisfies $h_d(\Delta) = 1$ and the property of being the f-vector of a simplicial complex Γ, we infer that Γ has a $(d-1)$-dimensional face and hence $h_1(\Delta) \geq d$, which together with $h_1(\Delta) = f_0(\Delta) - d$ gives $f_0(\Delta) \geq 2d$. Moreover we observe the following.

Lemma 2.13. [KN16b, Essentially Proposition 6.1] *Let Δ be a balanced $(d-1)$-dimensional pure simplicial complex with color classes V_i for $i \in [d]$. If there exists $i \in [d]$ such that $|V_i| = 1$, then Δ is a cone, i.e., $\Delta = \mathrm{st}_\Delta(v) = \langle v\rangle * \mathrm{lk}_\Delta(v)$, where $V_i = \{v\}$.*
Assume Δ is a pseudomanifold. If there exists $i \in [d]$ such that $|V_i| = 2$, then Δ is a suspension, i.e., $\Delta = \langle\{v\},\{w\}\rangle * \mathrm{lk}_\Delta(v)$, where $V_i = \{v,w\}$.

Proof. If Δ is pure, then every facet contains exactly one vertex per color class, which in turn implies that v is contained in every facet and Δ is a cone. Moreover if Δ is a pseudomanifold then every $(d-2)$-face G containing neither v nor w is contained in two facets $G \cup \{v\}$ and $G \cup \{w\}$ which concludes the proof. $\square$

Inductively Lemma 2.13 implies that any $(d-1)$-dimensional balanced pseudo-manifold on $2d$ vertices is isomorphic to $\partial\mathcal{C}_d$. The boundary of the cross-polytope plays indeed a very special role in the balanced setting, as we will see in the rest of this thesis.

Corollary 2.14. *Let Δ be a balanced homology $(d-1)$-manifold on $V(\Delta) = \dot{\bigcup}_{i=1}^{d} V_i$ that is not an homology sphere. Then, $|V_i| \geq 3$ for every $i \in [d]$.*

Proof. The claim follows from Lemma 2.13 together with the fact that vertex links of homology manifolds are homology spheres. $\square$

Lemma 2.13 has the following immediate consequence.

Corollary 2.15. *There is no balanced homology $(d-1)$-sphere on $2d+1$ vertices.*

Proof. By Lemma 2.13 such a simplicial complex would be isomorphic to the $(d-2)$-fold suspension over a 1-sphere on 5 vertices. Such a sphere, which appears as the link of a $(d-3)$-face, cannot be balanced, since balanced 1-spheres are precisely boundaries of polygons with an even number of vertices. $\square$

Corollary 2.16. *The set of h-vectors of balanced 2-spheres is $\{(1, n-3, n-3, 1) : n = 6 \text{ or } n \geq 8\}$.*

Proof. Corollary 2.15 shows that $(1,4,4,1)$ cannot be the h-vector of a balanced 2-sphere, while Lemma 2.13 establishes the inequality $n \geq 6$. If $n \geq 6$ is even then the suspension over an $(n-2)$-gon is a balanced 2-sphere Δ with $h(\Delta) = (1, n-3, n-3, 1)$, while if $n \geq 9$ is odd we can consider the suspension over an $(n-5)$-gon and replace one of its facets F with $\partial\mathcal{C}_3 \setminus F$. □

The combinatorial type of a 2-sphere is uniquely determined by its graph (see e.g., [Grü03]), which is known to be any graph dual to a planar 3-regular 3-connected graph. For balanced 2-spheres we can observe a similar result.

Proposition 2.17. *Balanced* 2*-spheres on* n *vertices are in bijection with planar* 3*-regular* 3*-connected bipartite graphs on* $2n-4$ *vertices.*

Proof. Since every 2-sphere is polytopal, the dual graph (see Definition 6.5) of a 2-sphere is the graph G of a simple 3-polytope and as such uniquely determines the polytope itself. Even more: a classical theorem of Steinitz [Grü03, Theorem 13.1.1] states that graphs of simple 3-polytopes are precisely 3-connected planar 3-regular graphs. Moreover since in the graph of a balanced 2-sphere every vertex has even degree, (i.e., such a graph is *Eulerian*) and a graph is Eulerian if and only if its dual graph is bipartite, the claim follows. □

Planar graphs as in Proposition 2.17 can be efficiently generated (see [BM07]) and the number of isomorphism classes of these graphs appears in [Slo, A007083]. In the following table we compare the number of combinatorial types of 2-spheres and balanced 2-spheres on n vertices.

n	$\lvert$2-spheres on $[n]\rvert$	$\lvert$Balanced 2-spheres on $[n]\rvert$
4	1	0
5	1	0
6	2	1
7	5	0
8	14	1
9	50	1
10	233	2
11	1249	2
12	7595	8
13	49566	8
14	339722	32
15	2406841	57
16	17490241	185
17	129664753	466
18	977526957	1543
19	7475907149	4583
20	57896349553	15374

TABLE 2.1: A table comparing the number of simplicial 2-spheres and balanced simplicial 2-spheres on n vertices.

2.3 Face numbers of balanced spheres and manifolds

We turn now our attention to face numbers of higher dimensional balanced spheres. In the last ten years this topic has been the target of active research, and many of the classical results stated in Chapter 1 turned out to have balanced analogs. In this section we present and comment on some of these results starting from the balanced lower bound theorem.

Remark 2.18. If Δ and Γ are two pure balanced simplicial complexes of the same dimension, then their connected sum $\Delta \#_{\varphi} \Gamma$ as defined in Definition 1.39 is balanced. Indeed the bijection $\varphi : F \to G$ defines an identification between the color class of v and that of $\varphi(v)$ for every $v \in F$

Definition 2.19. Any simplicial $(d-1)$-sphere that can be obtained as the connected sum of $k-1$ copies of $\partial \mathcal{C}_d$ for some $k \geq 2$ is called a *cross-polytopal stacked* $(d-1)$-*sphere* on kd vertices. For $k \geq 4$ there are several combinatorial types of such spheres (see Figure 2.3), and we denote by $\mathcal{ST}^{\times}(kd, d)$ the set of cross-polytopal stacked $(d-1)$-spheres on kd vertices.

By Remark 2.18 cross-polytopal stacked spheres are balanced complexes, and it is not hard to see that they are polytopal. Their f- and h-vector can be easily computed, and shown to depend only on k and d. We denote by $f(\mathcal{ST}^{\times}(kd, d))$ and $h(\mathcal{ST}^{\times}(kd, d))$ the f- and h-vector of any sphere in $\mathcal{ST}^{\times}(kd, d)$. Via Lemma 1.40 it is straightforward to observe that we have

$$\begin{aligned} f_{i-1}(\mathcal{ST}^{\times}(kd, d)) &= (k-1) f_{i-1}(\partial \mathcal{C}_d) - (k-2) f_{i-1}(\Delta_{d-1}) \\ &= ((k-1)2^i - (k-2)) \binom{d}{i}, \end{aligned}$$

for $i = 1, \ldots, d-1$ and $f_{d-1}(\mathcal{ST}^{\times}(kd, d)) = (k-1) f_{d-1}(\partial \mathcal{C}_d) - 2(k-2) f_{d-1}(\Delta_{d-1}) = 2^d(k-1) - 2(k-2)$. The description of the h-vector is rather simple, since again via Lemma 1.40 we obtain

$$h_i(\mathcal{ST}^{\times}(kd, d)) = (k-1) \binom{d}{i}$$

for every $i = 1, \ldots, d-1$. In analogy with the case of arbitrary normal pseudomani-

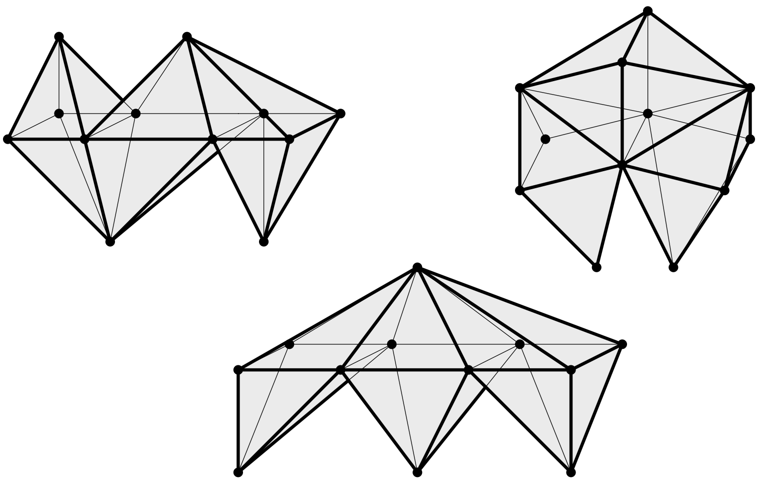

FIGURE 2.3: Three spheres in $\mathcal{ST}^{\times}(12, 3)$ that are not combinatorially isomorphic.

folds, there is a similar, though stronger, inequality involving the numbers h_1 and h_2 for the balanced case. We refer to this result as the *Balanced Lower Bound Theorem (BLBT)*.

Theorem 2.20. [GKN11; BK11; KN16b] (BLBT) *Let Δ be a balanced normal $(d-1)$-pseudomanifold with $d \geq 3$. Then $2h_2(\Delta) \geq (d-1)h_1(\Delta)$. In particular, if $f_0(\Delta) = kd$ for some $k \geq 2$, then $f_i(\Delta) \geq f_i(\mathcal{ST}^{\times}(kd, d))$ for every $i \geq 0$.*

The BLBT represents a non-trivial strengthening of Theorem 1.43. Theorem 2.20 was first stated by Goff, Klee and Novik [GKN11] for balanced 2-Cohen-Macaulay complexes (a class which includes homology spheres) and later extended to balanced Buchsbaum* complexes (a class which includes homology manifolds) by Browder and Klee [BK11]. The validity for balanced normal pseudomanifold was finally established by Klee and Novik [KN16b]. Their proof relies on the fact that for a balanced normal $(d-1)$-pseudomanifold with $d \geq 3$, and for a subset $S \subseteq V(\Delta)$ with $|S| = 3$, the rank selected subcomplex Δ_S is *generically d-rigid*, a property which ensures the following.

Lemma 2.21. [KN16b, Lemma 3.5] *Let Δ be a balanced normal $(d-1)$-pseudomanifold with $d \geq 3$ and $S \subseteq [d]$ with $|S| = 3$. Then:*

$$f_1(\Delta_S) \geq f_0(\Delta_S)d - \binom{d+1}{2},$$

or, equivalently, $h_2(\Delta_S) \geq h_1(\Delta_S)$.

In order to convert the inequalities for the subcomplexes Δ_S into one for Δ we proceed with elementary identities involving the h-numbers of rank-selected subcomplexes. Observe that any subset $T \subseteq [d]$ with $|T| = i$ is contained is exactly $\binom{d-i}{k-i}$ many subsets $S \subseteq [d]$ with $|S| = k$, and hence it follows that

$$\sum_{|S|=k} h_i(\Delta_S) = \sum_{|S|=k} \sum_{\substack{T \subseteq S \\ |T|=i}} h_i(\Delta_T) = \sum_{|T|=i} \binom{d-i}{k-i} h_i(\Delta_T) = \binom{d-i}{k-i} h_i(\Delta). \tag{2.1}$$

In particular for $k = 3$ and $i = 1, 2$ we obtain

$$\sum_{|S|=3} h_1(\Delta_S) = \binom{d-1}{2} h_1(\Delta) \quad \text{and} \quad \sum_{|S|=3} h_2(\Delta_S) = (d-2)h_2(\Delta).$$

The power of Lemma 2.21 is then to establish that $(d-2)h_2(\Delta) \geq \binom{d-1}{2}h_1(\Delta)$, from which Theorem 2.20 follows. For the equivalent statement in terms of the f-numbers, combining Lemma 1.7 and Lemma 2.4, we observe that the family of balanced normal pseudomanifolds is closed under taking links, and hence the standard MPW-reduction argument still holds.

Remark 2.22. If $d \geq 4$ does not divide the number n of vertices, [KN16b, Theorem 4.1] shows that the inequality in Theorem 2.20 is never attained by a balanced normal $(d-1)$-pseudomanifold. Indeed the authors proved that $2h_2(\Delta) = (d-1)h_1(\Delta)$ if and only if Δ is a stacked cross-polytopal sphere.

As in the previous chapter we can go one step further and investigate how the first Betti number can be employed to sharpen the bound in Theorem 2.20. Juhnke-Kubitzke, Murai, Novik and Sawaske [JK+18] proved a balanced analog of Theorem 1.49.

Theorem 2.23. [JK+18](BLBT for manifolds) *Let Δ be a balanced $\mathbb{F}$-homology $(d-1)$-manifold with $d \geq 3$. Then $2h_2(\Delta) - (d-1)h_1(\Delta) \geq 4\binom{d}{2}(\widetilde{\beta}_1(\Delta;\mathbb{F}) - \widetilde{\beta}_0(\Delta;\mathbb{F}))$.*

As in Theorem 1.49 the proof relies on the theory of Buchsbaum rings, which we do no treat here. We point out that in [JK+18] the authors obtain similar inequalities as the one in Theorem 2.23 for all numbers $h_i(\Delta)$, under the assumption that the links of faces in a certain dimension satisfy the weak Lefschetz property.
In the next paragraphs we discuss the case of equality in Theorem 1.49. It is worth remarking the analogy with the case of arbitrary homology manifolds presented in Chapter 1.

Remark 2.24. Consider a bijection $\varphi : F \to G$ between two facets of a pure $(d-1)$-simplicial complex Δ as in Definition 1.46. Assume moreover that Δ is balanced with coloring map κ. If the bijection φ is color preserving (i.e., $\kappa(v) = \kappa(\varphi(v))$) for every $v \in F$, then the handle addition on Δ is a balanced complex. Under this condition we refer to this operation as the *balanced handle addition.*

Definition 2.25 (Balanced Walkup class)**.** The *balanced Walkup class* $\mathcal{BH}_d$ is the set of all $(d-1)$-dimensional simplicial complexes which can be obtained through a sequence of connected sums of disjoint copies of $\partial\mathcal{C}_d$ and balanced handle addition.

Proposition 2.26. *Let Δ be a connected balanced $\mathbb{F}$-homology $(d-1)$-manifold for $d \geq 5$. Then the following are equivalent:*

i. $\Delta \in \mathcal{BH}_d$;

ii. $2h_2(\Delta) - (d-1)h_1(\Delta) = 4\binom{d}{2}\widetilde{\beta}_1(\Delta; \mathbb{F})$;

iii. $\mathrm{lk}_\Delta(v) \in \mathcal{ST}^\times(k(d-1), d-1)$ *for some* $k \geq 2$ *and for every vertex* $v \in \Delta$.

The implications $i \Rightarrow ii$ and $i \Rightarrow iii$ follow simply from the definition, while the proofs of $ii \Rightarrow i$ and $iii \Rightarrow i$ can be found [JK+18] and [KN16b] respectively. Note that the assumption $d \geq 5$ leaves open the case $d = 4$, for which Proposition 2.26 remains a conjecture (see [KN16b, Conjecture 4.14]).
Recall that for a simplicial d-polytope P it holds that $h_{i-1}(\partial P) \leq h_i(\partial P)$, for $1 \leq d \leq \lfloor \frac{d}{2} \rfloor$. The *balanced generalized lower bound theorem (BGLBT)* for polytopes shows that the balanced case is once more subject to stricter inequalities. Interestingly the numbers $h_i(\partial\mathcal{C}_d) = \binom{d}{i}$ seem to play a normalizing role in the result.

Theorem 2.27. [JKM17, Theorem 1.3] (BGLBT for polytopes) *Let ∂P be a balanced $(d-1)$-dimensional polytopal sphere. Then*

$$\frac{h_0(\partial P)}{\binom{d}{0}} \leq \frac{h_1(\partial P)}{\binom{d}{1}} \leq \cdots \leq \frac{h_{\lfloor \frac{d}{2} \rfloor}(\partial P)}{\binom{d}{\lfloor \frac{d}{2} \rfloor}}.$$

The main ingredient for the proof of Theorem 2.27 lies in the following Lefschetz-type result.

Lemma 2.28. [JKM17] *Let ∂P be a balanced $(d-1)$-dimensional polytopal sphere. For every $S \subseteq [d]$ there exists an l.s.o.p. Θ_S of $\mathbb{F}[\partial P_S]$ and a linear form ω such that the map induced by multiplication*

$$\times\omega^{|S|-2i} : (\mathbb{F}[\partial P_S]/\Theta_S\mathbb{F}[\partial P_S])_i \longrightarrow (\mathbb{F}[\partial P_S]/\Theta_S\mathbb{F}[\partial P_S])_{|S|-i}$$

is injective for every $i \leq \frac{|S|}{2}$.

Proof of Theorem 2.27. Note that it follows from elementary linear algebra that $\times\omega$ is also an injective map. In particular since by Lemma 2.11 the subcomplexes ∂P_S are Cohen-Macaulay a direct consequence of Lemma 2.28 is that

$$h_{i-1}(\partial P_S) = \dim_{\mathbb{F}}(\mathbb{F}[\partial P_S]/\Theta_S\mathbb{F}[\partial P_S])_{i-1} \leq \dim_{\mathbb{F}}(\mathbb{F}[\partial P_S]/\Theta_S\mathbb{F}[\partial P_S])_i = h_i(\partial P_S)$$

for $i \leq \frac{|S|+1}{2}$ and for every $S \subseteq [d]$. Using (2.1) with $k = 2i - 1$ we obtain

$$\binom{d-i+1}{i} h_{i-1}(\partial P) \leq \binom{d-i}{i-1} h_i(\partial P)$$

for $i \leq \lfloor \frac{d}{2} \rfloor$, which is equivalent to the statement of Theorem 2.27. □

To extend the analogy between the balanced and the standard setting it would be natural to try to formulate a balanced upper bound theorem for spheres in the spirit of the result of McMullen and Stanley. Unfortunately there does not seem to be an equally satisfying statement. Let us call a $(d-1)$-dimensional balanced simplicial complex Δ *balanced k-neighborly* if all the k-subsets of $V(\Delta)$ containing at most one element in each color class are faces of Δ. Clearly the number of edges of a balanced 2-neighborly complex is an upper bound for the number of edges of all balanced simplicial complexes with the same vertex partition in color classes. Observe that the boundary of the d-cross-polytope is a balanced d-neighborly $(d-1)$-sphere, and it follows from Kuratowski's obstruction for planar graphs that $\partial\mathcal{C}_3$ is the unique balanced 2-neighborly 2-sphere. Recently the problem of finding balanced k-neighborly spheres was studied by Zheng [Zhe16].

Proposition 2.29. [Zhe16] *The following statements hold.*

- *The color classes of a balanced k-neighborly homology $(2k-1)$-sphere have all the same size.*
- *There is no balanced 2-neighborly homology 3-sphere on 12 vertices.*
- *There is no balanced 2-neighborly homology 4-sphere on 15 vertices.*
- *There exists a balanced 2-neighborly homology 3-sphere on 16 vertices. In particular, taking suspensions, there exists a balanced 2-neighborly homology $(3+m)$-sphere on $16+2m$ vertices for every $m \geq 0$.*

The result above suggests that the case of equipartitioned color classes, which implies that the number of vertices is a multiple of d, plays a special role. Currently no other balanced neighborly spheres are known in literature, nor it is known whether the sphere in the last statement of Proposition 2.29 is polytopal, a problem which can be phrased in terms of realizability of the complete multipartite graph as the graph of a simplicial polytope.

Chapter 3

A balanced non-partitionable Cohen-Macaulay complex

3.1 The partitionability conjecture

The study of the possible decompositions plays a fundamental role in the combinatorics of simplicial complexes, as presented in Chapter 1. The focus of this chapter is yet on another way to decompose simplicial complexes, namely *partitionability*.

Definition 3.1. A pure simplicial complex Δ with facets $F_1, \ldots, F_n$ is called *partitionable* if there exists a partitioning of Δ into pairwise disjoint Boolean intervals

$$\Delta = \dot{\bigcup_{i=1}^{n}} [R_i, F_i],$$

where $[R_i, F_i] = \{G \in \Delta \ : \ R_i \subseteq G \subseteq F_i\}$.

It was shown by Stanley [Sta96, Proposition III.2.3] that the h-vector of a partitionable simplicial complex Δ has a combinatorial interpretation analog to that of shellable complexes:

$$h_i(\Delta) = |\{1 \le j \le n \ : \ |R_j| = i\}|\,. \tag{3.1}$$

In particular, all h-vector entries are non-negative in this case. Another interesting feature of partitionability is that it can be tested efficiently by solving the mixed integer linear program

$$\begin{array}{lll} \text{maximize} & \sum\limits_{(R,F)} 2^{\dim(F)-\dim(R)} x_{(R,F)} & \\ \text{subject to} & x_{(R,F)} + x_{(R',F')} \le 1 & \text{if } [R,F] \cap [R',F'] \neq \emptyset \\ & x_{(R,F)} \in \{0,1\} & \text{for every facet } F \text{ and } R \subseteq F \end{array} \tag{3.2}$$

A d-dimensional simplicial complex Δ is partitionable if and only if the optimal value of the linear program in (3.2) equals $\sum_{i=-1}^{d} f_i(\Delta)$.

Example 3.2. Let Δ be a shellable simplicial complex. The intervals $[R_i, F_i]$, where R_i is the restriction face of the facet F_i give a partition of Δ, and hence shellability implies partitionability.

It is natural to ask how partitionability is related to constructibility and Cohen-Macaulayness. More than 30 years ago Björner observed that there are partitionable simplicial complexes that are not Cohen-Macaulay and hence neither constructible nor shellable (see Figure 3.1). In particular we can observe using the identity (3.1) that the h-vector of the complex in Figure 3.1 is (1,3,0,1), which is clearly not the h-vector of any Cohen-Macaulay complex. On the opposite direction the *Partitionability*

Conjecture due to Stanley [Sta79] (for all Cohen-Macaulay complexes) and Garsia (for Cohen-Macaulay posets) remained open for a long time.

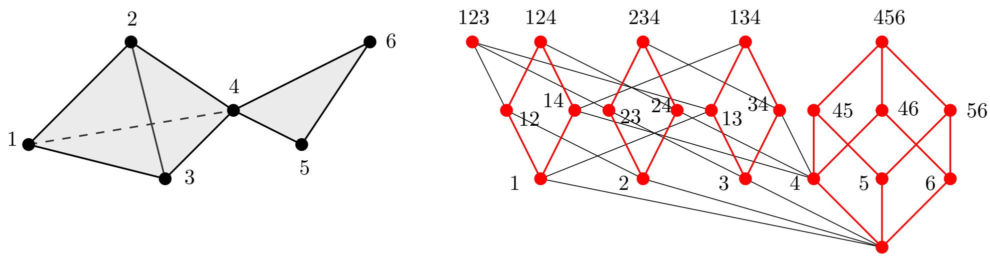

FIGURE 3.1: Björner's example of a partitionable complex that is not Cohen-Macaulay.

Conjecture 3.3. [Sta79] Every Cohen-Macaulay simplicial complex is partitionable.

Recently the work of Duval, Goeckner, Klivans and Martin [Duv+16] disproved this conjecture by exhibiting three dimensional non-partitionable Cohen-Macaulay complexes, which, after taking iterated cones, yields counterexamples in higher dimensions.

Theorem 3.4. [Duv+16] *There exists a* 3*-dimensional Cohen-Macaulay simplicial complex which is not partitionable.*

This result still leaves open Stanley's conjecture for some special classes, such as balanced Cohen-Macaulay simplicial complexes. Indeed the construction in [Duv+16] does not result in a complex that is 4-colorable, and since balancedness poses constraints on how dense the face poset can be it looks plausible to ask the following question.

Question 3.5. [Duv+16] Is every balanced Cohen-Macaulay simplicial complex partitionable?

In the rest of this chapter we answer this question in the negative. Using the technique introduced in [Duv+16], we construct an infinite family of balanced non-partitionable Cohen-Macaulay complexes. The main idea is to start with the counterexample in Theorem 3.4 and to *remove* the obvious obstructions to balancedness. We will make this idea more precise in Section 3.3. Indeed, our counterexample can be obtained from the one in Theorem 3.4 by performing a finite number of edge subdivisions. As in [Duv+16], our counterexample is not only Cohen-Macaulay but even constructible. As such, it is the first example of a balanced constructible non-partitionable simplicial complex [Hac00b, §4].

3.2 Relative simplicial complexes

A key ingredient in the proof of Theorem 3.4 is the use of relative simplicial complexes. A *relative simplicial complex* is a pair (Δ, Γ) of simplicial complexes, where $\Gamma \subseteq \Delta$ is a subcomplex of Δ. As for simplicial complexes, elements of $\Delta \setminus \Gamma$ are called *faces* of (Δ, Γ) and the *dimension* of (Δ, Γ) is the maximal dimension of a face in $\Delta \setminus \Gamma$. Other notions from arbitrary simplicial complexes carry over to relative simplicial complexes in exactly the same way. If (Δ, Γ) is a relative simplicial complex and Ω is a simplicial complex, then the pair $(\Delta \cup \Omega, \Gamma \cup \Omega)$ represents the same relative simplicial complex as (Δ, Γ) and every pair of simplicial complexes representing (Δ, Γ)

arises in this way. In particular, every relative simplicial complex (Δ, Γ) has a unique minimal representation $(\overline{\Omega}, \overline{\Omega} \setminus \Omega)$, where $\Omega = \Delta \setminus \Gamma$ and

$$\overline{\Omega} = \{F \ : \ F \subseteq G \text{ for some } G \in \Omega\}$$

is the minimal simplicial complex containing Ω. We also call $\overline{\Omega}$ the *combinatorial closure* of Ω. We will make use of this minimal representation of relative simplicial complexes in the construction of our counterexample.

Precisely as in the non-relative case the *f-vector* of a $(d-1)$-dimensional relative simplicial complex Φ is $f(\Phi) = (f_{-1}(\Phi), f_0(\Phi), \ldots, f_{d-1}(\Phi))$, where $f_i(\Phi)$ denotes the number of i-dimensional faces of Φ. The *h-vector* $h(\Phi) = (h_0(\Phi), h_1(\Phi), \ldots, h_d(\Phi))$ of Φ is defined by the relation

$$\sum_{i=0}^{d} f_{i-1}(\Phi)(t-1)^{d-i} = \sum_{i=0}^{d} h_i(\Phi) t^{d-i}.$$

Relative Stanley-Reisner theory has been recently successfully employed in [AS16] to prove the upper bound theroem for Minkowski sums, and face numbers of Cohen-Macaulay and shellable relative complexes have been studied in [CKS17].

Since our construction of a balanced non-partitionable Cohen-Macaulay complex is essentially a balanced version of the example from [Duv+16], we now recall this construction.

Example 3.6. The construction in [Duv+16] starts with a particular subcomplex Q of Ziegler's famous example of a non-shellable 3-ball on 10 vertices, labeled $0, \ldots, 9$ [Zie98]. More precisely, the subcomplex Q is the combinatorial closure of the following set of facets

$$\begin{aligned} \mathcal{F}(Q) = \{&\{1,2,4,9\}, \{1,2,6,9\}, \{1,5,6,9\}, \{1,5,8,9\}, \{1,4,8,9\}, \\ &\{1,4,5,8\}, \{1,4,5,7\}, \{4,5,7,8\}, \{1,2,5,6\}, \{0,1,2,5\}, \\ &\{0,2,5,6\}, \{0,1,2,3\}, \{1,2,3,4\}, \{1,3,4,7\}\}. \end{aligned}$$

Let $A = Q_{\{0,2,3,4,6,7,8\}}$ be the induced subcomplex of Q on vertex set $\{0,2,3,4,6,7,8\}$, i.e., A is the combinatorial closure of

$$\{\{0,2,6\}, \{0,2,3\}, \{2,3,4\}, \{3,4,7\}, \{4,7,8\}\}.$$

The complexes A and Q are depicted in Figure 3.2. Theorem 3.4 in [Duv+16] shows that glueing together 25 copies of Q along the subcomplex A produces a non-partitionable, constructible, hence Cohen-Macaulay simplicial complex. In fact, Theorem 3.5 in [Duv+16] shows that already 3 copies of Q, identified along A, yield such an example. However, since Q is not balanced, neither are those examples. One important fact, that was used intensively in [Duv+16] and that we will also employ in the next section, is that $\tau = (0\,7)(2\,4)(6\,8)$ is an automorphism of Q.

3.3 The balanced construction

In this section, we provide our construction of a balanced non-partitionable Cohen-Macaulay simplicial complex.

The following main tool from [Duv+16] is crucial for this construction.

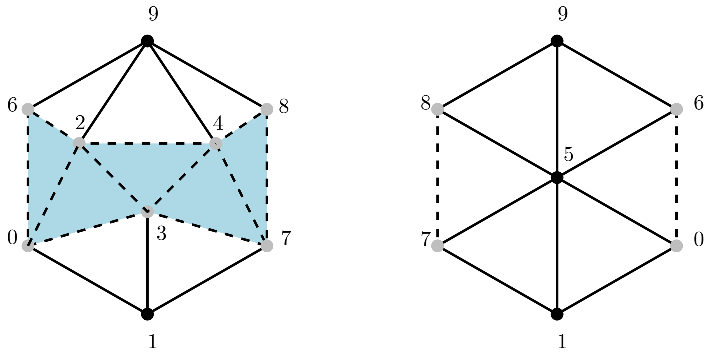

FIGURE 3.2: Front (*left*) and back (*right*) view of Q. The blue and dashed faces belong to A.

Theorem 3.7. [Duv+16, Theorem 3.1] *Let $X = (Q, A)$ be a relative simplicial complex such that:*

i. Q and A are Cohen-Macaulay;

ii. A is an induced subcomplex of Q of codimension at most 1;

iii. X is not partitionable.

For a positive integer N let C_N denote the simplicial complex obtained by identifying N disjoint copies of Q along A. If N is larger than the total number of faces of A, then C_N is Cohen-Macaulay and not partitionable.

Remark 3.8. Our construction makes use of a property of pure balanced simplicial complexes exhibited in Lemma 2.4: the links of a pure balanced simplicial complex are pure and balanced. Note that the converse of this statement is not necessarily true (e.g., cycles with an odd number of vertices).

In the following, we call a vertex v (or its vertex link $\mathrm{lk}_\Delta(\{v\})$) *critical* if $\mathrm{lk}_\Delta(\{v\})$ is not balanced and *uncritical* otherwise.

Before proceeding to our construction, we describe its underlying idea. If we look at the simplicial complex Q from Example 3.6, we easily see that vertices $0, 3, 7$ are uncritical, whereas all the other vertices are critical. Hence, by Lemma 2.4, those are obvious obstructions that prevent Q from being balanced. The idea now is to perform some (possibly few) edge subdivisions that make the critical vertex links balanced without affecting balancedness of other uncritical vertex links and without altering the symmetry of the simplicial complex Q. Luckily, – though this is not guaranteed by Remark 3.8 – it will turn out, that the simplicial complex, obtained in this way, is already balanced. We now make this idea more precise. We perform the following subdivision steps:

Step 1: We first subdivide the edge $\{2, 4\}$ by introducing a new vertex 10. In this way, the link of the former critical vertex 9 becomes the cone over a 6-gon and as such is balanced. Figure 3.3 shows the link of the vertex 9 before and after the subdivision. Moreover, the permutation $\tau = (0\,7)(2\,4)(6\,8)$ is still an automorphism of the subdivided complex (see Example 3.6).

Step 2: In the next step, we subdivide the edge $\{5, 9\}$ by adding a vertex 11. It is easy to check that the vertices 6 and 8 are now uncritical and that τ is still an

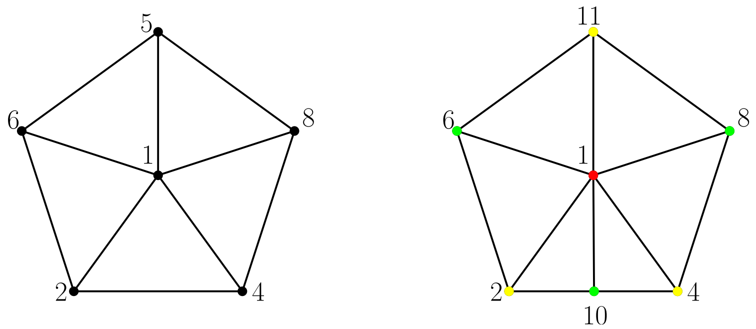

FIGURE 3.3: The link of 9 before (*left*) and after (*right*) the edge subdivision of $\{2,4\}$ and $\{5,9\}$. On the right the vertices are properly colored.

automorphism of this new complex.

Step 3: Subdividing the edges $\{0,6\}$ and $\{7,8\}$, the vertices 2 and 4, respectively become uncritical. Moreover, also 5 has an uncritical link now. Labeling the new vertex on the edge $\{0,6\}$ with 12 and the one on $\{7,8\}$ with 13, we also see that the permutation $\tau' = (0\,7)(2\,4)(6\,8)(12\,13)$ is an automorphism of the subdivided complex.

We call the simplicial complex obtained from the just described edge subdivisions Q^*. The top row of Figure 3.4 depicts the front and back view of Q^*. It is easy to check that – though we did not treat the critical vertex 1 separately – the simplicial complex Q^* has only uncritical vertices. We even have the following:

Lemma 3.9. *The simplicial complex Q^* constructed above is balanced.*

Proof. The bottom row of Figure 3.4 shows the 3-dimensional simplicial complex Q^* together with the proper 4-coloring obtained by partitioning the vertices as

$$\kappa^{-1}(\{1\}) = \{0,6,7,8,10\}, \qquad \kappa^{-1}(\{2\}) = \{1,12,13\},$$
$$\kappa^{-1}(\{3\}) = \{2,4,11\}, \qquad \kappa^{-1}(\{4\}) = \{3,5,9\}.$$

□

Another reasonable approach to construct a balanced counterexample to the partitionability conjecture could have been to start with a balanced non-shellable ball and then to try to apply the technique from [Duv+16]. However, all examples of balanced non-shellable balls are relatively big and it is hard to see, which subcomplex one should choose then. Indeed to the best of our knowledge the smallest balanced non-shellable 3-sphere known in the literature is the one presented in Chapter 6 of this thesis, which has 28 vertices. We also want to remark that applying the same edge subdivisions as above directly to Ziegler's non-shellable ball, does not produce a balanced ball.

The following simple remark will be useful later.

Remark 3.10. If Δ is a balanced simplicial complex, then any simplicial complex built from Δ by taking a certain number of copies of Δ and identifying them along a fixed subcomplex, is balanced.

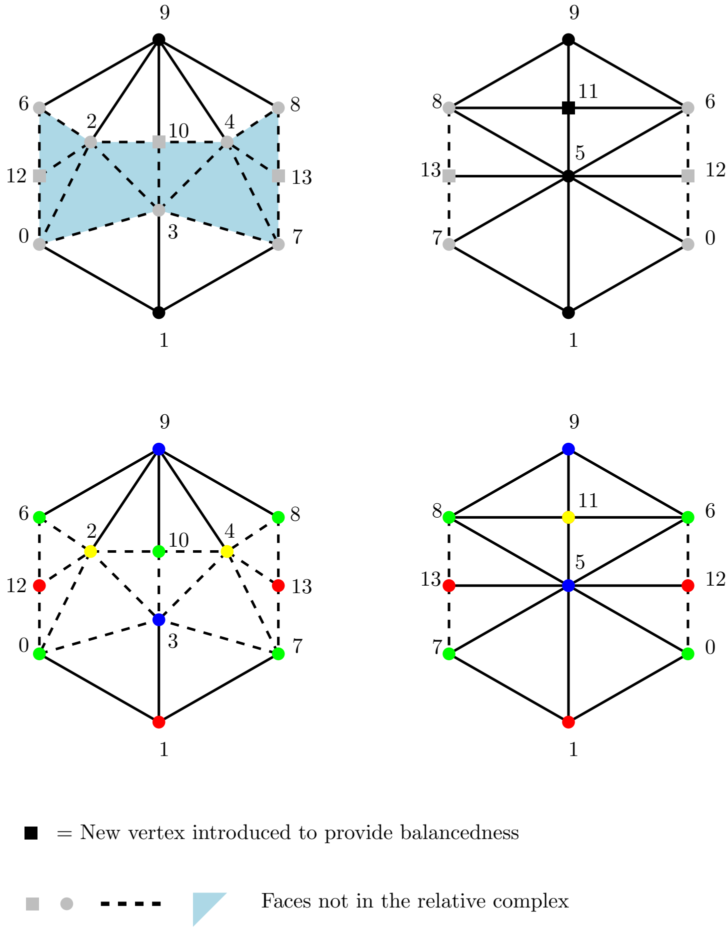

FIGURE 3.4: Front and back view of Δ.

We define a subcomplex A^* of Q^* to be the induced subcomplex $Q^*_{\{0,2,3,4,6,7,8,10,12,13\}}$. Note that $\dim A^* = 2$ and that A^* can be obtained from A in the same way we constructed Q^* from Q; namely, by subdividing the edges $\{2,4\}$, $\{0,6\}$ and $\{7,8\}$. (We do not subdivide $\{5,9\}$ since it is not present in A.) As edge subdivisions preserve shellability, we get the following lemma:

Lemma 3.11. *The simplicial complexes Q^* and A^* are shellable, hence constructible and Cohen-Macaulay.*

Our final goal is to apply Theorem 3.7 to the relative simplicial complex (Q^*, A^*). The only ingredient missing to be able to do so is to verify that condition *iii* of Theorem 3.7 is fulfilled. Indeed, we have the following statement:

Theorem 3.12. *The relative simplicial complex $X := (Q^*, A^*)$ is not partitionable.*

Proof. The proof uses similar ideas as the ones employed in the proof of [Duv+16, Theorem 3.3]. In fact, some parts are even verbatim the same but we include them for sake of completeness.

Assume by contradiction that X is partitionable. We will show that, in this case, the vertex 5 has to be contained in at least two intervals of any partitioning, which gives a contradiction.

For the sake of clearness we list the facets of both Q^* and A^*.

$$\begin{aligned}\mathcal{F}(Q^*) = \{&\{1,2,9,10\},\{1,4,9,10\},\{1,2,6,9\},\{1,5,6,11\},\{1,6,9,11\},\{1,5,8,11\},\\&\{1,8,9,11\},\{1,4,8,9\},\{1,4,5,8\},\{1,4,5,7\},\{4,5,7,13\},\{4,5,8,13\},\\&\{1,2,5,6\},\{0,1,2,5\},\{0,2,5,12\},\{2,5,6,12\},\{0,1,2,3\},\{1,2,3,10\},\\&\{1,3,4,10\},\{1,3,4,7\}\}\\ \mathcal{F}(A^*) = \{&\{0,2,3\},\{4,7,13\},\{3,4,10\},\{0,2,12\},\{3,4,7\},\{2,3,10\},\{2,6,12\},\\&\{4,8,13\}\}\end{aligned}$$

Given a partitioning $\mathcal{P}$ of X and a facet $F \in \mathcal{F}(Q^*)$, we denote by I_F the interval of $\mathcal{P}$ with top element F.
As $\{1,4,8,9\}$ is the only facet containing the triangle $\{4,8,9\}$, we have $\{4,8,9\} \in I_{\{1,4,8,9\}}$. If also $\{1,4,8\} \in I_{\{1,4,8,9\}}$, it follows that $I_{\{1,4,8,9\}}$ must contain $\{4,8\} = \{4,8,9\} \cap \{1,4,8\}$, which is a contradiction since $\{4,8\} \in A^*$. Therefore, $\{1,4,8\} \notin I_{\{1,4,8,9\}}$ and since $\{1,4,5,8\}$ is the only other facet of Q^* containing $\{1,4,8\}$, we conclude that $\{1,4,8\} \in I_{\{1,4,5,8\}}$. Again, as $\{4,8\} \in A^*$ and $\{4,8\} = \{1,4,8\} \cap \{4,5,8\}$, it must hold that $\{4,5,8\} \notin I_{\{1,4,5,8\}}$ and hence also $\{4,5\} \notin I_{\{1,4,5,8\}}$. The other facets of X containing $\{4,5\}$ are

$$\{4,5,8,13\},\{4,5,7,13\},\{1,4,5,7\}. \tag{3.3}$$

Using that τ' is an automorphism of Q^* and A^*, the same line of arguments applied to $\{2,6,9\}$ yields that the edge $\{2,5\}$ has to be contained in an interval with one of the following top elements:

$$\{2,5,6,12\},\{0,2,5,12\},\{0,1,2,5\}.$$

We now distinguish four cases:
Case 1: $\{4,5\} \in I_{\{4,5,8,13\}}$ and $\{2,5\} \in I_{\{2,5,6,12\}}$
As $\{4,5,8,13\}$ is the only facet containing $\{5,8,13\}$, we must have $\{5,8,13\} \in \{4,5,8,13\}$. As $\{5\} = \{4,5\} \cap \{5,8,13\}$ we infer that that $\{5\} \in I_{\{4,5,8,13\}}$. Similarly, using again that τ' is an automorphism of Q^* and A^*, we get that $\{5\} \in I_{\{2,5,6,12\}}$. Hence $\{5\}$ is contained in two intervals, which is a contradiction.

Case 2: $\{4,5\} \notin I_{\{4,5,8,13\}}$ and $\{2,5\} \notin I_{\{2,5,6,12\}}$
As $\{4,5\} \notin I_{\{4,5,8,13\}}$ it follows from (3.3) that $\{4,5\} \in I_{\{4,5,7,13\}}$ or $\{4,5\} \in I_{\{1,4,5,7\}}$. Since $\{4,5,7,13\}$ and $\{1,4,5,7\}$ are also the only facets of Q^* containing $\{4,5,7\}$ and since $\{4,5\},\{5,7\} \subseteq \{4,5,7\}$, it follows that they have to lie in the same interval, together with $\{5\} = \{4,5\} \cap \{5,7\}$. Therefore, we either have

$$\{5\} \in I_{\{1,4,5,7\}} \qquad \text{or} \qquad \{5\} \in I_{\{4,5,7,13\}}. \tag{3.4}$$

Applying the automorphism τ' to the above argument yields

$$\{5\} \in I_{\{0,1,2,5\}} \qquad \text{or} \qquad \{5\} \in I_{\{0,2,5,12\}}.$$

Hence, $\{5\}$ belongs to two intervals, which is a contradiction.

Case 3: $\{4,5\} \notin I_{\{4,5,8,13\}}$ and $\{2,5\} \in I_{\{2,5,6,12\}}$
As $\{4,5\} \notin I_{\{4,5,8,13\}}$, the argument of Case 2 shows that (3.4) holds. We now show that $\{5\}$ has to lie in a second interval.

Note that the only two facets containing $\{5, 12\}$ are $\{2, 5, 6, 12\}$ and $\{0, 2, 5, 12\}$. Since both of these contain $\{2, 5, 12\}$ and $\{5, 12\} \subseteq \{2, 5, 12\}$, it follows that $\{5, 12\}$ and $\{2, 5, 12\}$ have to belong to the same interval. Moreover, since $\{2, 5\} \in I_{\{2,5,6,12\}}$ by assumption, we must have $\{2, 5, 12\} \in I_{\{2,5,6,12\}}$ and hence $\{5, 12\} \in I_{\{2,5,6,12\}}$. Finally, this implies $\{5\} = \{2, 5\} \cap \{5, 12\} \in I_{\{2,5,6,12\}}$ and therefore again $\{5\}$ lies in two intervals, which is a contradiction.

Case 4: $\{4, 5\} \in I_{\{4,5,8,13\}}$ and $\{2, 5\} \notin I_{\{2,5,6,12\}}$
We reach a contradiction in this case by applying the automorphism τ' to the arguments of Case 3. This finishes the proof. □

The (relative) simplicial complexes Q^*, A^* and $X = (Q^*, A^*)$ have the following f-vectors:

$$f(Q^*) = (1, 14, 45, 52, 20)$$
$$f(A^*) = (1, 10, 17, 8)$$
$$f(X) = (0, 4, 28, 44, 20).$$

In particular, the subcomplex A^* has a total number of 36 faces. Theorem 3.7, Theorem 3.12, Lemma 3.11 and Remark 3.10 therefore imply our main result:

Theorem 3.13. *The simplicial complex C_{37} constructed from 37 disjoint copies of Q^* and identifying them along A^* is balanced, Cohen-Macaulay and not partitionable.*

Analogous to the situation in [Duv+16, Theorem 3.5] we note that a much smaller counterexample to the balanced partitionability conjecture can be found by glueing together only 3 copies of Q^*.

Theorem 3.14. *The simplicial complex C_3 obtained by taking 3 disjoint copies of Q^* and identifying them along A^* is balanced, Cohen-Macaulay and not partitionable.*

We omit the proof of the above theorem since it is verbatim the same as the one of Theorem 3.5 in [Duv+16], if one exchanges the automorphism τ by τ'.

The f-vector of the simplicial complex C_3 is $f(C_3) = (1, 22, 101, 140, 60)$. As Q^* and A^* are both constructible by Lemma 3.11, it follows by definition that C_3 is also constructible, and hence we have a balanced counterexample for a conjecture in [Hac00b, Section 4].

Corollary 3.15. *The simplicial complex C_3 is balanced, non-partitionable and constructible.*

We conclude commenting on some open questions. First of all we do not know if C_3 is the smallest balanced simplicial complex that is Cohen-Macaulay but not partitionable. However, it is possible to see, e.g., by solving the linear program (3.2), that C_2 is partitionable. Moreover, it is not possible to turn Q into a balanced simplicial complex by fewer than 4 edge subdivisions. On the other hand, if one finds a counterexample to the partitionability conjecture, which is smaller than the one of [Duv+16], then it might well be the case that one can also construct a counterexample to the balanced partitionability conjecture that is smaller than C_3. It is an open question if every barycentric subdivision of a Cohen-Macaulay simplicial complex is partitionable. This is indeed a particular case of the conjecture of Garsia [Gar80], which states that every Cohen-Macaulay poset (a ranked poset whose order complex is Cohen-Macaulay) is partitionable. Again solving (3.2) we verify that the barycentric subdivision of the simplicial complex C_3 from [Duv+16] is partitionable.

Chapter 4

Balanced combinatorial manifolds with boundary

4.1 Connecting manifolds with boundary

An *(elementary) shelling* is the removal of a facet F from a simplicial complex Δ with the additional requirement that the set

$$\{G \subseteq F \ : \ G \notin \Delta \setminus F\}$$

has a unique minimal element. A pure d-dimensional simplicial complex is *shellable* if it can be reduced to the d-simplex by a sequence of shellings. Shellability naturally extends to more general objects, such as polyhedral complexes and simplicial posets, and it has become an important concept not only in topological combinatorics [Bjö95] and polyhedral theory [BM71] but also in piecewise linear topology [Bin83; RS82], algebraic combinatorics and combinatorial commutative algebra [Sta96] as well as poset theory [BW96; BW97; Wac07]. Prominent examples of shellable simplicial complexes comprise e.g., triangulations of 2-spheres [DK78] and boundary complexes of polytopes, as shown by Brugesser and Mani [BM71]. The latter was used by McMullen in his proof of the Upper Bound Theorem, providing tight upper bounds on the face numbers of convex polytopes [McM70]. Shellability also places strong conditions on the topology of a simplicial complex as we saw in Proposition 1.11. In particular, any shellable pseudomanifold is homeomorphic to a ball or a sphere. Interestingly, there exists combinatorial balls and spheres that are non-shellable (see e.g., [Rud58; Zie98] and [HZ00; Lic91]).

Allowing not only shellings but also the inverse operations, Pachner [Pac91, Theorem 6.3] could show the following result:

Theorem 4.1. *Two combinatorial manifolds with boundary are PL homeomorphic if and only if they are related by a sequence of shellings and inverse shellings.*

Once again our focus lies on balanced simplicial complexes, introduced in Chapter 2. In particular, if Δ and Γ are balanced PL homeomorphic manifolds, the sequence provided by Theorem 4.1 might contain non-balanced simplicial complexes in intermediate steps. Our main result shows that this obstruction can be avoided.

Theorem 4.2. *Two balanced combinatorial manifolds Δ and Γ with boundary are PL homeomorphic if and only if they can be connected by a sequence of shellings and inverse shellings that preserves balancedness in each step.*

Theorem 4.2 provides a positive answer to Problem 1 in [IKN17], posed by Izmestiev, Klee and Novik. Our proof technique combines ideas of Pachner's proof of Theorem 4.1 and methods developed and employed in [IKN17]. As a key step, we use those ideas together with a result by Casali [Cas95, Proposition 4] to show the following:

Theorem 4.3. *Let Δ and Γ be balanced combinatorial manifolds with $\partial\Delta \cong \partial\Gamma$. Assume moreover that the isomorphism preserves the coloring. Then Δ and Γ are PL homeomorphic if and only if they are related by a sequence of cross-flips.*

The previous result provides an analog of Theorem 1.2 in [IKN17], where the corresponding statement was shown for closed manifolds. Roughly speaking a cross-flip is the balanced analog of a bistellar flip and it substitutes a subcomplex of the boundary of the cross-polytope with its complement (we defer more details and the precise definitions to Section 4.2). With Theorem 4.3 in hand, the strategy to show Theorem 4.2 is to first reduce to the situation that Δ and Γ have the same boundary and then to convert each cross-flip needed to transform Δ into Γ into a sequence of shellings and inverses. The latter requires two ingredients: first, the construction of shellings for particular subcomplexes of the boundary of the cross-polytope, relative to their boundaries (Theorem 4.38), and second, building a “collar” around a manifold in order to protect its boundary – an idea which goes back to Pachner (Theorem 4.34).

It was shown in [IKN17] that in order to relate any two closed balanced PL homeomorphic manifolds it is enough to consider a restricted set of moves, referred to as *basic* cross-flips (see Section 4.2.2 for the precise definition). [IKN17, Problem 2] asks for a description and the number of combinatorially distinct basic cross-flips. We provide an answer to this question, which can be summarized as follows (see Theorems 4.42 and 4.53 for the detailed statements):

Theorem 4.4. *There are $2^{d+1} - 1$ combinatorially distinct basic cross-flips in dimension d, out of which 2^d are sufficient to relate any two d-dimensional PL homeomorphic balanced manifolds without boundary or with the same boundary.*

The enumeration of combinatorially distinct cross-flips relies on a detailed study of their combinatorics. For the proof of the second part of Theorem 4.4 we construct a set M of basic cross-flips with $|M| = 2^d$ such that any other basic cross-flip can be expressed as a combination of cross-flips in M.

The layout of this chapter is as follows. Section 4.2 provides necessary background on simplicial complexes and the combinatorics of local moves. Section 4.3 contains the proof of Theorem 4.2. In Section 4.4 we prove the statements of Theorem 4.4. We end the chapter with some open problems in Section 4.5.

4.2 Moves on simplicial complexes

In this section we describe and define various moves (including stellar subdivisions, bistellar flips and cross-flips) on combinatorial manifolds. The last part of this section discusses shellability and (inverse) shellings.

4.2.1 Stellar moves and bistellar flips

In this section we define different local moves on simplicial complexes and state well known results on the equivalence classes determined by such moves.

Given a simplicial complex Δ and a face $F \in \Delta$, the *stellar subdivision* of Δ at F is the simplicial complex

$$\mathrm{sd}_F(\Delta) = \Delta \setminus F \cup (\langle v \rangle * \partial F * \mathrm{lk}_\Delta(F))$$

where $v \notin \Delta$ is a new vertex. If two simplicial complexes can be transformed one into the other by a sequence of stellar subdivisions and their inverses (stellar welds), we say that they are *stellarly equivalent*. Clearly, neither subdivisions nor welds do affect the topology of a simplicial complex. Indeed the following classical result was shown by Alexander:

Theorem 4.5. [Ale30, Theorem 10:3] *Two simplicial complexes Δ and Γ are PL homeomorphic if and only if they are stellarly equivalent.*

Several other results in the same flavor exist, e.g., Alexander [Ale30] and Newman [New31] independently showed that edge subdivisions and welds suffice, and Ludwig and Reitzner provided a "geometric" version of this result for polytopes [LR06]. Moreover, Lutz and Nevo [LN16] proved that PL homeomorphic flag simplicial complexes can be transformed into each other by a sequence of edge subdivisions and welds such that flagness is preserved in each step.

As the number of facets added by a stellar subdivision at a face $F \in \Delta$ depends on the combinatorics of the link $\mathrm{lk}_\Delta(F)$, unfortunately there are infinitely many combinatorially different stellar subdivisions even if both the dimension of Δ and the dimension of F are fixed. The following set of moves, that was introduced by Pachner [Pac78], remedies this situation by providing finitely many moves for each dimension.

Let Δ be a d-dimensional simplicial complex and assume that there exists a face $A \in \Delta$ such that $\mathrm{lk}_\Delta(A) = \partial B$, for some $B \notin \Delta$. A *bistellar flip* (or *bistellar move*) on Δ is the operation $\chi_{A,B}$ defined by

$$\Delta \mapsto \chi_{A,B}(\Delta) = \Delta \setminus (\langle A \rangle * \partial B) \cup (\partial A * \langle B \rangle),$$

i.e., a bistellar flip exchanges $\langle A \rangle * \partial B$ with $\partial A * \langle B \rangle$. Clearly, the inverse of a bistellar flip $\chi_{A,B}$ is given by the bistellar flip $\chi_{B,A}$. Two simplicial complexes Δ and Γ are called *bistellar equivalent* if they are related by a sequence of bistellar flips. We write $\Delta \overset{\mathrm{bst}}{\approx} \Gamma$ in this case. Bistellar moves admit a nice and simple geometric description: indeed the bistellar flip $\chi_{A,B}$ just replaces the subcomplex $\langle A \rangle * \partial B$ that is isomorphic to a subcomplex D of $\partial \Delta_{d+1}$ which is a d-ball, with the complex $\partial A * \langle B \rangle$ that is isomorphic to the complement of D in $\partial \Delta_{d+1}$.

As $\partial \Delta_{d+1}$ has exactly $d+1$ combinatorially different pure d-dimensional subcomplexes (that are all d-balls), there are exactly $d+1$ distinct bistellar flips in dimension d. Figure 4.1 depicts all bistellar flips in dimension 2. The following analog of Theorem 4.5 is due to Pachner (see also [Lic99] for a proof).

Theorem 4.6. [Pac91, Theorem 5.5] *Two closed combinatorial manifolds Δ and Γ are PL homeomorphic if and only if $\Delta \overset{\mathrm{bst}}{\approx} \Gamma$.*

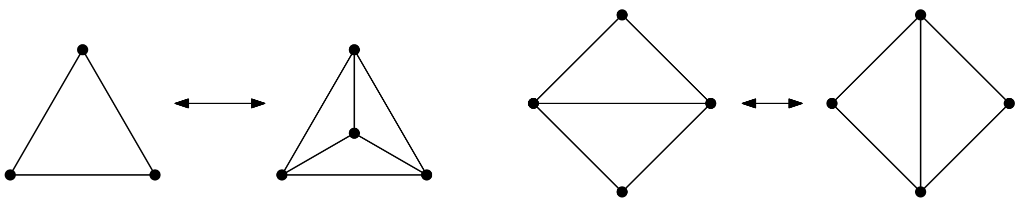

FIGURE 4.1: Bistellar flips for $d = 2$.

Since any bistellar flip can be written as a composition of a stellar subdivision and a weld, the "If"-part of Theorem 4.6 is a direct consequence of Theorem 4.5. However, for closed manifolds stellar and bistellar equivalence even turn out to be equally strong. Indeed, Pachner [Pac91, Lemma 4.8] showed that every stellar subdivision at a face F in the interior of a simplicial complex Δ such that $\mathrm{lk}_\Delta(F)$ is shellable can be realized by a sequence of bistellar flips. Casali [Cas95, Proposition 4] could improve on this result by proving that the shellability assumption is not necessary. As an almost immediate consequence, she obtained the following analog of Theorem 4.6 for manifolds with boundary.

Proposition 4.7. [Cas95, Main Theorem] *Let Δ and Γ be combinatorial d-manifolds with isomorphic boundaries, i.e., $\partial\Delta \cong \partial\Gamma$. Then Δ and Γ are PL homeomorphic if and only they are bistellar equivalent.*

Observe that, since bistellar flips do not affect the boundary, all manifolds constructed from the sequence of bistellar flips, guaranteed by the previous proposition, have the same boundary. We will use Proposition 4.7 together with this simple observation in Section 4.3.2.

4.2.2 Bistellar moves and cross-flips on balanced complexes

A natural question that arises from the previous sections is whether there are analogs of Theorem 4.5 and Theorem 4.6 for balanced complexes.

It is easy to see that stellar subdivisions might destroy balancedness of a simplicial complex. In fact, even if the resulting complex after applying such a move is balanced, in general the vertex colors are not preserved. As the balanced analog of stellar subdivisions, Fisk [Fis77a; Fis77b] introduced balanced stellar subdivisions (see also [IJ03]) and recently, Murai and Suzuki [MS18] showed that even in dimension 2 Theorem 4.5 does not have a balanced analog. See [KMS19] for further results in the 2-dimensional case.

At first glance the situation appears similar for bistellar subdivisions. In general balancedness is not maintained and, even if it is, the vertex colors might change. Nevertheless, Izmestiev, Klee and Novik obtained the following colored version of Theorem 4.6.

Theorem 4.8. [IKN17, Theorem 1.1] *Let Δ and Γ be closed combinatorial d-manifolds that are PL homeomorphic. Assume that Δ and Γ are properly m-colored, $m \geq d+2$. Then there exists a sequence of bistellar flips that transforms Δ into Γ such that each intermediate complex is properly m-colored and the flips preserve the vertex colors.*

Note that the last result does not cover the case of balanced combinatorial manifolds and indeed, as remarked before, balancedness might be destroyed by bistellar flips. The proof of Theorem 4.8 makes use of a so-called m-colorable *pseudo-cobordism* that connects Δ and Γ by a sequence of shellings and inverse shellings. The latter sequence is turned into a sequence of bistellar flips between Δ and Γ. However, the coloring of the pseudo-cobordism requires at least $d+2$ colors. A similar idea will be used in Section 4.3.5 to prove our main result Theorem 4.2.

For balanced complexes the right analog of bistellar flips are so-called *cross-flips*, introduced in [IKN17, Definition 2.6]. Recall that a bistellar flip can be defined by substituting a d-ball in $\partial\Delta_{d+1}$ by its complement. For balanced complexes it has turned out that the boundary of the $(d+1)$-dimensional cross-polytope plays the same role as $\partial\Delta_{d+1}$ does for arbitrary simplicial complexes (see e.g., [IKN17; JKM17; JK+18; KN16b]). The definition of cross-flips combines those two insights. More precisely, we consider the boundary complex $\partial\mathcal{C}_{d+1}$ of the $(d+1)$-dimensional cross-polytope. We call a pure subcomplex $D \subseteq \partial\mathcal{C}_d$ *co-shellable* if $\partial\mathcal{C}_d \setminus D$ is a shellable simplicial complex (see Definition 1.10).

Definition 4.9. Let Δ be a balanced d-dimensional simplicial complex and let $D \subsetneq \Delta$ be an induced d-dimensional subcomplex that is isomorphic to a shellable and co-shellable subcomplex of $\partial\mathcal{C}_d$. The operation χ_D^* given by

$$\Delta \mapsto \chi_D^*(\Delta) = (\Delta \setminus D) \cup (\partial\mathcal{C}_{d+1} \setminus D)$$

is called a *cross-flip* on Δ. If two balanced simplicial complexes Δ and Γ are connected by a sequence of cross-flips, we write $\Delta \overset{\text{crs}}{\approx} \Gamma$.

The fact that D is shellable and co-shellable directly implies that a cross-flip exchanges a combinatorial d-ball by another combinatorial d-ball sharing the same boundary. In particular, Δ and $\chi_D^*(\Delta)$ are PL homeomorphic. It is easy to see that $\chi_D^*(\Delta)$ is balanced and that the coloring is preserved. Moreover, the inverse of the cross-flip χ_D^* is given by the cross-flip $\chi^*_{\partial\mathcal{C}_{d+1}\setminus D}$, which justifies the notation $\Delta \overset{\text{crs}}{\approx} \Gamma$. It is important to underline that the shellability of D implies its co-shellability if and

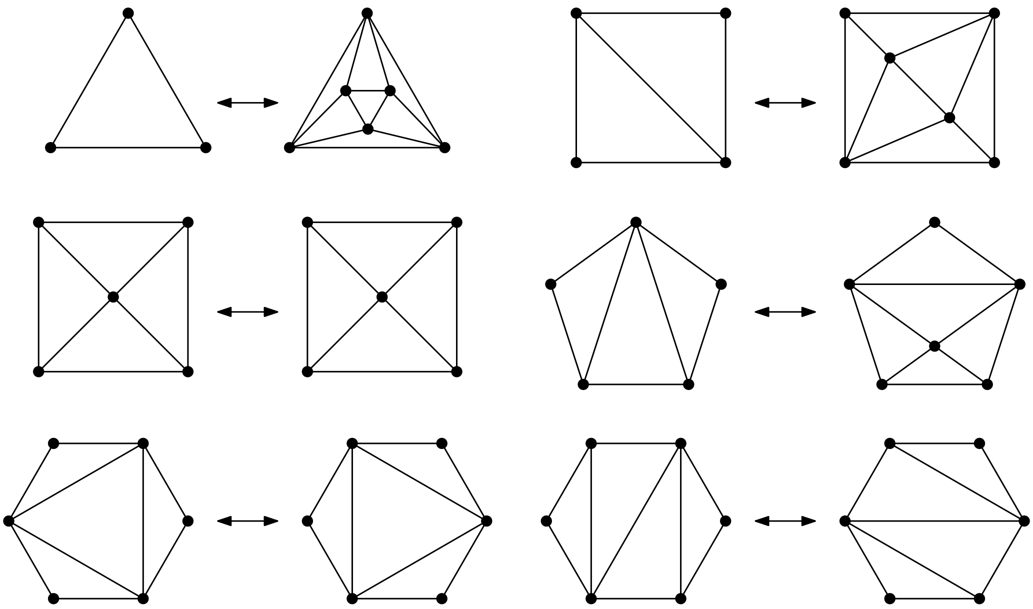

FIGURE 4.2: Cross-flips for $d = 2$.

only if $\partial\mathcal{C}_{d+1}$ is extendably shellable. For $d \geq 11$, Hall showed $\partial\mathcal{C}_{d+1}$ is not extendably shellable [Hal04].

The following result provides the balanced analog of Pachner's Theorem 4.6.

Theorem 4.10. [IKN17, Theorem 1.2] *Two closed balanced combinatorial manifolds Δ and Γ are PL homeomorphic if and only if $\Delta \overset{\text{crs}}{\approx} \Gamma$.*

Clearly, there are only finitely many d-dimensional cross-flips. However, compared to the number of d-dimensional bistellar flips, their number is considerable. It is therefore natural to ask if Theorem 4.10 can be improved by showing that a particular subset of cross-flips suffices. Indeed, in [IKN17] it is shown that such a set is provided by the so-called *basic cross-flips*. We now recall this construction, which uses the so-called *diamond operation*.

For simplicity, we assume that the vertices of the $(d+1)$-simplex Δ_{d+1} are labeled by $0, \dots, d+1$. Let $\Gamma \subseteq \partial\Delta_{d+1}$ be a pure and d-dimensional subcomplex. Following [IKN17, Section 3.3], we define another combinatorial d-ball $\Diamond(\Gamma)$ that is a subcomplex of $\partial\mathcal{C}_{d+1}$ in the following way: For $0 \leq i \leq d$: If $F_i = \{i+1, \dots, d+1\}$ is a face of Γ, then recursively stellar subdivide Γ at F_i and label the newly introduced vertex by v_i. In particular, the vertex $F_d = \{d+1\}$ gets renamed with v_d in this procedure. Note that $\Diamond(\partial\Delta_{d+1})$ gives the boundary $\partial\mathcal{C}_{d+1}$ of the $(d+1)$-dimensional cross-polytope on vertex set $\{0,\ldots,d\} \cup \{v_0,\ldots,v_d\}$ (see [IKN17, Lemma 3.6]). As stellar subdivisions preserve shellability, $\Diamond(\Gamma)$ is a shellable and co-shellable d-ball if $\Gamma \subsetneq \partial\Delta_{d+1}$ is a d-ball. Those are exactly the balls one considers in the definition of basic cross-flips.

Definition 4.11. Let $\Gamma \subsetneq \partial\Delta_{d+1}$ be a d-ball. The cross-flip $\chi^*_{\Diamond(\Gamma)}$ is called a *basic cross-flip.*

The inverse of the basic cross-flip $\chi^*_{\Diamond(\Gamma)}$ is again a basic cross-flip, namely $\chi^*_{\partial\mathcal{C}_{d+1}\setminus\Diamond(\Gamma)}$. We also want to remark that the diamond operation can easily be extended to any pure balanced $(d+1)$-dimensional simplicial complex Δ with a specified coloring by first interpreting the vertex colors as vertex labels and then applying the diamond operation to the boundary of every $(d+1)$-simplex. In this way, a balanced simplicial complex can be converted into a cross-polytopal complex. We will use this idea from [IKN17] in Section 4.3.

In [IKN17] (essentially the proof of Theorem 3.10; cf., [IKN17, Remark 3.12]), the following improvement of Theorem 4.10 was proven:

Theorem 4.12. *Two closed balanced combinatorial manifolds are PL homeomorphic if and only if they can be obtained from each other by a sequence of basic cross-flip.*

Though the number of basic cross-flip is already considerably smaller than the number of cross-flips, a priori there are still about $2^{d+2}-2$ many (one for each proper subcomplex of Δ_{d+1} that is a d-ball). However, not all of those are combinatorially different. In dimension 2 there are 11 combinatorially distinct cross-flips (see Figure 4.2), but only the 7 in the first two lines of Figure 4.2 turn out to be basic. Moreover, the move in the middle row on the left is the trivial move, which does not have any effect.

It is hence natural to raise the following problem:

Problem 4.13. [IKN17, Problem 2] Give an explicit description of basic cross-flips. How many combinatorially distinct basic cross-flips are there?

Theorem 4.42 will provide a solution to this question.

4.2.3 Shellings and their inverses

It is worth remarking that all operations considered so far leave the boundary of a simplicial complex unchanged. Hence, if one wants to connect PL homeomorphic manifolds with boundary another set of moves is needed. This set is provided by shellings and inverse shellings. Recall from Definition 1.10 that a pure d-dimensional simplicial complex Δ is shellable if there exists an ordering $F_1,\ldots,F_m$ of the facets of Δ such that for every $1 \le i \le m$ the set $\{G \subseteq F_i \,:\, G \nsubseteq F_j \text{ for } 1 \le j \le i-1\}$ has a unique minimal element R_i, the so-called restriction face of F_i. The ordering $F_1,\ldots,F_m$ is called a shelling of Δ. Here we present a slightly more restrictive definition of shelling, which will be used for the rest of the chapter, since we need to preserve the boundary.

Definition 4.14. Let Δ be a pure d-dimensional simplicial complex and let $F \in \Delta$ be a facet. Assume that F can be written as $F = A \cup R$, where

i. $\dim A \ge 0$, $\dim R \ge 0$,

ii. $A \in \overset{\circ}{\Delta}$,

iii. $\partial A * \langle R \rangle \subseteq \partial\Delta$.

The operation

$$\Delta \overset{\mathsf{sh}}{\longmapsto} \Delta \setminus F$$

is called an (elementary) *shelling* on Δ. The inverse operation is referred to as *inverse shelling*.

If two pure simplicial complexes Δ and Γ are related by a sequence of shellings and inverse shellings, we write $\Delta \overset{\mathsf{sh}}{\approx} \Gamma$.

Now let $F_1, \dots, F_m$ be a shelling of a pure d-dimensional simplicial complex Δ and let us define $\Delta_i = \langle F_1, \dots, F_i \rangle$ for $1 \leq i \leq m$. It follows directly from the definition that for every facet F_i we either have $F_i = R_i$, or F_i can be decomposed as in Definition 4.14 (with Δ_i in place of Δ). In particular, it follows that shellable combinatorial manifolds with boundary are exactly those simplicial complexes that can be transformed into a simplex by a sequence of shellings (without inverses), which implies that any shellable combinatorial manifold with boundary is a combinatorial ball. Similarly, any shellable combinatorial manifold without boundary is a combinatorial sphere. Once again, shellings and their inverses preserve the PL homeomorphism type and Pachner showed that also the converse is true.

Theorem 4.15. [Pac91, Theorem 6.3] *Two combinatorial manifolds with boundary Δ and Γ are PL homeomorphic if and only if $\Delta \overset{sh}{\approx} \Gamma$.*

Following the line of discussion of the previous section it is natural to ask, what happens if in Theorem 4.15 one assumes Δ and Γ to be balanced. On the one hand, shellings are rather harmless, since no new edges are created and since the resulting complex is a subcomplex of the starting complex. On the other hand, inverse shellings with $\dim(R) = 1$ create new edges and those might be monochromatic. In particular, balancedness is destroyed in this case. This motivates the following question by Izmestiev, Klee and Novik [IKN17, Problem 1]:

Question 4.16. Can any two PL homeomorphic balanced combinatorial manifolds with boundary be related by a sequence of elementary shellings and inverse shellings, such that balancedness (and the coloring) is preserved in each intermediate step?

If two balanced combinatorial manifolds with boundary Δ and Γ can be connected by such a sequence, we write $\Delta \overset{\mathsf{bsh}}{\approx} \Gamma$. Inverse shellings that preserve balancedness will also be referred to as *balanced inverse shellings* in the following. It is not hard to see that Question 4.16 has an affirmative answer if Δ and Γ are balanced shellable balls, since in this case they can be reduced to the simplex only using shellings. However, as already mentioned, there are combinatorial balls that are non-shellable. Our main result Theorem 4.2 answers Question 4.16 in the positive in full generality and thereby provides a balanced analog of Theorem 4.15. In particular, together with Theorem 4.6, Theorem 4.10 and Theorem 4.15 it completes the picture, telling us which moves are necessary to relate any two PL homeomorphic manifolds (with or without) boundary in the balanced as well as in the non-balanced case.

4.3 Balanced shellings for combinatorial manifolds with boundary

The aim of this section is to prove our main result Theorem 4.2. The proof will require several intermediate steps and we start with a brief outline of the proof strategy, which should serve as a golden thread in this section.

Let us assume that Δ and Γ are balanced PL homeomorphic combinatorial manifolds with boundary.

Step 1: First, via shellings we convert Δ into a balanced manifold Δ' such that Δ' and Γ have isomorphic boundaries with the same coloring. (This is Proposition 4.18.)

Step 2: It follows from Proposition 4.7 that Δ' and Γ can be connected by a sequence of bistellar flips. Adapting Theorem 4.8, Lemma 5.2 and Corollary 5.3 from [IKN17] to our situation, we encode this sequence of bistellar flips by a shellable pseudo-cobordism (Ω, ϕ, ψ) between Δ' and Γ. (This is Corollary 4.25.)

Step 3: Applying the diamond operation to Ω yields a cross-polytopal complex. As a result, every bistellar flip is converted into a basic cross-flip. (This is Theorem 4.26.)

Step 4: The last step consists of converting every cross-flip into a sequence of shellings, followed by a sequence of balanced inverse shellings (see Theorem 4.38). This step also requires building a balanced collar around a balanced manifold with boundary, an idea already appearing in the proof of Theorem 4.15 by Pachner. (This is Theorem 4.34.)

Step 2 and 3 provide the proof of Theorem 4.3 by adapting the proof of Theorem 1.2 in [IKN17] to our setting.

4.3.1 Step 1: Restricting to manifolds with the same boundary

We consider two balanced PL homeomorphic manifolds Δ and Γ of dimension d. Our aim is to show that, using shellings and balanced inverse shellings, we can transform them in such a way that they have isomorphic boundary complexes that moreover have the same induced coloring.

First note that the boundary complexes $\partial\Delta$ and $\partial\Gamma$ are closed $(d-1)$-dimensional manifolds that are properly $(d+1)$-colorable. In fact, those boundaries might even be d-colorable and as such balanced. By Theorem 4.8, we know that there is a sequence of bistellar flips connecting $\partial\Delta$ with $\partial\Gamma$ such that each intermediate complex is properly $(d+1)$-colored. It now remains to encode this sequence of bistellar flips on $\partial\Delta$ as a sequence of shellings and balanced inverse shellings on Δ. The next lemma fulfills this task in the non-balanced situation.

Lemma 4.17. *Let Δ be a combinatorial d-manifold with boundary. Let $A \in \partial\Delta$ such that $\mathrm{lk}_{\partial\Delta}(A) = \partial B$ for some $B \notin \partial\Delta$. Then there exists a combinatorial d-manifold with boundary Δ' that is obtained from Δ by a single shelling, or inverse shelling, and $\partial\Delta' = \chi_{A,B}(\partial\Delta)$.*

Combining the previous result with Theorem 4.8, we can now prove the main result of this section.

Proposition 4.18. *Let Δ and Γ be balanced combinatorial d-manifolds with boundary that are PL homeomorphic. Then there exists a balanced combinatorial d-manifold with boundary Δ' such that*

i. $\Delta \overset{\mathsf{bsh}}{\approx} \Delta'$,

ii. there exists a simplicial isomorphism $\varphi : \partial\Delta' \to \partial\Gamma$ that preserves the coloring, i.e., if κ' and κ are proper $(d+1)$-colorings of Δ' and Γ, respectively, then $\kappa'(v) = \kappa(\varphi(v))$ for all $v \in V(\partial(\Delta'))$.

Proof. Since Δ and Γ are balanced PL homeomorphic combinatorial d-manifolds with boundary, it follows that their boundaries, $\partial\Delta$ and $\partial\Gamma$, are closed combinatorial $(d-1)$-manifolds that are PL homeomorphic and properly $(d+1)$-colorable. By Theorem 4.8 there exists a sequence of bistellar flips from $\partial\Delta$ to $\partial\Gamma$ such that each intermediate complex is properly $(d+1)$-colored, and the flips preserve the vertex colors. Due to Lemma 4.17, this sequence induces a sequence of shellings and inverse shellings from Δ to some PL homeomorphic manifold Δ' whose boundary is isomorphic to $\partial\Gamma$. Moreover, Theorem 4.8 ensures that the vertex colors of $\partial\Delta'$ are preserved under this isomorphism. To see that none of the inverse shellings in the constructed sequence destroys balancedness, it is enough to remark that newly created edges lie in the boundary, which is itself properly $(d+1)$-colored. $\square$

In general, even if two balanced manifolds have the same boundary, their colorings restricted to the boundary might be different. The first row of Figure 4.3 shows an example of this phenomenon. However, Theorem 4.8 guarantees that one can enforce a particular coloring just using bistellar flips. An illustration for this is given in the second row of Figure 4.3, where the boundary of a triangle is related to the boundary of a square with a prescribed proper 2-coloring through a sequence of bistellar flips. If we drop any requirement on the coloring, the first move already suffices. Proposition 4.18 enables us to convert balanced manifolds Δ and Γ into

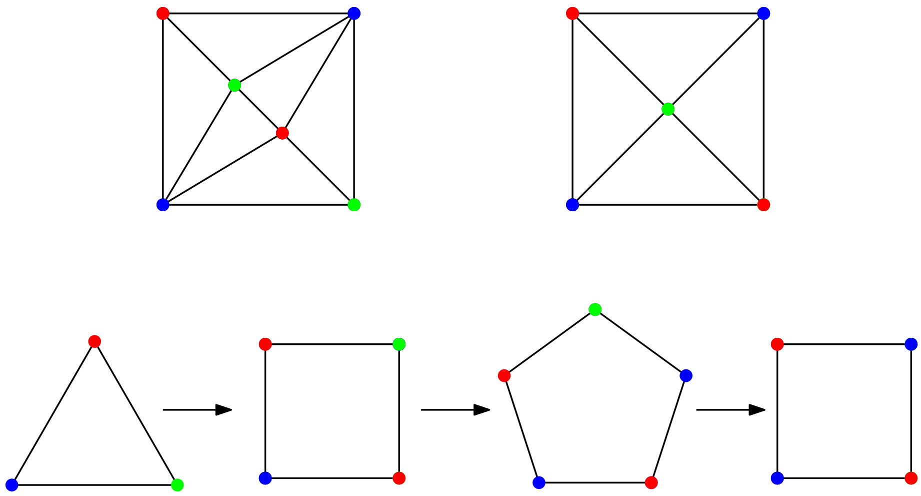

FIGURE 4.3: *First row*: two balanced 2-balls with isomorphic, though differently colored, boundaries. *Second row*: a sequence of bistellar flips connecting a triangle and a square with a prescribed coloring.

manifolds with the same boundary and then to glue those two manifolds along their boundary via the map φ. In this way, we will obtain a simplicial poset. This idea will be made more precise in the next section.

4.3.2 Step 2: Constructing a shellable pseudo-cobordism

The aim of this section is to prove that analogs of Theorem 4.8, Lemma 5.2 and Corollary 5.3 of [IKN17] hold for combinatorial manifolds that have the same boundary. This will require to generalize the notion of a pseudo-cobordism from [IKN17], that is for closed manifolds Δ and Γ, to non-closed manifolds with isomorphic boundaries. The proofs are almost verbatim the same as the ones in [IKN17] and we will therefore only describe the overall strategy and indicate the differences.

Before giving the definition of a pseudo-cobordism, we need to recall some notions concerning simplicial posets. A *simplicial poset* is a finite poset Ω with a unique minimal element $\emptyset$ such that for each $F \in \Omega$ the interval $[\emptyset, F] = \{G \in \Omega \ : \ \emptyset \leq G \leq F\}$ is isomorphic to a Boolean lattice. Here, we denote by $\leq$ the order relation on Ω. A *relative simplicial poset* is a pair of posets (Ω, Σ), such that $\Sigma \subseteq \Omega$ is a lower order ideal of Ω, i.e., $\sigma \in \Sigma$ and $\tau < \sigma$ implies $\tau \in \Sigma$. Note that a simplicial poset Ω can be identified with the relative simplicial poset $(\Omega, \emptyset)$. The set of (relative) simplicial posets contains the set of (relative) simplicial complexes, but this inclusion is strict, see e.g., Figure 4.4 for an example of a simplicial poset that is not a simplicial complex.

Faces and *facets* of a relative simplicial poset (Ω, Γ) are elements and inclusion-maximal elements of $\Omega \setminus \Gamma$, respectively. The *dimension* of a face $F \in \Omega \setminus \Gamma$ is defined as $\dim F = \mathrm{rk}([\emptyset, F]) - 1$, where $\mathrm{rk}([\emptyset, F])$ denotes the rank of the Boolean interval $[\emptyset, F]$. The *dimension* of (Ω, Σ) is the maximal dimension of its facets and we say that (Ω, Σ) is *pure* if all facets have the same dimension. A pure relative simplicial poset (Ω, Σ) is called *shellable* if there exists an ordering $F_1, \ldots, F_m$ of the facets of (Ω, Σ) such that for every $1 \leq i \leq m$ the set

$$\{G \in [\emptyset, F_i] \ : \ G \notin \bigcup_{j=1}^{i-1} [\emptyset, F_j] \cup \Sigma\}$$

has a unique minimal element R_i.

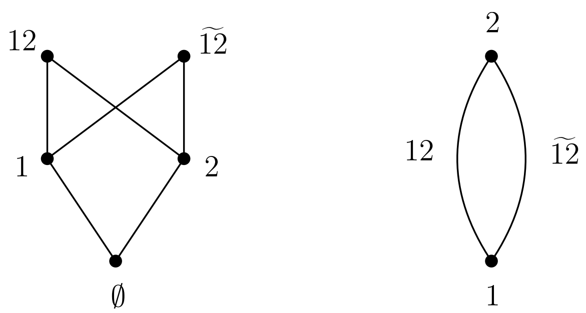

FIGURE 4.4: A simplicial poset that is not a simplicial complex (*left*) and its corresponding geometric realization (*right*).

The definition of a *balanced* simplicial poset is completely analogous to the one of a balanced simplicial complex and is natural in the sense that a simplicial poset might have multiple edges but no loops. A $(d+1)$-dimensional simplicial poset Ω is a *nonpure pseudomanifold* if every d-face is properly contained at most two facets. The *pseudoboundary* $\tilde{\partial}\Omega$ of Ω is the subposet of Ω induced by the d-faces contained in zero or one $(d+1)$-face. We are now ready to define a pseudo-cobordism.

Definition 4.19. Let Δ and Γ be combinatorial d-manifolds with $\partial\Delta \cong \partial\Gamma$. (We also allow $\partial\Delta = \partial\Gamma = \emptyset$.) A *pseudo-cobordism* (Ω, φ, ψ) between Δ and Γ is a nonpure pseudomanifold Ω together with two simplicial embeddings $\varphi : \Delta \hookrightarrow \Omega$ and $\psi : \Gamma \hookrightarrow \Omega$ such that:

i. $\varphi(\Delta) \cup \psi(\Gamma) = \tilde{\partial}\Omega$,

ii. a d-face $F \in \Omega$ lies in $\varphi(\Delta) \cap \psi(\Gamma)$ if and only if F is not contained in any $(d+1)$-face of Ω,

iii. $\varphi(\partial\Delta) = \psi(\partial\Gamma)$.

Note that if Δ and Γ are closed manifolds then the assumption $\partial\Delta \cong \partial\Gamma$ is trivially satisfied and condition *iii* is vacuous. In this case, we thus recover the definition of a pseudo-cobordism from [IKN17]. Also observe that condition *iii* already implies $\partial\Delta \cong \partial\Gamma$, so that one might omit this assumption in the definition. However, we decided to keep it in order to emphasize that the definition is only for manifolds satisfying this condition. We will mostly be interested in *shellable* pseudo-cobordisms:

Definition 4.20. Let Δ and Γ be combinatorial d-manifolds such that $\partial\Delta \cong \partial\Gamma$. A pseudo-cobordism (Ω, φ, ψ) between Δ and Γ is *shellable* if there is an ordering $F_1, \dots, F_t$ of the $(d+1)$-faces of Ω such that

i. $F_1, \dots, F_t$ is a shelling order on the relative simplicial poset $(\Omega, \varphi(\Delta))$,

ii. $F_t, \dots, F_1$ is a shelling order on the relative simplicial poset $(\Omega, \psi(\Gamma))$.

The simplest example of a shellable pseudo-cobordism is provided by a bistellar flip. More precisely, given a simplicial complex Δ and a face $A \in \Delta$ such that $\mathrm{lk}_\Delta(A) = \partial B$, for some $B \notin \Delta$, the simplicial complex $\Delta \cup (\langle A \rangle * \langle B \rangle)$ is called an *elementary pseudo-cobordism*. Indeed, using that a bistellar flip does not modify the boundary, it is not difficult to see that if Δ is a combinatorial manifold, then $\Delta \cup (\langle A \rangle * \langle B \rangle)$ is a shellable pseudo-cobordism between Δ and $\chi_{A,B}(\Delta)$ with the obvious embeddings. The following characterization of a shellable pseudo-cobordism was shown in [IKN17, Proposition 4.7].

Proposition 4.21. *A pseudo-cobordism is shellable if and only if it can be represented as a composition of elementary pseudo-cobordisms.*

Though the statement in [IKN17] is only for pseudo-cobordisms between closed manifolds, their proof carries over verbatim to the situation of non-closed manifolds with isomorphic boundaries. Indeed, the "Only-if"-part relies on a series of lemmas that only use part *i* and *ii* of Definition 4.19 but nowhere that the manifolds are assumed to be closed. The "If"-part follows from the fact that the composition of shellable pseudo-cobordisms is again a shellable pseudo-cobordism [IKN17, Lemma 4.6]. To see that this statement is true in our setting, it is enough to note that condition *iii* of Definition 4.19 is preserved under composition.

As a corollary of Proposition 4.21 one now obtains that Theorem 4.8 of [IKN17] remains true in our setting (with the same proof).

Theorem 4.22. *Let Δ and Γ be combinatorial manifolds such that $\partial\Delta \cong \partial\Gamma$. Then Δ and Γ are bistellar equivalent if and only if there exists a shellable pseudo-cobordism between Δ and Γ.*

Example 4.23. Let Δ be a 1-dimensional simplicial complex consisting of 3 consecutive edges. Let Γ be the simplicial complex obtained from Δ by first applying a bistellar flip to the middle edge and then performing the inverse move, as depicted in the first row of Figure 4.5. Note that we have $\Delta = \Gamma$. This sequence of bistellar flips can be encoded in a shellable pseudo-cobordism between Δ and Γ. This is shown in the second row of Figure 4.5. In the bottom right picture, the complex $\varphi(\Delta) \cap \psi(\Gamma)$ is depicted in green, while the blue and the red segment are respectively $\varphi(\Delta) \setminus \psi(\Gamma)$ and $\psi(\Gamma) \setminus \varphi(\Delta)$. We refer to [IKN17, Section 4] for the precise construction of the pseudo-cobordism.

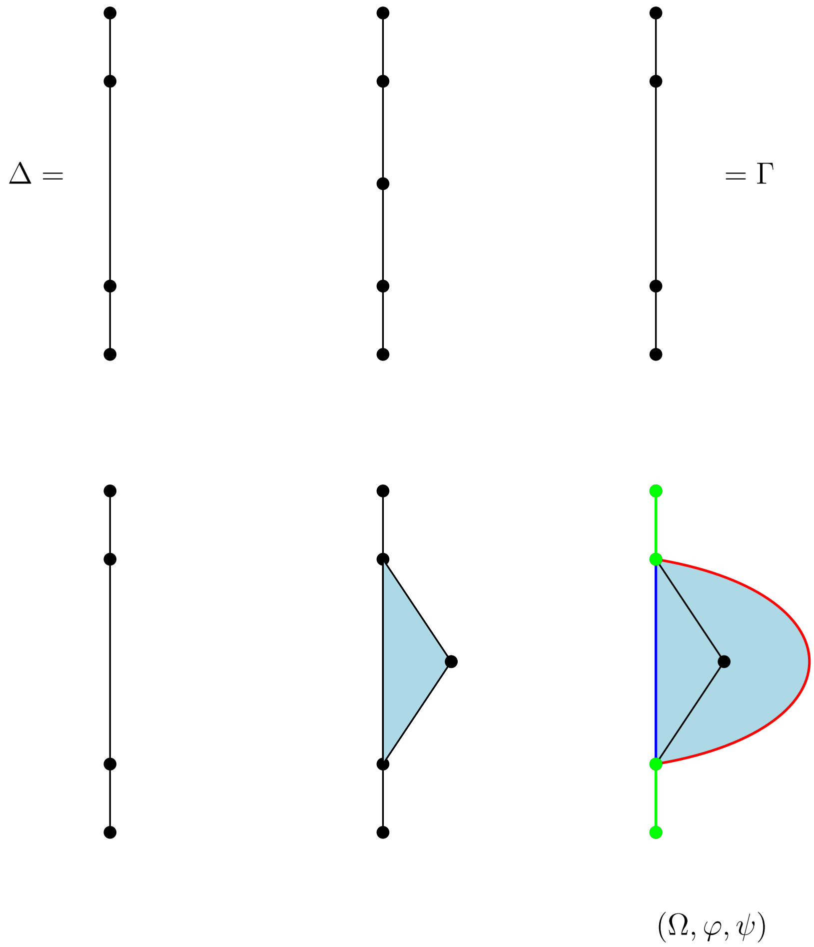

FIGURE 4.5: A sequence of 1-dimensional bistellar flips (*first row*) encodes a sequence of elementary pseudo-cobordisms (*second row*) and vice versa.

In [IKN17, Corollary 5.3] it is shown that, if there exists a shellable pseudo-cobordism between Δ and Γ, then it can always be chosen in such a way that Δ and Γ embed disjointly. This is done by constructing a shellable pseudo-cobordism from Δ to some manifold Δ', such that $F \notin \Delta'$ for a given face $F \in \Delta$ ([IKN17, Lemma 5.2]). Iterating this procedure over all vertices of Δ, and then composing the obtained shellable pseudo-cobordism from Δ to Δ' with the shellable pseudo-cobordism between Δ' and Γ yields the desired result. In our setting, we have the following analog:

Lemma 4.24. *Let Δ be a combinatorial d-manifold with boundary and let $F \in \overset{\circ}{\Delta}$ be a face in the interior of Δ. Then there exists a combinatorial d-manifold Δ' with boundary and a $(d+1)$-dimensional nonpure pseudomanifold Ω such that*

i. $\partial\Delta' \cong \partial\Delta$, *ii.* Δ *and* Δ' *are PL homeomorphic,*

iii. Ω *is a shellable pseudo-cobordism between* Δ *and* Δ'*,*

iv. $F \notin \Delta'$.

We only comment on some specific points of the proof, since it is basically the same as the one of [IKN17, Lemma 5.2]. There, the first ingredient is that the link of any face F of a closed combinatorial d-manifold is a combinatorial $(d - |F|)$-sphere. In our setting this is true, since the face F lies in the interior of Δ. In the next step, the proof proceeds by constructing Δ' and Ω explicitly. In our situation, one immediately sees that this construction satisfies $\partial\Delta \cong \partial\Delta'$ and correspondingly for the embeddings, which is why *i* and *iii* hold. Finally, instead of using Theorem 4.6 and [IKN17, Theorem 4.8], we use Proposition 4.7 and Theorem 4.22 to conclude that *ii* holds.

Using the same arguments as in the proof of [IKN17, Corollary 5.3], we obtain the main result of this section.

Corollary 4.25. *Let* Δ *and* Γ *be balanced PL homeomorphic combinatorial* d*-manifolds with boundary. Assume that* $\partial\Delta \cong \partial\Gamma$ *and that this isomorphism preserves the coloring. Then there exists a shellable pseudo-cobordism* (Ω, φ, ψ) *between* Δ *and* Γ*, such that* $\varphi(\Delta) \cap \psi(\Gamma) = \varphi(\partial\Delta) = \psi(\partial\Gamma)$*. Moreover, the pseudoboundary* $\tilde{\partial}\Omega$ *is a balanced simplicial poset.*

The “Moreover”-part is immediate from the fact that Δ and Γ are both balanced and that their colorings coincide on their boundaries. We also want to remark that the pseudoboundary $\tilde{\partial}\Omega = \varphi(\Delta) \cup \psi(\Gamma)$ is a simplicial poset but not necessarily a simplicial complex.

4.3.3 Step 3: Converting bistellar flips into cross-flips

The aim of this section is to prove Theorem 4.3, providing an analog of Theorem 1.2 of [IKN17] for PL homeomorphic manifolds with isomorphic boundaries (see also Theorem 4.3).

Theorem 4.26. *Let* Δ *and* Γ *be balanced combinatorial* d*-manifolds with boundary that are PL homeomorphic. Assume moreover that* $\partial\Delta \cong \partial\Gamma$ *and that the colorings coincide on the boundary. Then, there exists a sequence of basic cross-flips connecting* Δ *and* Γ*.*

We need the following definition.

Definition 4.27. A *cross-polytopal complex* of dimension d is a pure regular CW-complex, all whose maximal cells are combinatorially isomorphic to $\partial\mathcal{C}_{d+1}$ and whose $(d-1)$-skeleton is a simplicial complex.

Remark 4.28. In [CV19] Codenotti and the author show that for every balanced 3-polytope P there exists a *geometric cross-polytopal complex* Δ, i.e., a pure polytopal complex in which all maximal cells are combinatorially isomorphic to $\mathcal{C}_3$, such that the union of the cells in Δ equals P. In other words every balanced 3-polytope can be geometrically decomposed into octahedra, and if two such octahedra intersect in a non-empty face, then the intersection has exactly one maximal face.

The proof of Theorem 4.26 is analogous to the last steps of the one of Theorem 1.2 in [IKN17]. We include it as a service to the reader.

Proof. We apply Corollary 4.25 to obtain a shellable pseudo-cobordism (Ω, φ, ψ) between Δ and Γ with $\varphi(\Delta) \cap \psi(\Gamma) = \varphi(\partial\Delta) = \psi(\partial\Gamma)$. Though Ω might not be balanced, it follows from [IKN17, Corollary 3.2] that there exists a balanced non-pure $(d+1)$-pseudomanifold Ω' obtained by stellar subdivision of interior faces of Ω, which is a pseudo-cobordism between Δ and Γ. Since stellar subdivisions preserve shellability, Ω' is shellable. (This follows from Proposition 5.7 in [IKN17], which is only stated for closed manifolds but whose proof also goes through in our setting.) Applying the diamond operation to the simplicial poset Ω' leads to a cross-polytopal $(d+1)$-complex $\Diamond(\Omega')$, and the shelling order on (Ω', φ, ψ) induces an order on the $(d+1)$-cells of $\Diamond(\Omega')$, which encodes a sequence of cross-flips between Δ and Γ. For more details on this part, we refer to the proof of Theorem 3.10 in [IKN17] and to the next example as an illustration. □

Example 4.29. As in Example 4.23 we let $\Delta = \Gamma$ be the 1-dimensional ball consisting of 3 consecutive edges whose vertices are colored alternately with red and blue. The pseudo-cobordism provided in Example 4.23 does not satisfy the conditions in Corollary 4.25 since the embeddings of Δ and Γ do not intersect just in two vertices, but also along two edges. However using stellar subdivisions we can construct a shellable pseudo-cobordism (Ω, φ, ψ) that meets the requirements of Corollary 4.25. Such a pseudo-cobordism, though not a minimal one, is depicted in the top left image of Figure 4.6. The labels on the 2-faces encode a shelling order $F_1, \ldots, F_{10}$ of $(\Omega, \varphi(\Delta))$. Applying the diamond operation, we obtain a cross-polytopal complex. In this case, the diamond operation subdivides all 1-faces not containing vertices of color 0 (which is blue in the figure), i.e., all edges whose vertices are colored with red and green. The 2-cells of $\Diamond(\Omega)$ are 2-dimensional cross-polytopes. We now describe how an inverse shelling gets encoded into a cross-flip, taking F_3 as an example. If we identify $\Diamond(F_3)$ with $\partial\mathcal{C}_2$ and consider the decomposition $F_3 = A_3 \cup R_3$ given by the restriction face R_3 and $A_3 = F_3 \setminus R_3$, we notice that adding F_3 to the simplicial poset corresponds to replacing $\Diamond(A_3)$ with $\Diamond(\langle R_3\rangle * \partial A_3) = \partial\mathcal{C}_2 \setminus \Diamond(A_3)$, which describes the application of the cross-flip $(\Diamond(A_3), \partial\mathcal{C}_2 \setminus \Diamond(A_3))$, see the bottom right of Figure 4.6).

4.3.4 Step 4: Building a collar and shelling cross-flips

The last ingredient we need for the proof of Theorem 4.2 is a way to convert every cross-flip into a sequence of shellings and balanced inverse shellings. Having achieved this, the basic strategy is the following: whenever a cross-flip is performed in the interior of a manifold, we will first “enter” it from an arbitrary boundary facet, “dig” into the manifold by removing facets (using shellings) until we meet a facet that is involved in the cross-flip to be carried out. In the next step, we shell the subcomplex to be removed, add its complement with respect to the boundary of the cross-polytope using inverse shellings and finally close the path we built before using inverse shellings. To avoid weird and undesirable side effects, when shelling the cross-flip, we need to make sure that the only facet involved in the cross-flip that meets the boundary of the manifold in codimension 1 is the facet that was first hit when digging the tunnel into the manifold. This can be achieved by building a collar around the manifold (using shellings and balanced inverse shelling), an idea going back to the proof of Theorem 4.15 by Pachner.

The following results will be crucial.

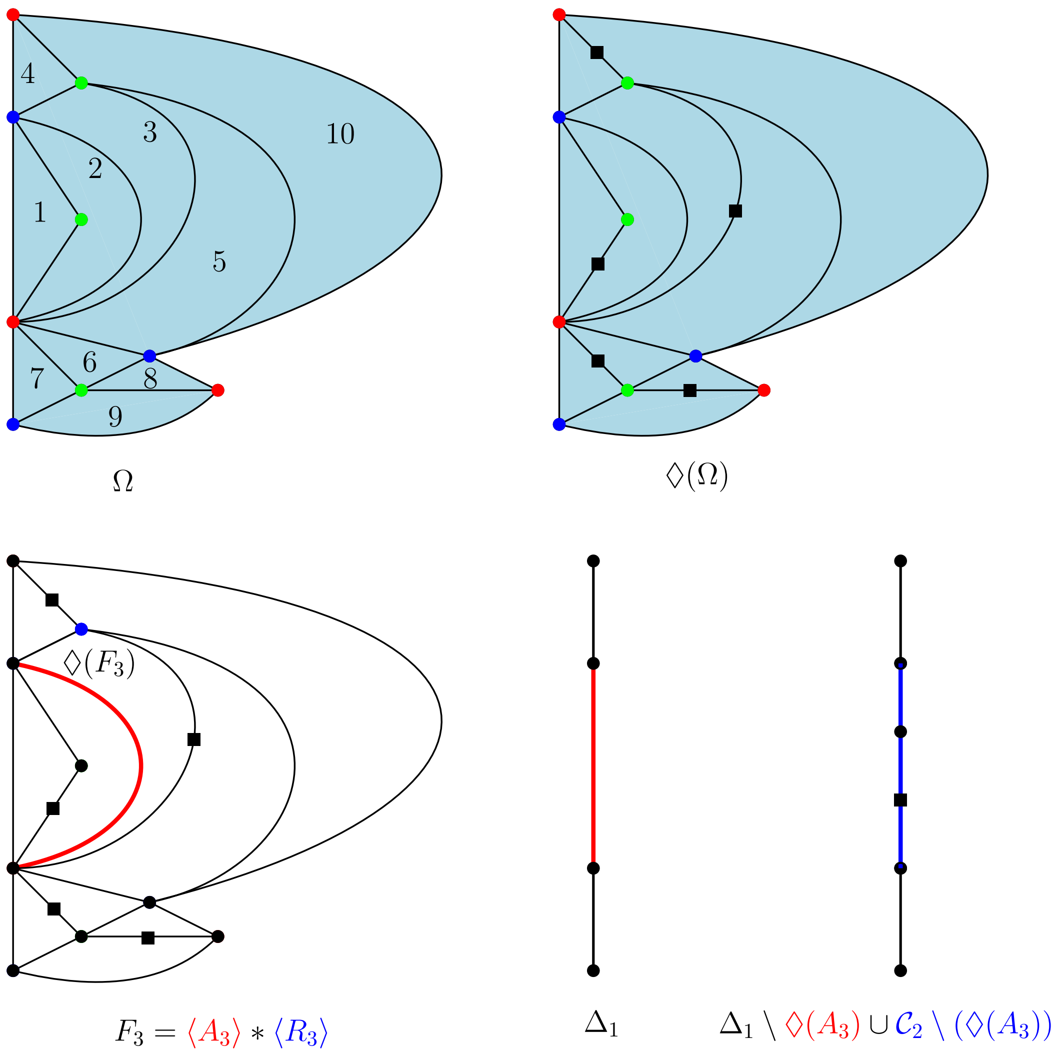

FIGURE 4.6: *First row*: a shellable pseudo-cobordism (Ω, φ, ψ) and the cross-polytopal complex $\lozenge(\Gamma)$. *Second row*: a shelling on Ω encodes a cross-flip between the intermediate steps.

Lemma 4.30. [Pac91, Theorem 5.8] *Every combinatorial d-sphere Δ is the boundary of a shellable combinatorial $(d+1)$-ball Ω. Moreover, Ω can be chosen so that Δ is an induced subcomplex of Ω.*

Theorem 4.31. [IKN17, Theorem 3.1] *Let Ω be a d-dimensional simplicial complex and $\Delta \subsetneq \Omega$ a subcomplex. Let $\kappa : V(\Delta) \longrightarrow \{0, \dots, m-1\}$ be a proper m-coloring of Δ. Then there is a stellar subdivision Ω' of Ω s.t.:*

i. Δ is a subcomplex of Ω'.

ii. The coloring κ extends to a proper coloring $\kappa' : V(\Omega') \longrightarrow \{0, \dots, \max\{m-1, d\}\}$ such that all vertices not in Δ receive colors in $\{0, \dots, d\}$.

Remark 4.32. Let Δ be a combinatorial d-sphere and let $\kappa : V(\Delta) \to \{0, \dots, m-1\}$ be a proper m-coloring of Δ. Then, by Lemma 4.30, there exists a shellable $(d+1)$-ball Ω with $\partial\Omega = \Delta$ and such that Δ is an induced subcomplex of Ω. Theorem 4.31 further yields a $(d+1)$-ball Ω' such that:

i. $\partial\Omega' = \Delta$,

ii. Ω' is shellable,

iii. Ω' is properly $(\max\{m, d+2\})$-colored and this extends the coloring κ of Δ,

iv. Δ is an induced subcomplex of Ω'.

Conditions *ii* and *iii* hold since stellar subdivisions preserve both shellability and the property of being an induced subcomplex.

We state a further lemma.

Lemma 4.33. [Pac91, Lemma 4.7] *Let Δ be a combinatorial d-manifold and $\Omega \subseteq \partial\Delta$ a shellable $(d-1)$-ball such that $\Omega \subseteq \mathrm{lk}_\Delta(v)$ for some vertex $v \in \mathring{\Delta}$. Then there is a sequence of shellings converting Δ into $\Delta \setminus (\langle v\rangle * \mathring{\Omega})$.*

Putting the previous results together allows us to build a collar:

Theorem 4.34. *Let Δ be a balanced combinatorial d-manifold with boundary and let F be a facet of $\partial\Delta$. Then there exists a balanced combinatorial d-manifold Δ' such that*

i. Δ' can be transformed into Δ by a sequence of shellings,

ii. Δ is an induced subcomplex of Δ',

iii. $\partial\Delta \cap \partial\Delta' = \langle F\rangle$.

Proof. We fix a coloring κ of Δ. Let $v \in V(\partial\Delta)$ with $v \notin F$. Then $\mathrm{lk}_{\partial\Delta}(v)$ is a combinatorial $(d-2)$-sphere, which is properly d-colored (though not necessarily balanced). By Remark 4.32 we can choose a balanced, shellable $(d-1)$-ball Ω, whose boundary is $\mathrm{lk}_{\partial\Delta}(v)$ and the latter is an induced subcomplex of Ω. Moreover, there is a proper d-coloring κ' of Ω that restricts to κ on $\mathrm{lk}_{\partial\Delta}(v)$. Let us consider $\widetilde{\Delta} := \Delta \cup (\langle v\rangle * \Omega)$. Since $\partial\widetilde{\Delta} = (\partial\Delta \setminus (\mathrm{st}_{\partial\Delta}(v))) \cup \Omega$, the open star $\mathring{\mathrm{st}}_{\partial\Delta}(v)$ is contained in $\mathring{\widetilde{\Delta}}$. As $\mathrm{lk}_{\partial\Delta}(v)$ is an induced subcomplex of Ω, the same is true for Δ, considered as a subcomplex of $\widetilde{\Delta}$. Since the coloring κ' of Ω does not use $\kappa(v)$ and since $\kappa'|_{\mathrm{lk}_{\partial\Delta}(v)} = \kappa$, we conclude that $\Omega * \langle v\rangle$ and thus also $\widetilde{\Delta}$ is balanced. Finally, it follows from Lemma 4.33 that there is a sequence of shellings from $\widetilde{\Delta}$ to Δ. We now apply the described construction to every vertex $v \in V(\partial\Delta) \setminus F$ to obtain a simplicial complex Δ'. As any face $G \in \partial\Delta$ with $G \not\subseteq F$ lies in the open star $\mathring{\mathrm{st}}_{\partial\Delta}(v)$ of some vertex $v \in V(\partial\Delta) \setminus F$, it follows that Δ' satisfies condition *iii*. □

As the only missing ingredient for the proof of Theorem 4.2 we need to convert cross-flips into shellings and their inverses in such a way that balancedness is preserved. This is done in the remaining part of this section.

In the following, we assume that Δ is a combinatorial d-manifold with boundary, $D = \lozenge(\Gamma) \subseteq \Delta$ is an induced subcomplex for some d-ball $\Gamma \subseteq \partial\Delta_{d+1}$ and $D \cap \partial\Delta = \langle F\rangle$ for a $(d-1)$-face $F \subseteq \partial\Delta$. We let $\Delta' = \chi^*_D(\Delta)$. Our aim is to construct a sequence of shellings from Δ to $\Delta \setminus D$ and a sequence of inverse shellings from $\Delta \setminus D$ to $\Delta' = \Delta \setminus D \cup (\partial\mathcal{C}_{d+1} \setminus D)$ that preserves balancedness. Since we know that D and $\partial\mathcal{C}_{d+1} \setminus D$ are both shellable, an obvious choice might be a reversed shelling order on D, followed by a shelling order on $\partial\mathcal{C}_{d+1} \setminus D$. However, in general this will not work, since faces of D and $\partial\mathcal{C}_{d+1} \setminus D$ might also intersect $\Delta \setminus D$, which can cause obstructions to shellability. An instance for this behavior is given in Figure 4.7. On the one hand, the ordering of the facets, indicated by their labels, is a reversed shelling for the designated subcomplex. On the other hand, taking into account the large complex, we are not allowed to remove the facet labeled 6 (once facets $1, \dots, 5$ have been removed) since it intersects the boundary of the given manifold in a nonpure subcomplex, consisting of an edge and an isolated vertex (shown in red in the right picture of Figure 4.7). So, what we need to construct is a shelling of the relative simplicial complex $(\Delta, \Delta \setminus D)$, which is defined as for relative simplicial posets (see Section 4.3.2).

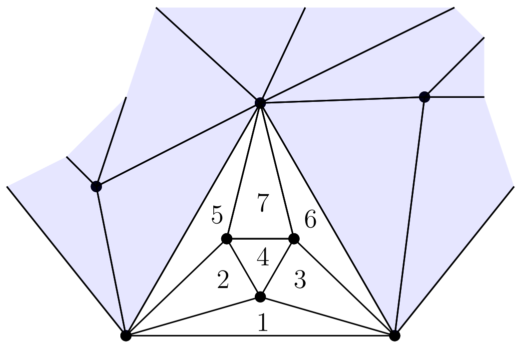

FIGURE 4.7: A shelling order on D (*left*) that is not a shelling for $(\Delta, \Delta \setminus D)$ (*right*).

In the sequel, we will define an ordering on the facets of D and $\partial\mathcal{C}_{d+1} \setminus D$, that provides a sequence of shellings for Δ and Δ', respectively. In this way, we can relate Δ and Δ' by the sequence for D followed by the reversed sequence for $\partial\mathcal{C}_{d+1} \setminus D$. First, we need some preparations. Let Δ_{d+1} be the $(d+1)$-simplex on vertex set $\{0, 1, \dots, d+1\}$ and for $0 \leq i \leq d+1$ let $\Gamma_i = \{0, \dots, \hat{\imath}, \dots, d+1\}$ be the facet not containing the vertex i. We start with a description of the facets of $\Diamond(\Gamma_i)$ for $0 \leq i \leq d$, where we follow the notation of the paragraph preceding Definition 4.11.

Lemma 4.35. *For $0 \leq i \leq d-1$ let $\partial\mathcal{C}_{d-i-1}(i+1, \dots, d)$ be the boundary complex of the $(d-i-1)$-dimensional cross-polytope on vertex set $\{i+1, \dots, d\} \cup \{v_{i+1}, \dots, v_d\}$ and set $\partial\mathcal{C}_{-1} = \{\emptyset\}$. Then,*

$$\Diamond(\Gamma_i) = \begin{cases} \langle\{0, \dots, i-1, v_i\}\rangle * \partial\mathcal{C}_{d-i-1}(i+1, \dots, d) & \textit{for } 0 \leq i \leq d, \\ \langle\{0, \dots, d\}\rangle & \textit{for } i = d+1. \end{cases}$$

Proof. First note that the facet Γ_{d+1} is not subdivided by the diamond operation. Let $0 \leq i \leq d$. In this case, the diamond operation will successively subdivide the faces $F_j = \{j+1, \dots, d+1\}$ for $i \leq j \leq d$, i.e.,

$$\Diamond(\Gamma_i) = \mathrm{sd}_{F_d} \circ \mathrm{sd}_{F_{d-1}} \circ \cdots \circ \mathrm{sd}_{F_i}(\Gamma_i).$$

We prove the more general statement that for any $i \leq j \leq d$,

$$\mathrm{sd}_{F_j} \circ \mathrm{sd}_{F_{j-1}} \circ \cdots \circ \mathrm{sd}_{F_i}(\Gamma_i) = \langle\{0, \dots, i-1, v_i\}\rangle * \partial\langle\{v_{i+1}, i+1\}\rangle * \cdots * \partial\langle\{v_j, j\}\rangle * \partial F_j.$$

We proceed by induction on j. For $j = i$, we have

$$\mathrm{sd}_{F_i}(\Gamma_i) = \langle\{0, \dots, i-1\}\rangle * \langle\{v_i\}\rangle * \partial F_i,$$

which implies the desired statement. For $j > i$, it holds that

$$
\begin{aligned}
&\mathrm{sd}_{F_j} \circ \mathrm{sd}_{F_{j-1}} \circ \cdots \circ \mathrm{sd}_{F_i}(\Gamma_i) = \\
&\mathrm{sd}_{F_j}(\langle\{0,\ldots,i-1,v_i\}\rangle * \partial\langle\{v_{i+1},i+1\}\rangle * \cdots * \partial\langle\{v_{j-1},j-1\}\rangle * \partial F_{j-1}) = \\
&\langle\{0,\ldots,i-1,v_i\}\rangle * \partial\langle\{v_{i+1},i+1\}\rangle * \cdots * \partial\langle\{v_{j-1},j-1\}\rangle * \mathrm{sd}_{F_j}(\partial F_{j-1}) = \\
&\langle\{0,\ldots,i-1,v_i\}\rangle * \partial\langle\{v_{i+1},i+1\}\rangle * \cdots * \partial\langle\{v_{j-1},j-1\}\rangle * \partial\langle\{v_j,j\}\rangle * \partial F_j,
\end{aligned}
$$

where we use that

$$\mathrm{sd}_G(\Gamma * \Delta) = \Gamma * \mathrm{sd}_G(\Delta) \text{ for } G \in \Delta$$

and

$$\mathrm{sd}_{F_j}(\partial F_{j-1}) = \partial\langle\{v_j,j\}\rangle * \partial F_j.$$

The claim now follows. □

Figure 4.8 shows an illustration of the decomposition of $\lozenge(\Gamma_i)$ as provided by Lemma 4.35.

Remark 4.36. As a consequence of Lemma 4.35 we can describe the boundary complex of $\lozenge(\Gamma_i)$ as

$$
\partial\lozenge(\Gamma_i) = \begin{cases} \partial\langle\{0,\ldots,i-1,v_i\}\rangle * \partial\mathcal{C}_{d-i-1}(i+1,\ldots,d) & \text{for } 0 \le i \le d, \\ \partial\langle\{0,\ldots,d\}\rangle & \text{for } i = d+1. \end{cases}
$$

Consequently, it follows that for $0 \le i < k \le d+1$,

$$
\begin{aligned}
&\partial\lozenge(\Gamma_i) \cap \partial\lozenge(\Gamma_k) \\
&= \begin{cases} \langle\{0,\ldots,i-1,i+1,\ldots,k-1,v_k\}\rangle * \partial\mathcal{C}_{d-k+1}(k+1,\ldots,d) & \text{for } 1 \le k \le d, \\ \partial\langle\{0,\ldots,i-1,i+1,\ldots,d\}\rangle & \text{for } k = d+1. \end{cases}
\end{aligned}
$$

In particular, $\lozenge(\Gamma_i)$ and $\lozenge(\Gamma_k)$ intersect in a pure $(d-1)$-dimensional subcomplex of their boundaries. Since any facet F in $\lozenge(\Gamma_k)$ is of the form $\{0,\ldots,k-1,v_k\} \cup F'$ for a facet F' of $\partial\mathcal{C}_{d-k-1}(k+1,\ldots,d)$, we can further conclude that for any $F \in \mathcal{F}(\lozenge(\Gamma_k))$ and any $0 \le i < k \le d+1$ there exists $G \in \mathcal{F}(\lozenge(\Gamma_i))$ such that $\dim(F \cap G) = d-1$. Namely, $G = F \setminus \{i\} \cup \{v_i\}$. Observe that we can further infer that for any $0 \le k < i \le d+1$ there exists a facet $F \in \lozenge(\Gamma_k)$ and a facet $G \in \lozenge(\Gamma_i)$ such that $\dim(F \cap G) = d-1$.

As before, let Δ be a balanced combinatorial d-manifold with boundary. Let $\Gamma = \langle\Gamma_{i_1},\ldots,\Gamma_{i_k}\rangle$, where the i_j are pairwise distinct and $i_2 < \cdots < i_k$, and let $D = \lozenge(\Gamma)$ be an induced subcomplex of Δ such that $\partial\Delta \cap D = \langle F\rangle$ for a $(d-1)$-face F. Without loss of generality, assume $F \in \lozenge(\Gamma_{i_1})$. Let G be the unique facet of $\lozenge(\Gamma_{i_1})$ containing F. We now describe a shelling for $(\Delta, \Delta \setminus D)$, starting with the facets of $\lozenge(\Gamma_{i_1})$ followed by the facets of $\lozenge(\Gamma_{i_2})$, …, $\lozenge(\Gamma_{i_k})$.
We need some further notation. For $2 \le \ell \le k$, we let $1 \le m(\ell) \le \ell$ such that $i_{m(\ell)} = \min\{i_j : 1 \le j \le \ell \text{ and } i_j \ge i_\ell\}$. For $1 \le \ell \le k$, we define the *initial facet* $F_{\mathrm{in}}^{(\ell)}$ of $\lozenge(\Gamma_{i_\ell})$ (with respect to Γ and F) as

$$
F_{\mathrm{in}}^{(\ell)} = \begin{cases} G & \text{for } \ell = 1, \\ \{0,\ldots,i_\ell-1,v_{i_\ell}\} \cup \{i_\ell+1,\ldots,i_{m(\ell)}-1\} \cup \{v_{i_{m(\ell)}},\ldots,v_d\} & \text{for } 2 \le \ell \le k. \end{cases}
$$

We have the following simple observation:

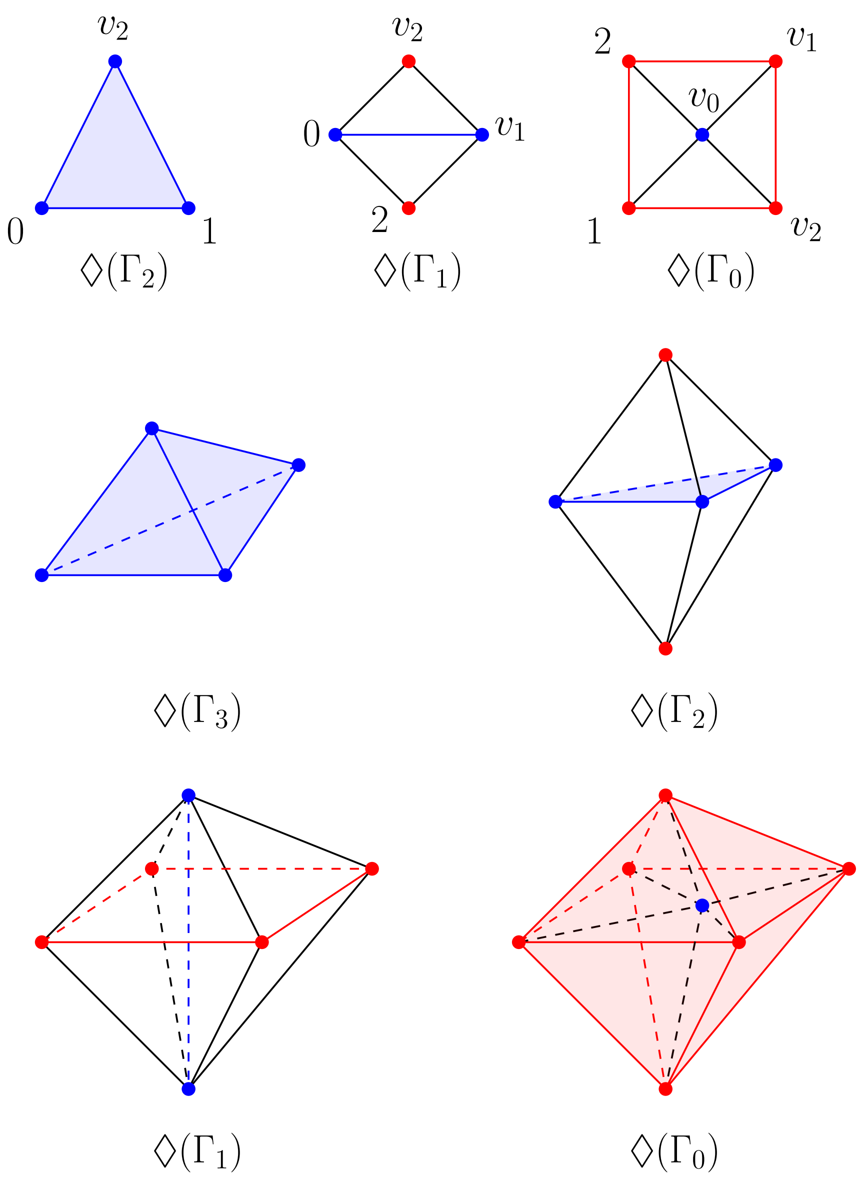

FIGURE 4.8: All the subcomplexes $\Diamond(\Gamma_i)$ for $d = 2$ (*first row*) and $d = 3$ (*second row*). The coloring here indicates the decomposition $\Diamond(\Gamma_i) = \Delta_i * \partial\mathcal{C}_{d-i-1}(i+1, \dots, d)$.

Lemma 4.37. *For any $1 \le \ell \le k$, the initial facet $F_{\text{in}}^{(\ell)}$ of $\Diamond(\Gamma_{i_\ell})$ intersects the boundary of $\Delta_\ell = \Delta \setminus \left(\bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j})\right)$ in a pure subcomplex of dimension $d-1$. In particular, the operation $\Delta_\ell \mapsto \Delta_\ell \setminus F_{\text{in}}^{(\ell)}$ is an elementary shelling.*

Proof. The statement is clear for i_1.
Let $\ell \ge 2$. We have

$$F_{\text{in}}^{(\ell)} \cap \left(\Delta \setminus \left(\bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j})\right)\right) = F_{\text{in}}^{(\ell)} \cap \partial \left(\bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j})\right) =$$
$$F_{\text{in}}^{(\ell)} \cap \left(\bigcup_{j=1}^{\ell-1} \partial\Diamond(\Gamma_{i_j})\right) = \bigcup_{j=1}^{\ell-1} \left(F_{\text{in}}^{(\ell)} \cap \partial\Diamond(\Gamma_{i_j})\right),$$

where the second equality follows from the fact that any $(d-1)$-face of Δ is contained in at most 2 facets. First assume $m_\ell \neq \ell$ and hence $i_{m_\ell} > i_\ell$. In this case,

$$F_{\text{in}}^{(\ell)} \cap \partial\Diamond(\Gamma_{i_j}) = \begin{cases} F_{\text{in}}^{(\ell)} \setminus \{i_j\} & \text{if } i_j < i_\ell \\ F_{\text{in}}^{(\ell)} \setminus \{v_{i_\ell}\} & \text{if } i_j = i_{m(\ell)} \\ F_{\text{in}}^{(\ell)} \setminus \{v_{i_\ell}, v_{i_{m(\ell)}}, \dots, v_{i_j-1}\} & \text{if } i_j > i_{m(\ell)}. \end{cases} \tag{4.0}$$

As the intersection in the last case is clearly contained in $F_{\text{in}}^{(\ell)} \cap \partial\Diamond(\Gamma_{i_{m(\ell)}})$, the claim follows.
If $m(\ell) = \ell$, then $i_j < i_\ell$ for all $1 \leq j \leq \ell - 1$ and the claim follows from (4.0).
The "In particular"-part is now immediate (see e.g., [BH93, Definition 5.1.11]). □

Lemma 4.37 in particular implies that for any $1 \leq \ell \leq k$ the initial facet $F_{\text{in}}^{(\ell)}$ is *free* with respect to $\Delta \setminus \left(\bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j})\right)$, meaning that it intersects the boundary of $\Delta \setminus \left(\bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j})\right)$ in a $(d-1)$-ball.

We can finally define an ordering on the facets of $\Diamond(\Gamma_{i_\ell})$ with respect to the initial facet $F_{\text{in}}^{(\ell)}$. This ordering will be inspired by the lexicographic ordering, that provides a shelling for $\partial\mathcal{C}_{d+1}$ (see e.g., [Lon13, Theorem 1.14]). By Lemma 4.35, any facet $F \in \Diamond(\Gamma_{i_\ell})$ is of the form $\{0, \dots, i_\ell - 1, v_{i_\ell}\} \cup F'$, where F' is a facet of $\partial\mathcal{C}_{d-i_\ell-1}(i_\ell + 1, \dots, d)$. We can therefore interpret F as an ordered $(d+1)$-tuple in $\{0\} \times \cdots \times \{i_{\ell-1}\} \times \{v_{i_\ell}\} \times \{i_\ell + 1, v_{i_\ell+1}\} \times \cdots \times \{d, v_d\}$. We write $F(j)$ for the j^{th} entry of F, i.e., $F(j) = F \cap \{j, v_j\}$ for $0 \leq i \leq d$. We use the same notation for tuples in $\{0,1\}^{d-i_\ell}$. We define the *characteristic function*

$$\varphi_F : \mathcal{F}(\Diamond(\Gamma_{i_\ell})) \to \{0,1\}^{d-i_\ell}$$

on $\mathcal{F}(\Diamond(\Gamma_{i_\ell}))$ with respect to F by setting

$$\varphi_F(G)(j) = \begin{cases} 0 & \text{if } G(i_\ell + 1 + j) = F(i_\ell + 1 + j), \\ 1 & \text{if } G(i_\ell + 1 + j) \neq F(i_\ell + 1 + j) \end{cases}$$

for $0 \leq j \leq d - i_\ell - 1$. We set $\deg_F(G) := \sum_{j=1}^{d_{i_\ell}} \varphi_F(G)(j)$ and call this the *degree* of G with respect to F. As $\deg_F(G)$ counts the number of elements in $G \setminus (F \cap G)$, we can interpret $\deg_F(G)$ as a measure of similarity between F and G. We now order the facets of $\Diamond(\Gamma_{i_\ell})$ by ordering $\{\varphi_{F_{\text{in}}^{(\ell)}}(F) \ : \ F \in \Diamond(\Gamma_{i_\ell})\}$ according to the degree lexicographic ordering, i.e., we set $F \prec F'$ if

(i) $\deg_{F_{\text{in}}^{(\ell)}}(\varphi(F)) < \deg_{F_{\text{in}}^{(\ell)}}(\varphi(F'))$, or

(ii) $\deg_{F_{\text{in}}^{(\ell)}}(\varphi(F)) = \deg_{F_{\text{in}}^{(\ell)}}(\varphi(F'))$ and there exists m such that $\varphi_{F_{\text{in}}^{(\ell)}}(F)(k) = \varphi_{F_{\text{in}}^{(\ell)}}(F')(k)$ for all $1 \leq k \leq m$ and $\varphi_{F_{\text{in}}^{(\ell)}}(F)(m+1) < \varphi_{F_{\text{in}}^{(\ell)}}(F')(m+1)$.

In the latter case, we must have $\varphi_{F_{\text{in}}^{(\ell)}}(F)(m+1) = 0$ and $\varphi_{F_{\text{in}}^{(\ell)}}(F')(m+1) = 1$. For simplicity, we call this ordering *degree lexicographic ordering* with respect to $F_{\text{in}}^{(\ell)}$.

Theorem 4.38. *Let Δ be a balanced combinatorial d-manifold with boundary. Let $\Gamma = \langle \Gamma_{i_1}, \dots, \Gamma_{i_k} \rangle$, where the i_j are pairwise distinct and $i_2 < i_3 < \cdots < i_k$, and let $\Diamond(\Gamma)$ be an induced subcomplex of Δ intersecting $\partial\Delta$ in a $(d-1)$-face $F \in \Diamond(\Gamma_{i_1})$. For $1 \leq \ell \leq k$, let $r_\ell = |\mathcal{F}(\Diamond(\Gamma_{i_\ell}))| - 1$ and let*

$$F_0^{(\ell)} = F_{\text{in}}^{(\ell)}, F_1^{(\ell)}, \dots, F_{r_\ell}^{(\ell)}$$

be the ordering of the facets of $\Diamond(\Gamma_{i_\ell})$ *according to the degree lexicographic ordering with respect to* $F_{\text{in}}^{(\ell)}$*. Then*

$$F_{r_k}^{(k)}, \ldots, F_1^{(k)}, F_0^{(k)}, F_{r_{k-1}}^{(k-1)}, \ldots, F_1^{(k-1)}, F_0^{(k-1)}, \ldots F_{r_0}^{(1)}, \ldots, F_1^{(1)}, F_0^{(1)}$$

is a shelling on $\left(\Delta, \Delta \setminus \left(\bigcup_{j=1}^{k} \Diamond(\Gamma_{i_j})\right)\right)$.

Proof. We show that for any $1 \leq \ell \leq k$ the ordering $F_{r_\ell}^{(\ell)}, \ldots, F_1^{(\ell)}, F_0^{(\ell)}$ is a shelling for the relative complex $\left(\Delta \setminus \left(\bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j})\right), \Delta \setminus \left(\bigcup_{j=1}^{\ell} \Diamond(\Gamma_{i_j})\right)\right)$. We treat the cases $\ell = 1$ and $\ell > 1$ separately, since they require slightly different arguments. However, a general remark is in order first. Let Δ' be a pure subcomplex of Δ with $\Delta \setminus \Diamond(\Gamma) \subseteq \Delta'$ and let $A \in \Diamond(\Gamma) \cap \Delta'$. Since $\partial\Delta \cap \Diamond(\Gamma) = \langle F \rangle$, the face A is in the interior of Δ' if and only if it is only contained in facets of Δ that also belong to Δ'.

First assume $\ell = 1$. For $0 \leq i \leq r_1$, we set $\Delta_i = \Delta \setminus \langle F_0^{(1)}, \ldots, F_{i-1}^{(1)} \rangle$. We need to show that the operation $\Delta_i \mapsto \Delta_i \setminus F_i^{(1)} = \Delta_{i+1}$ is an elementary shelling. For $F_0^{(1)}$ this follows from Lemma 4.37. For $1 \leq i \leq r_1$, we set

$$A_i^{(1)} = \{F_i^{(1)}(j) \ : \ \varphi_{F_{\text{in}}^{(1)}}(F_i^{(1)})(i_1 + 1 + j) = 1\}$$

and

$$R_i^{(1)} = \{0, \ldots, i_1 - 1, v_{i_1}\} \cup \{F_i^{(1)}(j) \ : \ \varphi_{F_{\text{in}}^{(1)}}(F_i^{(1)})(i_1 + 1 + j) = 0\}.$$

Note that the designated restriction face contains exactly the vertices that lie in the initial facet, whereas $A_i^{(1)}$ contains the vertices not in $F_{\text{in}}^{(1)}$. Let $i \geq 1$. First note that $A_i \neq \emptyset$ and $R_i \neq \emptyset$. Since the facets are ordered according to the degree lexicographic ordering with respect to $F_0^{(1)}$, the only facets of Δ containing $A_i^{(1)}$ are those that are larger than $F_i^{(1)}$ and which hence belong to Δ_i. Therefore, $A_i \in \mathring{\Delta}_i$. Consider a facet H of $\partial(A_i^{(1)}) * \langle R_i^{(1)} \rangle$, i.e., $H = A_i^{(1)} \setminus \{F_i^1(j)\} \cup R_i^{(1)}$ for some $i_\ell + 1 \leq j \leq d$ with $\varphi_{F_0^{(1)}}(F_i^{(1)})(j) = 1$. Then, $H' = H \cup \{F_0^{(1)}(j)\} \in \mathcal{F}(\Diamond(\Gamma_{i_1}))$ and $H' < F_i$, which implies that $H' \notin \Delta_i$. We conclude $H \in \partial\Delta_i$. The operation $\Delta_i \mapsto \Delta_i \setminus F_i$ is hence an elementary shelling.

Now let $\ell \geq 2$. We set $\Delta_0 = \Delta \setminus \Diamond(\Gamma_{i_1}, \ldots, \Gamma_{i_{\ell-1}})$ and $\Delta_i = \Delta_0 \setminus \langle F_0^{(\ell)}, \ldots, F_{i-1}^{(\ell)} \rangle$ for $1 \leq i \leq r_\ell$. It follows from Lemma 4.37 that the operation $\Delta_0 \mapsto \Delta \setminus F_0^{(\ell)}$ is an elementary shelling. To simplify notation we set $\varphi = \varphi_{F_{\text{in}}^{(\ell)}}$ and $F_i = F_i^{(\ell)}$ for $0 \leq i \leq r_\ell$. For $1 \leq i \leq r_\ell$, we let $t(i)$ be the first position at which F_i differs from the initial facet F_0, i.e., $t(i) = \min\{0 \leq j \leq d - i_\ell - 1 \ : \ \varphi(F_i)(j) = 1\} + i_\ell + 1$. Let $1 \leq i \leq r_\ell$ fixed and set $A' = \{i_j \ : \ 1 \leq j \leq \ell - 1 \text{ and } i_j < i_\ell\}$. We distinguish several cases:

Case 1: $i_1 < i_2$.

For $1 \leq i \leq r_\ell$ we define

$$A_i = A' \cup \{F_i(j) \ : \ \varphi(F_i)(j) = 1\},$$

and

$$R_i = (\{0, \ldots, i_\ell - 1\} \setminus A') \cup \{v_{i_\ell}\} \cup \{F_i(j) \ : \ \varphi(F_i)(j) = 0\}.$$

We will show that A_i and R_i satisfy conditions *i*–*iii* from Definition 4.14.

Condition *i*: First note that $A_i \neq \emptyset$ since there always exists j with $\varphi(F_i)(j) = 1$ (as $F_i \neq F_0$). Moreover, we have $R_i \neq \emptyset$ since $v_{i_\ell} \in R_i$. Thus, condition (1) holds.

Condition *ii*: We need to show that $A_i \in \overset{\circ}{\Delta}_i$. By the argument at the beginning of this proof, we need to prove that A_i is not contained in any facet H of $\Diamond(\Gamma_{i_1}, \ldots, \Gamma_{i_{\ell-1}}) \cup \langle F_0, \ldots, F_{i-1}\rangle$. Those come in two different types:

Type 1: $H \in \{F_0, \ldots, F_{i-1}\}$.

Type 2: $H \in \Diamond(\Gamma_{i_j})$ for some $1 \leq j \leq \ell - 1$ with $i_j < i_\ell$.

Since the facets are ordered lexicographically, F_i is the lexicographically smallest facet of $\Diamond(\Gamma_{i_\ell})$ containing $\{F_i(j) \ : \ \varphi(F_i)(j) = 1\}$ and in particular A_i. Therefore, A_i cannot be contained in a facet of Type 1. If $H \in \Diamond(\Gamma_{i_j})$ is of Type 2, then $i_j \in A' \subseteq A_i$ but $i_j \notin H$ by Lemma 4.35, which implies $A_i \not\subseteq H$.

Condition *iii*: We have to show that $\partial A_i * \langle R_i\rangle \subseteq \partial\Delta_i$. Let $v \in A_i$ and $B = (A_i \setminus \{v\}) \cup R_i \in \partial A_i * \langle R_i\rangle$ be a facet. If $v = i_j$ for some $i_j \in A'$, then $B = F_i \setminus \{i_j\} \subseteq F_i \setminus \{i_j\} \cup \{v_{i_j}\}$. Lemma 4.35 implies that the latter is a facet in $\Diamond(\Gamma_{i_j})$, which by construction does not lie in Δ_i. It follows that $B \subseteq \partial\Delta_i$. Suppose $v = F_i(j)$ for some $i_\ell + 1 \leq j \leq d$ with $\varphi(F_i)(j - i_\ell - 1) = 1$. In this case, we have $B = F_i \setminus \{F_i(j)\} \subseteq (F_i \setminus \{F_i(j)\}) \cup \{F_0(j)\}$. By Lemma 4.35, the latter is a facet in $\Diamond(\Gamma_{i_\ell})$ of a smaller degree than F_i and that is hence lexicographically smaller than F_i, meaning that $(F_i \setminus \{F_i(j)\}) \cup \{F_0(j)\} \notin \Delta_i$. Hence, $B \in \partial\Delta_i$.
Case 2: $i_1 > i_2$.
We have different subcases:

(2a) $t(i) \leq i_1$, or

(2b) $t(i) > i_1$.

In case (2a), we define A_i and R_i as in Case 1 and show that conditions *i*–*iii* from Definition 4.14 hold. The arguments are verbatim the same as in Case 1. Only for condition *ii* we need to verify that A_i does not lie in a facet of $\Diamond(\Gamma_{i_1})$. First assume $t(i) < i_1$. Then there exists $0 \leq j < i_1 - i_\ell - 1$ with $\varphi(F_i)(j) = 1$. Using the definition of F_0, we infer $F_i(j) = v_{j+i_\ell+1} \in A_i$. It follows from Lemma 4.35 that $v_j \notin \Diamond(\Gamma_{i_1})$ as $0 \leq j \leq i_1 - 1$. We conclude that A_i cannot be contained in a facet of $\Diamond(\Gamma_{i_1})$. Next assume $t(i) = i_1$. In this case, it follows that $F_i(i_1) = i_1 \in A_i$. As $i_1 \notin \Diamond(\Gamma_{i_1})$, the claim follows.

If we are in case (2b) we define

$$A_i = A' \cup \{v_{i_\ell}\} \cup \{F_i(j) \ : \ \varphi(F_i)(j) = 1\}$$

and

$$R_i = (\{0, \ldots, i_\ell - 1\} \setminus A') \cup \{F_i(j) \ : \ \varphi(F_i)(j) = 0\}.$$

Once more, we need to verify conditions *i*–*iii*.

Condition *i*: $t(i) > i_1$ implies that $F_i(i_1) = F_0(i_1) = v_{i_1}$ and hence $\{F_i(j) \ : \ \varphi(F_i)(j) = 0\} \neq \emptyset$. It follows that $R_i \neq \emptyset$. We also have $A_i \neq \emptyset$ since $v_{i_\ell} \in A_i$.

Condition *ii*: The same arguments as above show that A_i is not contained in a facet of type 1 or 2. Since $v_{i_\ell} \in A_i$ but $v_{i_\ell} \notin \Diamond(\Gamma_{i_1})$ (by Lemma 4.35), the face A_i cannot be contained in a facet of $\Diamond(\Gamma_{i_1})$.

Condition *iii*: The same arguments as above show that $F_i \setminus \{i_j\} \subseteq \partial\Delta_i$ for $2 \leq j \leq \ell - 1$ and $F_i \setminus \{F_i(j)\} \subseteq \partial\Delta_i$ for $i_\ell + 1 \leq j \leq d$ with $\varphi(F_i)(j) = 1$. It only remains to show that $F_i \setminus \{v_{i_\ell}\} = (A_i \setminus \{v_{i_\ell}\}) \cup R_i \subseteq \partial\Delta_i$. As $t(i) > i_1$, it holds that

$$\{0, \ldots, i_\ell - 1, i_\ell + 1, \ldots, i_1 - 1, v_{i_1}\} \subseteq F_i \setminus \{v_{i_\ell}\} \subseteq (F_i \setminus \{v_{i_\ell}\}) \cup \{i_\ell\}.$$

Lemma 4.35 then implies that $(F_i \setminus \{v_{i_\ell}\}) \cup \{i_\ell\} \in \Diamond(\Gamma_{i_1})$. The claim follows. □

Example 4.39. Figure 4.9 shows the shelling order from Theorem 4.38 for $\lozenge(\Gamma_0, \Gamma_1, \Gamma_2)$ within a 2-ball. First $\lozenge(\Gamma_1)$ is shelled, since it contains the first free facet. Afterwards, one continues with $\lozenge(\Gamma_0)$ and $\lozenge(\Gamma_1)$.

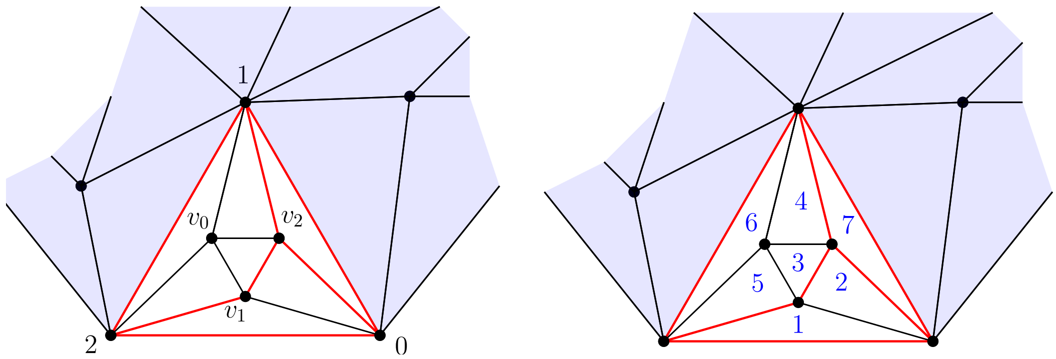

FIGURE 4.9: The blue numbers on the right represent the shelling order on $\lozenge(\Gamma_0, \Gamma_1, \Gamma_2)$. The red lines separate the three complexes $\lozenge(\Gamma_i)$.

4.3.5 Proof of Theorem 4.2

Combining the results from Sections 4.3.1 – 4.3.4 we can finally provide the proof of Theorem 4.2. An illustration of the proof is given in Figure 4.10.

Proof of Theorem 4.2. Let Δ and Γ be balanced combinatorial d-manifolds with boundary. If $\Delta \overset{\text{bsh}}{\approx} \Gamma$, then clearly Δ and Γ are PL homeomorphic.
Assume that Δ and Γ are PL homeomorphic. We need to show that Δ and Γ are related by a sequence of shellings and their inverses, preserving balancedness in each step. Using Proposition 4.18, we can assume that the boundaries of Δ and Γ are isomorphic and that this isomorphism respects the coloring. It follows from Theorem 4.26 that there exists a sequence of basic cross-flips connecting Δ and Γ. W.l.o.g. we can assume that Δ and Γ differ by a single cross-flip, i.e., $\Gamma = \chi^*_D(\Delta)$ with $D = \lozenge(\Phi)$, for some subcomplex $\Phi \subseteq \partial\Delta_{d+1}$. We can also assume that Δ (and Γ) are connected, which implies that they are strongly connected (i.e., their dual graph is connected) and that there exists a sequence of facets $F_0, \ldots, F_m \in \Delta$ such that F_i and F_{i+1} intersect in a common face of dimension $d-1$ (for $0 \leq i \leq m-1$) and F_0 and F_m intersect $\partial\Delta$ and D, respectively, in a face of dimension $d-1$ and d, respectively. Choosing such a sequence with minimal m, we can assure that $F_0, \ldots, F_{m-1}, F'_m$ – where F'_m is the unique face in $\partial\mathcal{C}_{d+1} \setminus D$ that intersects F_m in $F_m \cap \partial D$ – is a sequence of facets in Γ with the same properties. We now proceed by induction on m. Let $F = \partial\Delta \cap F_0$. By assumption we have that $\dim F = d-1$. Applying Theorem 4.34 to Δ, we can construct a sequence of inverse shellings that transforms Δ into a balanced manifold Δ' such that $\partial\Delta' \cap \partial\Delta = \langle F \rangle$. Since Δ and Γ have isomorphic boundaries, we can apply the same sequence of inverse shellings to Γ in order to obtain a balanced manifold Γ', whose boundary is isomorphic to $\partial\Delta'$. As Δ and Γ are induced subcomplexes of Δ' and Γ', respectively, we can apply the cross-flip χ^*_D and $\chi^*_{\partial\mathcal{C}_{d+1} \setminus D}$ to Δ' and Γ', respectively. In particular, we have $\Delta' \setminus D = \Gamma' \setminus (\partial\mathcal{C}_{d+1} \setminus D)$. If $m = 0$, then we have $D \cap \partial\Delta' = F$. Theorem 4.38 further yields a sequence of shellings transforming Δ' into $\Delta' \setminus D$ and a sequence of inverse shellings from $\Delta' \setminus D = \Gamma' \setminus (\partial\mathcal{C}_{d+1} \setminus D)$

to Γ':

$$\Delta' \mapsto \Delta_1 \mapsto \ldots \mapsto \Delta_s = \Delta' \setminus D = \Gamma' \setminus (\partial \mathcal{C}_{d+1} \setminus D) \mapsto \Gamma_r \mapsto \ldots \mapsto \Gamma_1 \mapsto \Gamma'. \quad (4.1)$$

Since Δ' and Γ' are both balanced and every Δ_i and Γ_i is a subcomplex of Δ' and Γ', respectively, every intermediate step in (4.1) is balanced. We conclude that $\Delta' \overset{\mathsf{bsh}}{\approx} \Gamma'$ and hence, by construction of Δ' and Γ', we also have $\Delta \overset{\mathsf{bsh}}{\approx} \Gamma$.
Now let $m \geq 1$. We set $\Delta'' = \Delta' \setminus F_0$ and $\Gamma'' = \Gamma' \setminus F_0$. As m is minimal, we infer that $D \subseteq \Delta''$ and $\partial \mathcal{C}_{d+1} \setminus D \subseteq \Gamma''$. Moreover, since Δ and Γ are induced subcomplexes of Δ' and Γ', respectively, it follows that we can apply the cross-flip χ_D^* and $\chi_{\partial \mathcal{C}_{d+1} \setminus D}^*$ to Δ'' and Γ'', respectively. Moreover, we have that $\partial \Delta'' \cong \partial \Gamma''$ and

$$\Gamma'' = \chi_D^*(\Delta'') = \chi_D^*(\Delta') \setminus F_0 = \Gamma' \setminus D.$$

Applying the induction hypothesis to Δ'' and Γ'' yields a sequence of shellings and inverse shellings from Δ'' to Γ''. It hence follows that $\Delta' \overset{\mathsf{bsh}}{\approx} \Gamma'$ and thus also $\Delta \overset{\mathsf{bsh}}{\approx} \Gamma$. $\square$

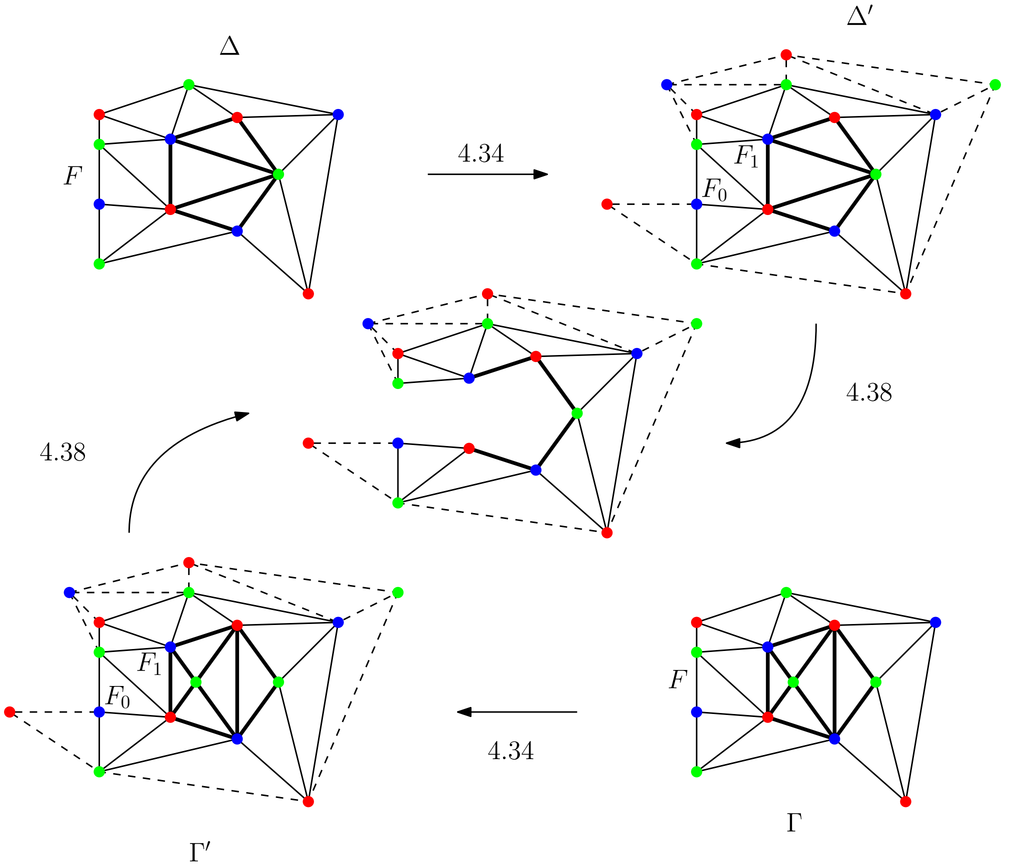

FIGURE 4.10: An illustration of Theorem 4.2 for $\Gamma = \chi_D^*(\Phi)$. The labeling of the faces follows the one in the proof.

4.4 Combinatorics of basic and reducible cross-flips

The aim of this section is threefold. In the first and second part, we will provide a solution to Problem 4.13 by determining the number of combinatorially different basic cross-flips (Theorem 4.42) and by computing the h-vectors of diamond complexes. However, it appears from some experiments that not all of those moves are really needed to transform two closed balanced PL homeomorphic manifolds into each other. A simple reason for this might be that one can write a certain cross-flip as a combination of others. In the last part, we will provide a set of such moves, which will be called *reducible* moves.

4.4.1 Counting basic cross-flips

As a warm-up, we consider the possible basic cross-flips in dimension 2. Figure 4.2 depicts all combinatorially different (not necessarily basic) cross-flips. The basic cross-flips can be seen in the first two rows of this figure. More precisely, the left picture in the first row shows the cross-flip that exchanges $\Diamond(\Gamma_2) \cong \Diamond(\Gamma_3)$ with $\Diamond(\Gamma_0, \Gamma_1, \Gamma_3) \cong \Diamond(\Gamma_0, \Gamma_1, \Gamma_2)$ (and its reverse). It follows from Lemma 4.35 that $\Diamond(\Gamma_d)$ and $\Diamond(\Gamma_{d+1})$ are always just d-simplices and in particular isomorphic. The isomorphism $\Diamond(\Gamma_0, \Gamma_1, \Gamma_3) \cong \Diamond(\Gamma_0, \Gamma_1, \Gamma_2)$ can also be generalized appropriately to higher dimensions and more complicated subcomplexes (see Lemma 4.41). In the right picture on the first line of Figure 4.2 one sees the cross-flip, that removes $\Diamond(\Gamma_1) \cong \Diamond(\Gamma_2, \Gamma_3)$ and adds $\Diamond(\Gamma_0, \Gamma_2, \Gamma_3) \cong \Diamond(\Gamma_0, \Gamma_1)$ (and vice versa). Those isomorphisms will also be instances of Lemma 4.41 below. The isomorphism $\Diamond(\Gamma_1) \cong \Diamond(\Gamma_2, \Gamma_3)$ is also shown in Figure 4.11. The left picture in the second row of Figure 4.2 shows the cross-flip (and its inverse) substituting $\Diamond(\Gamma_0)$ with $\Diamond(\Gamma_1, \Gamma_2, \Gamma_3)$, which turns out to be isomorphic to $\Diamond(\Gamma_0)$. We will refer to this move as the *trivial move* and such a move exists in each dimension. Finally, the right picture in the second row depicts the cross-flip interchanging $\Diamond(\Gamma_1, \Gamma_3) \cong \Diamond(\Gamma_1, \Gamma_2)$ and $\Diamond(\Gamma_0, \Gamma_2) \cong \Diamond(\Gamma_0, \Gamma_3)$ and vice versa. Again, the isomorphisms will follow from Lemma 4.41.

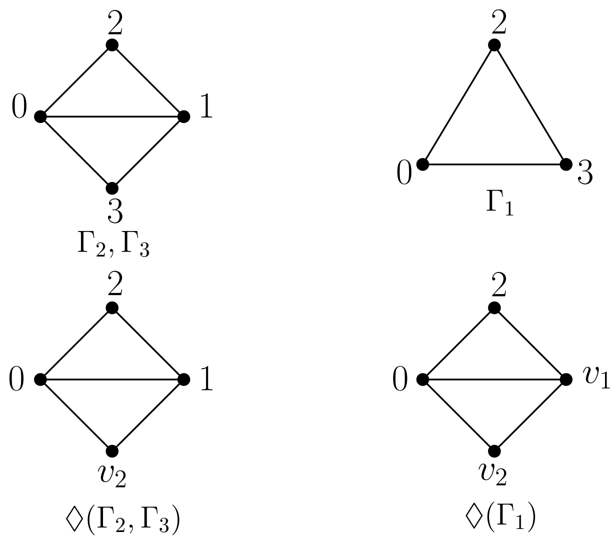

FIGURE 4.11: The isomorphism $\Diamond(\Gamma_2, \Gamma_3) \cong \Diamond(\Gamma_1)$ for $d = 2$.

We state a first simple observation.

Lemma 4.40. *Let* $0 \le k \le d$. *Then*

$$\Diamond(\Gamma_k) \cong \bigcup_{i=k+1}^{d+1} \Diamond(\Gamma_i).$$

Proof. The claimed isomorphism is provided by switching vertices k and v_k. The statement is then immediate since on the one hand, by Lemma 4.35, $F \in \Diamond(\Gamma_k)$ is a facet if and only $\{0,\dots,k-1,v_k\} \subseteq F$, and on the other hand $G \in \bigcup_{i=k+1}^{d+1} \Diamond(\Gamma_i)$ is a facet if and only if $\{0,\dots,k\} \subseteq G$. □

The previous lemma in particular provides an explanation for the isomorphism of the trivial move and also for $\Diamond(\Gamma_1) \cong \Diamond(\Gamma_2,\Gamma_3)$, as seen above. However, it is a priori not clear that this isomorphism is somehow preserved, when we add further subcomplexes, e.g., we cannot use it to explain the isomorphism $\Diamond(\Gamma_0,\Gamma_1) \cong \Diamond(\Gamma_0,\Gamma_2,\Gamma_3)$. This is the content of the next lemma.

Lemma 4.41. *Let $k \in \mathbb{N}$, $0 \le \ell \le d$ and $0 \le i_1 < \cdots < i_k < \ell$. Then,*

$$\Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k},\Gamma_\ell) \cong \Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k},\Gamma_{\ell+1},\dots,\Gamma_{d+1}).$$

Proof. For $k = 0$, the statement is Lemma 4.40. Now assume $k \ge 1$. We set $D = \Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k},\Gamma_\ell)$ and $D' = \Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k},\Gamma_{\ell+1},\dots,\Gamma_{d+1})$. Let further $\tilde{D} = \Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k})$. As $k \ge 1$, we deduce from Lemma 4.35 that $V(D) = V(D')$ and $\{\ell, v_\ell\} \subseteq V(D)$. Let $\psi_\ell : V(D) \to V(D')$ be the map that switches vertices ℓ and v_ℓ, i.e.,

$$\psi_\ell(v) = \begin{cases} v & \text{if } v \notin \{\ell, v_\ell\}, \\ \ell & \text{if } v = v_\ell, \\ v_\ell & \text{if } v = \ell. \end{cases}$$

We claim that ψ_ℓ induces a simplicial isomorphism between D and D'. As $\ell > i_k$ it follows from Lemma 4.35 that $G \cup \{\ell\}$ is a facet of $\tilde{D}$ if and only if $G \cup \{v_\ell\}$ is a facet of $\tilde{D}$. Hence, $F \in \mathcal{F}(D) \cap \mathcal{F}(\tilde{D})$ if and only if $\psi_\ell(F) \in \mathcal{F}(D') \cap \mathcal{F}(\tilde{D})$. Moreover, F is a facet in $\Diamond(\Gamma_\ell)$ if and only if $\{0,\dots,\ell-1,v_\ell\} \subseteq F$, if and only if $\{0,\dots,\ell-1,\ell\} \subseteq \psi_\ell(F)$, which is the case if and only if $\psi_\ell(F)$ is a facet of $\Diamond(\Gamma_{\ell+1},\dots,\Gamma_d)$ by Lemma 4.35. The claim follows. □

We state the main result of this section which provides a solution to Problem 4.13.

Theorem 4.42. *There are $2^{d+1} - 1$ combinatorially different basic cross-flips in dimension d.*

Proof. It follows from Lemma 4.35 that $f_d(\Diamond(\Gamma_\ell)) = 2^{d-\ell}$ for $0 \le \ell \le d$ and $f_d(\Diamond(\Gamma_{d+1})) = 1$. In particular, if $0 \le i_1 < i_2 < \cdots < i_k \le d$, then $f_d(\Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k})) = \sum_{\ell=1}^{k} 2^{d-i_\ell}$. As the representation of $f_d(\Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k}))$ as a sum of different powers of 2 is clearly unique, it follows that

$$f_d(\Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k})) \neq f_d(\Diamond(\Gamma_{j_1},\dots,\Gamma_{j_s})),$$

for distinct sequences $0 \le i_1 < \cdots < i_k \le d$ and $0 \le j_1 < \cdots < j_s \le d$. This implies that $\Diamond(\Gamma_{i_1},\dots,\Gamma_{i_k})$ and $\Diamond(\Gamma_{j_1},\dots,\Gamma_{j_s})$ cannot be isomorphic and the only isomorphisms we have to consider are those given by Lemma 4.41. Therefore, the number of combinatorially different basic cross-flips is given by the number of non-empty subsets of $\{0,\dots,d\}$ and the claim follows. □

4.4.2 Face numbers of basic cross-flips

In this section, our aim is to compute the face numbers of the diamond complexes that describe basic cross-flips. This will be done by showing that the degree lexicographic shelling order from Theorem 4.38 for $(\Delta, \Delta \setminus \Diamond(\Gamma))$ also provides a shelling order for $\Diamond(\Gamma)$.

Proposition 4.43. *Let* $\Gamma = \langle \Gamma_{i_1}, \ldots, \Gamma_{i_k} \rangle$, *where* $0 \leq i_1 < i_2 < i_3 < \cdots < i_k \leq d+1$. *For* $1 \leq \ell \leq k$, *let* $r_\ell = f_d(\Diamond(\Gamma_{i_\ell})) - 1$ *and let*

$$F_0^{(\ell)} = F_{\text{in}}^{(\ell)}, F_1^{(\ell)}, \ldots, F_{r_\ell}^{(\ell)}$$

be the ordering of the facets of $\Diamond(\Gamma_{i_\ell})$ *according to the degree lexicographic ordering with respect to* $F_{\text{in}}^{(\ell)}$. *Then*

$$F_0^{(1)}, F_1^{(1)}, \ldots, F_{r_1}^{(1)}, \ldots, F_0^{(k)}, F_1^{(k)}, \ldots, F_{r_k}^{(k)}$$

is a shelling order for $\Diamond(\Gamma)$.

Proof. Let $F_i^{(\ell)}$ be a facet. We claim that

$$R_i^{(\ell)} = \{i_1, \ldots, i_{\ell-1}\} \cup \{F_i^{(\ell)}(j) \ : \ 0 \leq j \leq d - i_\ell - 1 \text{ such that } \varphi_{F_0^{(\ell)}}(F_i^{(\ell)})(j) = 1\}$$

is the restriction face of $F_i^{(\ell)}$, i.e., the unique minimal face of $F_i^{(\ell)}$ not lying in $\Delta_i^{(\ell)} = \bigcup_{j=1}^{\ell-1} \Diamond(\Gamma_{i_j}) \cup \langle F_0^{(\ell)}, \ldots, F_{i-1}^{(\ell)} \rangle$. Let $H \in \langle F_i^{(\ell)} \rangle \setminus \Delta_i^{(\ell)}$. As $H \not\subseteq F_j^{(\ell)}$ for $0 \leq j \leq i-1$, we must have $\{F_i^{(\ell)}(j) \ : \ \varphi_{F_0^{(\ell)}}(F_i^{(\ell)})(j) = 1\} \subseteq H$. Otherwise, H would be contained in a degree lexicographically smaller facet than $F_i^{(\ell)}$. Moreover, $H \not\subseteq \Diamond(\Gamma_{i_j})$ for $1 \leq j \leq \ell - 1$ implies that $i_j \in H$ for $1 \leq H \leq \ell - 1$. We conclude that $R_i^{(\ell)} \subseteq H$. The same arguments also show that indeed $R_i^{(\ell)} \notin \Delta_i^{(\ell)}$. The claim follows. □

We use the shelling of Proposition 4.43 to compute the h-vectors of diamond complexes. In the following, we set $\binom{n}{k} = 0$ if $n < k$.

Proposition 4.44. *Let* $0 \leq i_1 < \cdots < i_k \leq d+1$ *and let* $\Diamond(\Gamma) = \Diamond(\Gamma_{i_1}, \ldots, \Gamma_{i_k})$. *Then, for* $0 \leq \ell \leq d$

$$h_\ell(\Diamond(\Gamma)) = \binom{d - i_1}{\ell} + \binom{d - i_2}{\ell - 1} + \cdots + \binom{d - i_k}{\ell - k + 1}.$$

The previous proposition in particular implies that $h_\ell(\Diamond(\Gamma)) = 0$ for $\ell > d - i_1$. Also note that for $k = 1$, the statement follows directly from Lemma 4.35 combined with the fact that $h_\ell(\partial\mathcal{C}_{d+1}) = \binom{d+1}{\ell}$.

Proof. By 1.2, to compute $h_\ell(\Diamond(\Gamma))$ we need to count restriction faces of size ℓ. It follows from the proof of Proposition 4.43 that for $1 \leq j \leq k$ the facets in $\Diamond(\Gamma_{i_j})$, whose restriction faces are of size ℓ are those that differ in $\ell - (j-1)$ positions from the initial facet $F_0^{(j)}$. Since there are $\binom{d-i_j}{\ell-j+1}$ such facets, one for each $(\ell-j+1)$-subset of $\{0, \ldots, d - i_j\}$, the claim follows. □

Using the following lemma, we can control the face numbers when applying a cross-flip.

Lemma 4.45. *Let* D *be a shellable and co-shellable subcomplex of* $\partial\mathcal{C}_d$ *that is a* d*-ball. Then, for* $0 \leq i \leq d+1$

$$h_i(D) + h_{d+1-i}(\partial\mathcal{C}_d \setminus D) = \binom{d+1}{i}.$$

Proof. We have seen in Chapter 2 that

$$h_i(\partial\mathcal{C}_d) = \binom{d+1}{i}$$

for all $0 \leq i \leq d$. Since D is both shellable and co-shellable, any shelling order on D can be extended to one of $\partial\mathcal{C}_d$ by adding a reverse order on $\partial\mathcal{C}_d \setminus D$. Reversing the order on $\partial\mathcal{C}_{d+1} \setminus D$ has the effect that a facet F, whose restriction face R was of size i before, now has the restriction face $F \setminus R$, which is of size $d+1-i$. The claim now follows from (1.2). $\square$

Example 4.46. We can use Lemma 4.45 to compute the h-vector of a cross-polytopal stacked d-sphere. We can build any $\Delta \in \mathcal{ST}^{\times}(n, d+1)$ from $\partial\mathcal{C}_{d+1}$ by replacing a facet F with its complement in $\partial\mathcal{C}_{d+1}$ and repeating this process $\left(\frac{n}{d+1} - 2\right)$-times. In other words, we apply the basic cross-flip $\chi^*_{\lozenge(\Gamma_d)}$ $\left(\frac{n}{d+1} - 2\right)$-times to $\partial\mathcal{C}_{d+1}$. By Lemma 4.45

$$h_{d+1-i}(\partial\mathcal{C}_{d+1} \setminus F) = \binom{d+1}{i}$$

for $0 \leq i \leq d$ and $h_{d+1}(\partial\mathcal{C}_{d+1} \setminus F) = 0$. Using a reversed shelling of $\partial\mathcal{C}_{d+1} \setminus F$, we can successively transform $\partial\mathcal{C}_{d+1}$ into $\mathcal{ST}^{\times}(n, d)$ and obtain

$$h_i(\mathcal{ST}^{\times}(n,d)) = \binom{d+1}{i} + \left(\frac{n}{d+1} - 2\right)\binom{d+1}{i} = \left(\frac{n}{d+1} - 1\right)\binom{d+1}{i}$$

for $0 < i < d$ and $h_0(\mathcal{ST}^{\times}(n,d)) = h_{d+1}(\mathcal{ST}^{\times}(n,d)) = 1$. Note that the h-vector of $\mathcal{ST}^{\times}(n,d)$ can be computed directly using Lemma 1.40, as we did in Chapter 2. However, we included this example as a nice application of Lemma 4.45.

4.4.3 Reducible cross-flips

In Section 4.4.1 we have seen that there are $2^{d+1} - 1$ combinatorially different cross-flips in dimension d. However, if we want to relate two balanced closed PL homeomorphic manifolds using basic cross-flips as in Theorem 4.10, then it is conceivable that not all of these are really needed. Indeed, it was shown in [MS18] that for 2-dimensional spheres, other than the boundary of the cross-polytope, it is sufficient to use the pentagon move and its inverse, which are the moves on the right in the second line of Figure 4.2. This is the motivation for this section.

Definition 4.47. Let BC_d the set of basic cross-flips in dimension d. We call a set $R \subseteq \mathrm{BC}_d$ of basic cross-flips *reducible* if every cross-flip in R can be expressed as a combination of cross-flips in $\mathrm{BC}_d \setminus R$.

Note that if R is a set of reducible cross-flips, then, in particular, the basic cross-flips contained in $\mathrm{BC}_d \setminus R$ suffice to relate any two balanced closed PL homeomorphic manifolds. By Lemma 4.41 we also know that for any basic cross-flip there is a representative of the form $\chi^*_{\lozenge(\Gamma)}$, where $\Gamma = \langle \Gamma_{i_1}, \ldots, \Gamma_{i_k} \rangle$, and $0 \leq i_1 < \cdots < i_k \leq d$. In the following, we will always use this representative. We need to introduce some further notation. For a set $I \subseteq \{0, \ldots, d\}$ and $a \in \mathbb{Z}$, we write $I + a$ for the set $\{i + a \ : \ i \in I\}$. Moreover, given $I = \{i_1, \ldots, i_k\}$ we use Γ_I to denote the d-ball $\langle \Gamma_{i_1}, \ldots, \Gamma_{i_k} \rangle$. Since the dimension of Γ_I is not clear from the notation a priori, we will add a superscript and write Γ_I^d from now on. Similarly, we write $\lozenge^d$ for the diamond operation in dimension d. Those distinctions will be important in the rest of this section.

We have the following observations. An instance is depicted in Figure 4.12.

Lemma 4.48. *Let $I \subseteq \{0, \dots, d\}$.*

i. If $d \notin I$, then $\lozenge^d(\Gamma_I^d) = \{d, v_d\} * \lozenge^{d-1}(\Gamma_I^{d-1})$.

ii. If $0 \notin I$, then $\lozenge^d(\Gamma_I^d) = \{0\} * \pi(\lozenge^{d-1}(\Gamma_{I-1}^{d-1}))$, where

$$\begin{aligned}\pi : \{0, \dots, d-1\} \cup \{v_0, \dots, v_{d-1}\} &\to \{1, \dots, d\} \cup \{v_1, \dots, v_d\} \\ i &\mapsto i+1 \\ v_i &\mapsto v_{i+1}.\end{aligned}$$

Proof. It follows from Lemma 4.35 that we have $\lozenge^d(\Gamma_j^d) = \{d, v_d\} * \lozenge^{d-1}(\Gamma_j^{d-1})$ for $0 \le j < d$. This implies i.

For ii, first note that 0 lies in every facet of $\lozenge^d(\Gamma_j^d)$ if $j \neq 0$. The statement is then immediate by Lemma 4.35. □

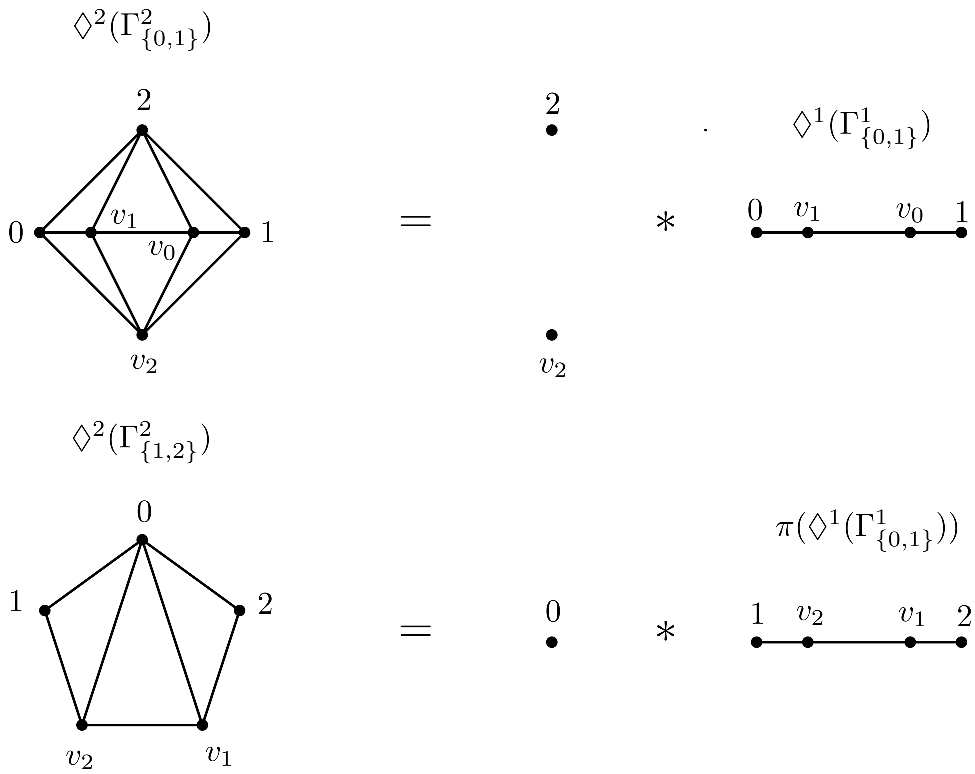

FIGURE 4.12: The isomorphism from Lemma 4.48.

Let $\psi : \{d, v_d\} \to \{d, v_d\}$ be the map that interchanges d and v_d and let

$$\begin{aligned}\rho : \{0, \dots, d\} \cup \{v_0, \dots, v_d\} &\to \{0, \dots, d\} \cup \{v_0, \dots, v_d\} \\ i &\mapsto i-1 \\ v_i &\mapsto v_{i-1},\end{aligned}$$

where we compute $i-1$ modulo $(d+1)$. We also set $\sigma = \psi \circ \rho$.

As a consequence of part i of the previous lemma, we obtain the following decomposition of $\Diamond^d(\Gamma_I^d)$ that will be crucial. Once again we include a 2-dimensional example in Figure 4.13 for the sake of clarity.

Corollary 4.49. *Let $I \subseteq \{0, \ldots, d\}$ with $d \notin I$. Then,*

$$\Diamond^d(\Gamma_I^d) = \rho(\Diamond^d(\Gamma_{I+1}^d)) \cup \sigma(\Diamond^d(\Gamma_{I+1}^d)), \tag{4.2}$$

where

$$\rho(\Diamond^d(\Gamma_{I+1}^d)) \cap \sigma(\Diamond^d(\Gamma_{I+1}^d)) = \Diamond^{d-1}(\Gamma_I^{d-1}).$$

Moreover, $\rho(\Diamond^d(\Gamma_{I+1}^d))$ and $\sigma(\Diamond^d(\Gamma_{I+1}^d))$ are induced subcomplexes of $\Diamond^d(\Gamma_I^d)$.

Proof. Since by assumption $d \notin I$ and clearly $0 \notin I+1$, by Lemma 4.48 i we have

$$\begin{aligned}\Diamond^d(\Gamma_I^d) =& \{d, v_d\} * \Diamond^{d-1}(\Gamma_I^{d-1}) \\ =& \left(\{d\} * \Diamond^{d-1}(\Gamma_I^{d-1})\right) \cup \left(\{v_d\} * \Diamond^{d-1}(\Gamma_I^{d-1})\right).\end{aligned}$$

Both complexes appearing in this decomposition are clearly induced and their intersection is $\Diamond^{d-1}(\Gamma_I^{d-1})$. Using Lemma 4.35 one easily verifies that

$$\{d\} * \Diamond^{d-1}(\Gamma_I^{d-1}) = \rho(\Diamond^d(\Gamma_{I+1}^d)) \quad \text{and} \quad \{v_d\} * \Diamond^{d-1}(\Gamma_I^{d-1}) = \sigma(\Diamond^d(\Gamma_{I+1}^d)).$$

□

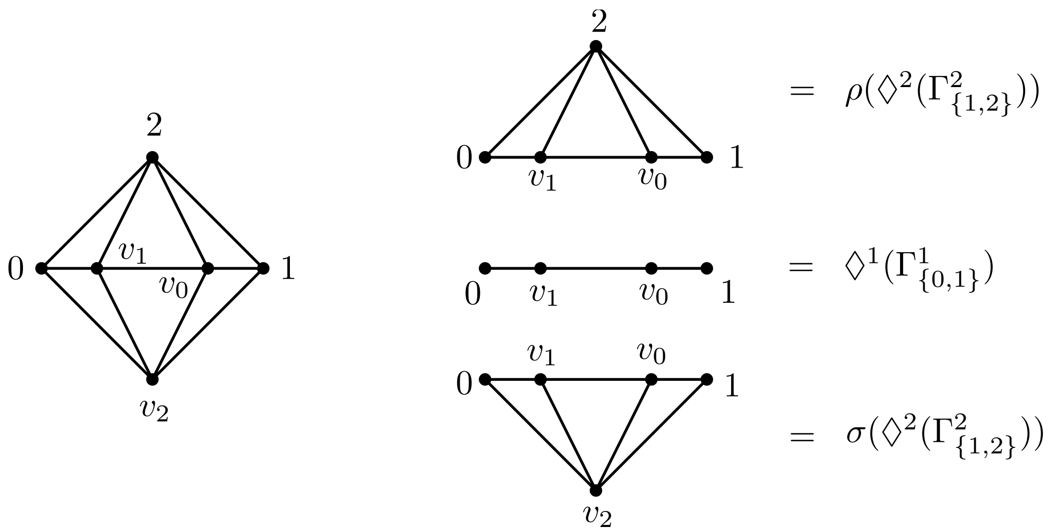

FIGURE 4.13: An instance of the decomposition given in Corollary 4.49.

The last ingredient, we need is the following decomposition that follows from part ii of Lemma 4.48.

Corollary 4.50. *Let $I \subseteq \{0, \ldots, d\}$ with $0 \in I$. Then,*

$$\Diamond^d(\Gamma_I^d) = \Diamond^d(\Gamma_{I\setminus\{0\}}^d) \cup \Diamond^d(\Gamma_0^d) \tag{4.3}$$

and

$$\Diamond^d(\Gamma_{I\setminus\{0\}}^d) \cap \Diamond^d(\Gamma_0^d) = \pi(\Diamond^{d-1}(\Gamma_{(I\setminus\{0\})-1}^{d-1})).$$

Moreover, $\Diamond^d(\Gamma_0^d)$ is an induced subcomplex of $\Diamond^d(\Gamma_I^d)$.

Proof. First note that
$$\Diamond^d(\Gamma_I^d) = \Diamond^d(\Gamma_{I\setminus\{0\}}^d) \cup \Diamond^d(\Gamma_0^d)$$
and that $\Diamond^d(\Gamma_0^d)$ is induced. Since $0 \notin I \setminus \{0\}$, it follows from Lemma 4.48 *ii* that
$$\Diamond^d(\Gamma_{I\setminus\{0\}}^d) = \{0\} * \pi(\Diamond^{d-1}(\Gamma_{I\setminus\{0\}-1}^{d-1})).$$
Combining this with Lemma 4.35 we obtain
$$\Diamond^d(\Gamma_{I\setminus\{0\}}^d) \cap \Diamond^d(\Gamma_0^d) = \pi(\Diamond^{d-1}(\Gamma_{(I\setminus\{0\})-1}^{d-1})).$$
□

Our aim is to show that the set of cross-flips that remove diamond complexes $\Diamond^d(\Gamma_I^d)$ with $d \notin I$ is a set of reducible cross-flips. For this, using Corollary 4.49, we decompose $\Diamond^d(\Gamma_I^d)$ into two copies of $\Diamond^d(\Gamma_{I+1}^d)$ that are glued together along a subcomplex of their boundaries, that is itself isomorphic to $\Diamond^{d-1}(\Gamma_I^{d-1})$. The idea is to first flip one of the subcomplexes that are isomorphic to $\Diamond^d(\Gamma_{I+1}^d)$, then to decompose again, using Corollary 4.50, and then to perform another flip. The next example makes this idea more precise.

Example 4.51. Let $d = 2$ and let us consider $\Diamond^2(\Gamma_1^2)$. We first decompose $\Diamond^2(\Gamma_1^2)$ into two copies of $\Diamond^2(\Gamma_2^2)$. In this example, this is just two triangles intersecting in an edge (see the left picture in the first row of Figure 4.14). We now flip the "upper" subcomplex (see the right picture in the first row of Figure 4.14). The flipped part is now decomposed again into a part that is isomorphic to $\Diamond^2(\Gamma_{\{1,2\}}^2)$ and a part isomorphic to $\Diamond^2(\Gamma_0^2)$ (shown in white and green respectively in the middle picture of Figure 4.14). The second component is grouped together with the untouched copy of $\Diamond^2(\Gamma_2^2)$. The union of those two subcomplexes is isomorphic to $\Diamond^2(\Gamma_{\{0,2\}}^2)$ (see left picture at the bottom of Figure 4.14) and is substituted by its complement in $\partial\mathcal{C}_2$. We obtain 2 copies of $\Diamond^2(\Gamma_{\{1,2\}}^2)$ glued together along a subcomplex of their boundaries, which is $\partial\mathcal{C}_2 \setminus (\Diamond^2(\Gamma_1^2)) = \Diamond^2(\Gamma_{\{0,1\}}^2)$, where the last equality follows from Lemma 4.41.

Remark 4.52. Let $I = \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, d\}$ with $i_1 < \cdots < i_k$ and $d \notin I$. The inverse of the cross-flip $\chi^*_{\Diamond^d(\Gamma_I^d)}$ is then given by $\chi^*_{\Diamond^d(\Gamma_{I^c}^d)}$, where here and thereafter we set $I^c = \{0, \ldots, d+1\} \setminus I$. However, it follows from Lemma 4.41 that the cross-flip $\chi^*_{\Diamond^d(\Gamma_J^d)}$ with
$$J = \{0, 1, \ldots, i_k\} \setminus \{i_1, \ldots, i_{k-1}\}$$
has the same effect. In particular, $d \notin J$. This observation will be useful in the proof of the main theorem of this section, which we now state.

Theorem 4.53. *The set of basic cross-flips*
$$\{\chi^*_{\Diamond^d(\Gamma_I^d)} \ : \ I \subseteq \{0, \ldots, d-1\}\}$$
is reducible. In other words, the set of basic cross-flips
$$\{\chi^*_{\Diamond^d(\Gamma_I^d)} \ : \ I \subseteq \{0, \ldots, d\},\ d \in I\}$$
is sufficient to relate any two balanced PL homeomorphic closed manifolds.

Proof. Let $I \subseteq \{0, \ldots, d-1\}$. We will show that the basic cross-flip $\chi^*_{\Diamond^d(\Gamma_I^d)}$ can be expressed as the combination of the two basic cross-flips $\chi^*_{\Diamond^d(\Gamma_{I+1}^d)}$ and $\chi^*_{\Diamond^d(\Gamma_{(I+1)\cup\{0\}}^d)}$.

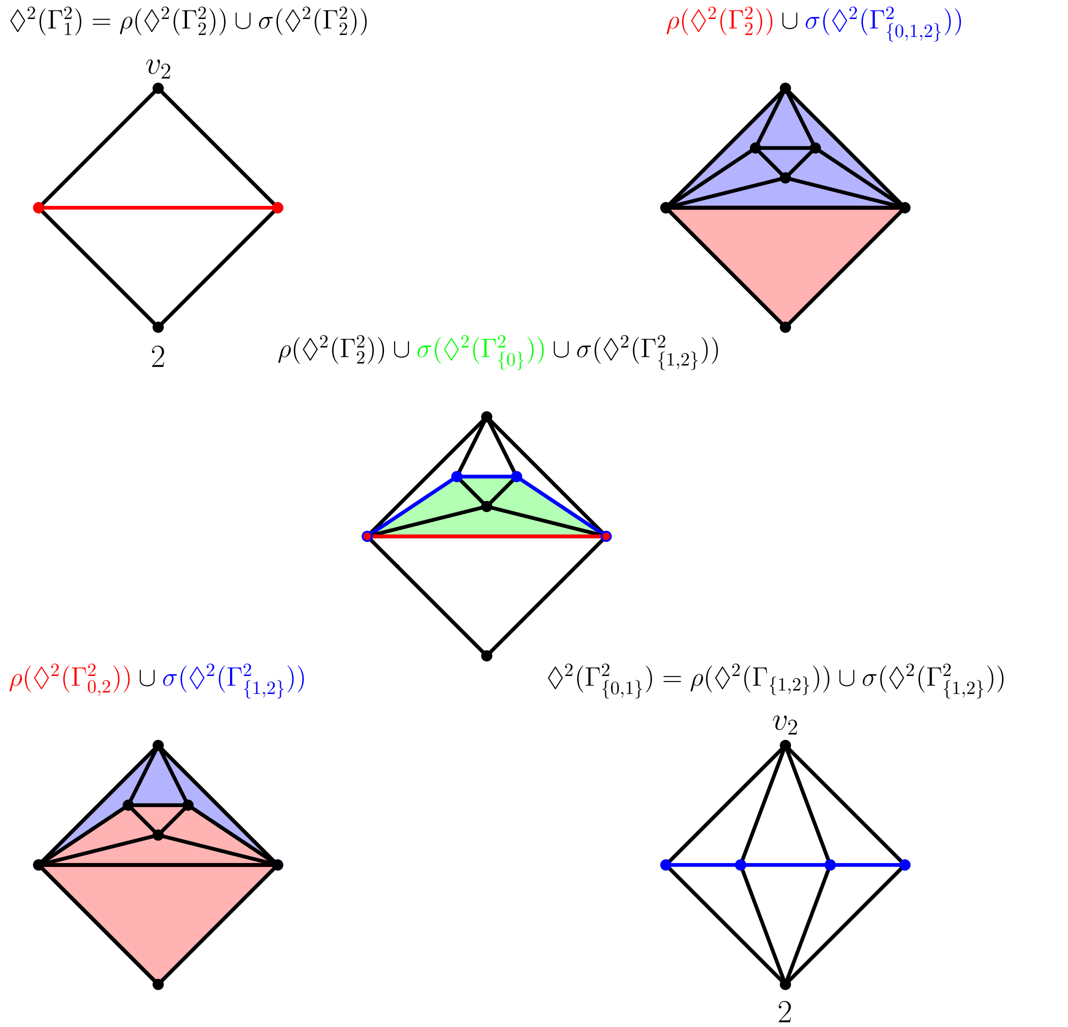

FIGURE 4.14: The reducibility of $\chi^*_{\Diamond^2(\Gamma^2_1)}$ and $\chi^*_{\Diamond^2(\Gamma^2_{\{0,1\}})}$. This particular reduction was already pointed out in [MS18, Remark 4.8].

The required statement then follows by applying this substitution process iteratively until $d \in I+1$.
By Corollary 4.49, we know that

$$\Diamond^d(\Gamma^d_I) = \rho(\Diamond^d(\Gamma^d_{I+1})) \cup \sigma(\Diamond^d(\Gamma^d_{I+1}))$$

and both subcomplexes on the right are induced. We can hence apply the cross-flip $\chi^*_{\Diamond^d(\Gamma^d_{I+1})}$ in order to replace one of the components, e.g., the first one, by its complement in $\partial\mathcal{C}_{d+1}$. Using that ρ is an isomorphism, that $0 \in (I+1)^c$ and Corollary 4.50, we can compute the complement of $\rho(\Diamond^d(\Gamma^d_{I+1}))$ via

$$\begin{aligned}\rho(\partial\mathcal{C}_{d+1}) \setminus (\rho(\Diamond^d(\Gamma^d_{I+1}))) =& \rho(\partial\mathcal{C}_{d+1} \setminus \Diamond^d(\Gamma^d_{I+1})) = \rho(\Diamond^d(\Gamma^d_{(I+1)^c}))\\ =& \rho(\Diamond^d(\Gamma^d_{(I+1)^c\setminus\{0\}})) \cup \rho(\Diamond^d(\Gamma^d_0)) = \rho(\Diamond^d(\Gamma^d_{I^c+1})) \cup \rho(\Diamond^d(\Gamma^d_0)).\end{aligned}$$

Note that both complexes, $\sigma(\Diamond^d(\Gamma^d_{I+1}))$ and $\rho(\Diamond^d(\Gamma^d_0))$ contain a vertex labeled v_d. In $\sigma(\Diamond^d(\Gamma^d_{I+1}))$, this is the vertex corresponding to 0 in $\Diamond^d(\Gamma^d_{I+1})$, in $\rho(\Diamond^d(\Gamma^d_0))$ this is the vertex corresponding to v_0 in $\Diamond^d(\Gamma^d_0)$. We relabel the vertex $v_d \in \rho(\Diamond^d(\Gamma^d_0))$ with $\tilde{v}_d$. As $\rho(\Diamond^d(\Gamma^d_0)) = \{\tilde{v}_d\} * \partial\mathcal{C}_{d-1}$ and $\sigma(\Diamond^d(\Gamma^d_{I+1})) = \{v_d\} * \Diamond^{d-1}(\Gamma^{d-1}_I)$, their intersection is simply

$$\sigma(\lozenge^d(\Gamma^d_{I+1})) \cap \rho(\lozenge^d(\Gamma^d_0)) = \lozenge^{d-1}(\Gamma^{d-1}_I).$$

As $0 \notin I+1$, it hence follows from Corollary 4.49 that

$$\sigma(\lozenge^d(\Gamma^d_{I+1})) \cup \rho(\lozenge^d(\Gamma^d_0)) = \tilde{\sigma}(\lozenge^d(\Gamma^d_{(I+1)\cup\{0\}})),$$

where $\tilde{\sigma}$ is the composition of σ, followed by a relabeling of the vertex d with $\tilde{v}_d$. As by Corollary 4.49 and Corollary 4.50 $\sigma(\lozenge^d(\Gamma^d_{I+1}))$ and $\rho(\lozenge^d(\Gamma^d_0))$ are induced, so is their union and we can apply the cross-flip $\chi^*_{\lozenge^d(\Gamma^d_{(I+1)\cup\{0\}})}$ substituting $\tilde{\sigma}(\lozenge^d(\Gamma^d_{(I+1)\cup\{0\}}))$ with its complement in $\tilde{\sigma}(\partial\mathcal{C}_{d+1})$, which is given by

$$\tilde{\sigma}(\partial\mathcal{C}_{d+1}) \setminus \tilde{\sigma}(\lozenge^d(\Gamma^d_{(I+1)\cup\{0\}})) = \tilde{\sigma}(\lozenge^d(\Gamma_{((I+1)\cup\{0\})^c})) = \tilde{\sigma}(\lozenge^d(\Gamma_{(I^c+1)\setminus\{0\}})).$$

It remains to show that the union of the two complexes $\rho(\lozenge^d(\Gamma^d_{I^c+1}))$ and $\tilde{\sigma}(\lozenge^d(\Gamma_{I^c+1}))$ is isomorphic to $\lozenge^d(\Gamma^d_{I^c})$. To see this, first note that by Remark 4.52 the basic cross-flip $\chi^*_{\lozenge^d(\Gamma^d_{I^c+1})}$ has a representative $\chi^*_{\lozenge(\Gamma_J)}$ with $d \notin J$. The claim then follows from Corollary 4.49. $\square$

Remark 4.54. As an immediate consequence of Theorem 4.53 it follows that 2^d basic cross-flips are sufficient to relate any two balanced PL homeomorphic manifolds without boundary, e.g., in dimension 2 the four cross-flips $\chi^*_{\lozenge^2(\Gamma^2_2)}$, $\chi^*_{\lozenge^2(\Gamma^2_{\{0,2\}})}$, $\chi^*_{\lozenge^2(\Gamma^2_{\{1,2\}})}$ and $\chi^*_{\lozenge^2(\Gamma^2_{\{0,1,2\}})}$ suffice. In Figure 4.2 the left picture in the first row depicts the interchange of $\lozenge^2(\Gamma^2_2)$ and $\lozenge^2(\Gamma^2_{\{0,1,2\}})$. The right picture in the second row depicts the interchange of $\lozenge^2(\Gamma^2_{\{1,2\}})$ and $\lozenge^2(\Gamma^2_{\{0,2\}})$.

4.5 Open problems

4.5.1 Connecting two balanced manifolds using few flips

The proof of Theorem 4.12 in [IKN17] does not provide information on the number of moves needed to connect two closed balanced combinatorial manifolds that are PL homeomorphic. On the other hand, especially from a computational point of view, an upper bound for the number of operations needed would be of interest. As an example let us assume that we start from a manifold Δ on n vertices and we perform *every* applicable cross-flip on Δ, obtaining a set T_1 of "target" manifolds. In the next step we repeat the procedure for each of the objects in T_1, and denote by T_2 the new targets. We proceed iteratively for a certain number of steps. What is the minimum number of steps that we need to obtain *all* balanced combinatorial manifolds that are PL-homeomorphic to Δ with n vertices? In a more general way, we formulate the following question.

Question 4.55. Fix two PL homeomorphic balanced combinatorial manifolds Δ and Γ. What is an upper bound (depending on $f_0(\Delta)$ and $f_0(\Gamma)$) for the minimal number of cross-flips needed to connect Δ and Γ?

4.5.2 The minimal set of sufficient basic cross-flips

In Section 4.4.1 we showed that for a fixed dimension d, there are precisely $2^{d+1}-1$ combinatorially distinct basic cross-flips, out of which 2^d suffice to relate any two balanced PL homeomorphic closed manifolds of dimension d. As remarked earlier for the case $d = 2$ it is proved in [MS18] that 3 flips actually suffice. Even more surprisingly, the authors show that it is possible to connect all *but a finite number* of balanced triangulations of a fixed surface using only 2 flips (see [MS18, Theorem 4.3]) and this exceptions have been investigated in [KMS19] in the case of surfaces S with Euler characteristic $-7 \leq \chi(S) \leq 2$. It turns out that for most of the surfaces considered there do not exist exceptional triangulations, and the graph of such triangulations is complete 3-partite. It is interesting to note that the set of 3 sufficient flips provided by Murai and Suzuki is not contained in the set of 4 flips given in Theorem 4.53. We hope that our detailed description of these moves will yield an answer to the following question:

Question 4.56. What is the cardinality of a minimal set of basic cross-flips that suffice to relate any two balanced combinatorial d-manifolds that are PL homeomorphic for $d > 2$?

Example 4.57. Figure 4.15 shows that we can write $\chi^*_{\Diamond^2(\Gamma^2_{\{1,2\}})}$ (or $\chi^*_{\Diamond^2(\Gamma^2_{\{0,2\}})}$) as a composition of two flips that belong to the set constructed in Theorem 4.53. Hence, for $d = 2$, the set in Theorem 4.53 can be further reduced to a set of cardinality 3. Moreover, it is not hard to notice by direct inspection that for $d = 2$ there are

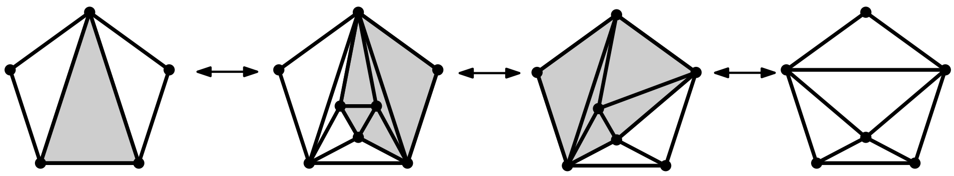

FIGURE 4.15: The flip $\chi^*_{\Diamond^2(\Gamma^2_{\{1,2\}})}$ as a composition of $\chi^*_{\Diamond^2(\Gamma^2_{\bar{2}})}$ and $\chi^*_{\Diamond^2(\Gamma^2_{\{0,2\}})}$. Reading the figure from right to left clearly shows that $\chi^*_{\Diamond^2(\Gamma^2_{\{0,2\}})}$ can be obtained from $\chi^*_{\Diamond^2(\Gamma^2_{\{0,1,2\}})}$ and $\chi^*_{\Diamond^2(\Gamma^2_{\{1,2\}})}$.

8 minimal sets of sufficient basic cross-flips, and they all have cardinality 3. More precisely, setting

$$
\begin{aligned}
\mathcal{B} = \{ &\{\{1\}, \{0,1\}, \{0,2\}\} \\
&\{\{1\}, \{0,1\}, \{1,2\}\} \\
&\{\{1\}, \{0,1,2\}, \{0,2\}\} \\
&\{\{1\}, \{0,1,2\}, \{1,2\}\} \\
&\{\{2\}, \{0,1\}, \{0,2\}\} \\
&\{\{2\}, \{0,1\}, \{1,2\}\} \\
&\{\{2\}, \{0,1,2\}, \{0,2\}\} \\
&\{\{2\}, \{0,1,2\}, \{1,2\}\}\},
\end{aligned}
$$

for every $B \in \mathcal{B}$ the set $\{\chi^*_{\Diamond^2(\Gamma^2_I)} : I \in B\}$ is a set of minimal sufficient cross-flips and there are no other minimal "generating sets".

Since the whole process of reduction seems to encode a notion of dependence among flips, we underline the following property of the set of minimal sufficient cross-flips for $d = 2$.

Lemma 4.58. *The set $\mathcal{B}$ is the set of bases of a matroid.*

Proof. It is straightforward to check that $\mathcal{B}$ satisfies the basis exchange axiom. In particular, the matroid $M(\mathcal{B})$ is isomorphic to a direct sum of 3 uniform matroids on 2 elements of rank 1: $M(\mathcal{B}) \cong U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$. $\square$

The previous lemma, though only providing weak evidence, clearly motivates the following question, which was also raised by Eran Nevo in personal communication:

Question 4.59. Do all the minimal sets of basic cross-flips that suffice to relate any two PL homeomorphic balanced combinatorial d-manifolds form the set of bases of a matroid, for $d \geq 2$?

Since the previous question appears to be rather ambitious, we also propose the following weaker question.

Question 4.60. Do all the minimal sets of basic cross-flips that suffice to relate any two PL homeomorphic balanced combinatorial d-manifolds have the same cardinality?

Chapter 5

Graded Betti numbers of balanced simplicial complexes

5.1 Preliminaries

Many of the results presented in the first two chapter of this thesis rely on algebraic properties of the Stanley-Reisner ring of simplicial complex, mostly though its Hilbert function. It is therefore natural to investigate further classical invariants of this ring, such as its minimal graded free resolution as a module over the polynomial ring. Our starting point are mainly two papers: In [MN03], Migliore and Nagel showed upper bounds for the graded Betti numbers of simplicial polytopes. More recently, building on those results, Murai [Mur15] established a connection between a specific property of a triangulation, so-called *tightness* and the graded Betti numbers of its Stanley-Reisner ring. Moreover, he employed upper bounds for graded Betti numbers to obtain a lower bound for the minimum number of vertices needed to triangulate a pseudomanifold with a given first (topological) Betti number, as well as an extension of the Lower Bound Theorem (Theorem 1.49) to all normal pseudomanifolds. It is conceivable that for more specific classes of simplicial complexes, better bounds (for the graded Betti numbers) hold, which then could be turned into lower bounds for the number of faces of such a family. This serves as the motivation for this chapter, where we will focus on the corresponding question for the case of balanced simplicial complexes. Our main results establish upper bounds for different cases, including arbitrary balanced simplicial complexes, balanced Cohen-Macaulay complexes and balanced normal pseudomanifolds. Along the way, we derive upper bounds on the graded Betti numbers of homogeneous ideals with a high concentration of generators in degree 2.

The structure of this chapter is the following:

- Section 5.1 is devoted to the definition of the minimal graded free resolution of a module and lex ideals.
- In Section 5.2 we use Hochster's formula to prove a first upper bound for the graded Betti numbers of an arbitrary balanced simplicial complex (see Theorem 5.7).
- We next restrict ourselves to the Cohen-Macaulay case, and provide two different upper bounds in this setting. The first approach provides a bound for graded Betti numbers of ideals with a high concentration of generators in degree 2, which immediately specializes to Stanley-Reisner ideals of balanced Cohen-Macaulay complexes (see Theorem 5.20). This is the content of Section 5.3.
- The second approach, presented is Section 5.4, employs the theory of *lex-plus-squares* ideals to bound the Betti numbers of ideals containing many generators

in degree 2, including the squares of the variables. Again the result on balanced complexes given in Theorem 5.31 follows as an immediate application.

- In Section 5.5, we focus on balanced normal pseudomanifolds. We use a result by Fogelsanger [Fog88] to derive upper bounds for the graded Betti numbers in the first strand of the graded minimal free resolution in this setting (see Theorem 5.37).

- Finally in Section 5.6 (Theorem 5.45) we compute the graded Betti numbers of cross-polytopal stacked spheres as in Definition 2.19, and show that, like their f- and h-numbers, they only depend on the number of vertices and on the dimension. The same behavior in known to occur for stacked spheres [TH97]. Moreover, we conjecture that the graded Betti numbers in the linear strand of their resolution provide upper bounds for the ones of any balanced normal pseudomanifold.

As a service to the reader, in particular to help comparing the different bounds, we use the same example to illustrate the predicted upper bounds: Namely, the toy example is a 3-dimensional balanced simplicial complex on 12 vertices with each color class being of cardinality 3. All computations and experiments have been carried out with the help of the computer algebra system Macaulay2 [GS].

5.1.1 Minimal graded free resolutions

Let $S = \mathbb{F}[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over an arbitrary field $\mathbb{F}$ and let $\mathfrak{m}$ be its maximal homogeneous ideal, i.e., $\mathfrak{m} = (x_1, \ldots, x_n)$. Denote with $\mathrm{Mon}_i(S)$ the set of monomials of degree i in S, and for $u \in \mathrm{Mon}_i(S)$ and a term order $<$, we let $\mathrm{Mon}_i(S)_{<u}$ be the set of monomials of degree i that are smaller than u with respect to $<$. Due to (1.3), the study of the Hilbert function of Stanley-Reisner rings is crucial for many of the results in face enumeration presented in the first two chapters of this thesis. A finer invariant can be obtained in the following way: consider a sequence of free finitely generated S-modules

$$F_\bullet : 0 \to F_p \xrightarrow{\varphi_p} \ldots \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} \ldots \xrightarrow{\varphi_1} F_0 \longrightarrow 0 \tag{5.1}$$

that is exact in all degrees but the 0th, with $\mathrm{coker}(\varphi_1) = R$. Such a sequence is called a *free resolution* of R. If we further assume that $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for every i, then $F_\bullet$ is called a *minimal free resolution*. If $S = \mathbb{F}[x_1, \ldots, x_n]$, then a free resolution exists for every finitely generated S-module and the minimal one is unique up to isomorphism of chain complexes. When $F_\bullet$ is a minimal free resolution of R the rank of F_i is called the *i-th Betti number of* R, denoted by $\beta_i^S(R)$ (we often drop the superscript in the notation, if there is no ambiguity between multiple module structures). Hence $F_i \cong S^{\beta_i(R)}$. If we apply $\_ \otimes_S \mathbb{F}$ to $F_\bullet$ we obtain a sequence of $\mathbb{F}$-vector spaces whose (finite) dimensions are still given by Betti numbers, but all the maps between them are zero. The homology of $F_\bullet \otimes_S \mathbb{F}$, which is a well studied functor in homological algebra, is then given by $\mathrm{Tor}_i^S(R, \mathbb{F}) := H_i(F_\bullet \otimes_S \mathbb{F}) \cong \mathbb{F}^{\beta_i(R)}$, which allows us to read the Betti numbers as $\beta_i^S(R) = \dim_{\mathbb{F}} \mathrm{Tor}_i^S(R, \mathbb{F})$. If, as in the rest of this chapter, the S-module R is a graded $\mathbb{F}$-algebra then we can impose that the maps in (5.1) are degree preserving by simply shifting the source module. More precisely if $M = \oplus_{j \geq 0} M_j$ is a graded S-module, we write $M(-j)$ for the graded module $M(-j)_k := M_{k+j}$. A sequence as in (5.1) in which all $F_i = \bigoplus_j S(-j)^{\beta_{i,i+j}}$ are graded and the maps are degree preserving is called a *minimal graded free resolution* of R. We define the *graded Betti numbers* of R to be the numbers

$$\beta^S_{i,i+j}(R) := \dim_{\mathbb{F}} H_i(F_\bullet \otimes_S \mathbb{F})_{i+j} = \dim_{\mathbb{F}} \operatorname{Tor}^S_i(R, \mathbb{F})_{i+j}.$$

Many important invariants in commutative algebra and algebraic geometry are encoded in these numbers. As an example the *Castelnuovo-Mumford* regularity of a graded S-module R is $\operatorname{reg}(R) := \max\{j : \beta^S_{i,i+j} > 0\}$. We refer to any commutative algebra book (e.g., [BH93]) for further properties of the graded minimal free resolution of a graded S-module.

Remark 5.1. The reason why graded Betti numbers are finer than the Hilbert function is that the Hilbert series can be computed from the graded minimal free resolution $F_\bullet$. Indeed consider the exact sequence $F_\bullet \to R \to 0$. Since the Hilber series is additive on exact sequences it implies that

$$\begin{aligned}
\operatorname{Hilb}(R,t) &= \sum_{i=0}^{p} (-1)^i \operatorname{Hilb}(F_i, t) \\
&= \sum_{i=0}^{p} (-1)^i \operatorname{Hilb}(\oplus_j S(-j)^{\beta_{i,i+j}}, t) \\
&= \sum_{i=0}^{p} (-1)^i \sum_j \beta^S_{i,i+j}(R) t^j \operatorname{Hilb}(S,t) \\
&= \frac{\sum_{i=0}^{p} (-1)^i \sum_j \beta^S_{i,i+j}(R) t^j}{(1-t)^n}
\end{aligned}$$

The next lemma will play a key role in the study of upper bounds for graded Betti numbers.

Lemma 5.2. *Let $R = S/I$ with I an homogeneous ideal and $\theta \in S_1$.*

i. [MN03, Corollary 8.5] *If the multiplication map $\times\theta : R_k \longrightarrow R_{k+1}$ is injective for every $k \le j$, then*

$$\beta^S_{i,i+k}(R) \le \beta^{S/\theta S}_{i,i+k}(R/\theta R),$$

for every $i \ge 0$ and $k \le j$.

ii. [BH93, Proposition 1.1.5] *If θ is not a zero divisor of R, then*

$$\beta^S_{i,i+j}(R) = \beta^{S/\theta S}_{i,i+j}(R/\theta R),$$

for every $i, j \ge 0$.

We report a short proof of the second statement.

Proof of ii. Since θ is not a zero divisor of R we have the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot\theta} R \longrightarrow R/\theta R \longrightarrow 0 \tag{5.2}$$

If $F_\bullet$ is a graded minimal free resolution of R we claim that $F_\bullet \otimes_S S/\theta S$ is a graded minimal free resolution of $R/\theta R$. We have that $\operatorname{coker}(F_1 \otimes_S S/\theta S \to F_0 \otimes_S S/\theta S) = R/\theta R$, so it suffices to show that $F_\bullet \otimes_S S/\theta S$ is exact. The homology of a free resolution of R tensored with $S/\theta S$ is $\operatorname{Tor}^S_\bullet(R, S/\theta S)$, which can be also computed as the homology of any free resolution of $S/\theta S$ tensored with R. We observe that $0 \to S \xrightarrow{\cdot\theta} S \to 0$ is indeed a free resolution of $S/\theta S$, and tensoring with R we obtain precisely (5.2), which is exact by the hypothesis. □

From Lemma 5.2 it immediately follows that modding out a regular sequence does not affect the graded Betti numbers.

5.1.2 Lex ideals

In order to show upper bounds for the graded Betti numbers we will make use of lexicographic ideals. As above, we let $S = \mathbb{F}[x_1, \ldots, x_n]$. Given a monomial ideal $I \subseteq S$ we denote by $G(I)$ its unique minimal set of monomial generators and we use $G(I)_j$ to denote those monomials in $G(I)$ of degree j. Let $>_{\text{lex}}$ be the *lexicographic order* on S with $x_1 >_{\text{lex}} \ldots, >_{\text{lex}} x_n$, i.e., we have $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} >_{\text{lex}} x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ if the leftmost non-zero entry of $(a_1 - b_1, \ldots, a_n - b_n)$ is positive. A monomial ideal $L \subseteq S$ is called a *lexicographic ideal* (or *lex ideal* for short) if for any monomials $u \in L$ and $v \in S$ of the same degree, with $v >_{\text{lex}} u$ it follows that $v \in L$. Macaulay [Mac27] showed that for any graded homogeneous ideal $I \subseteq S$ there exists a unique lex ideal, denoted by I^{lex}, such that S/I and S/I^{lex} have the same Hilbert function. In particular, the $\mathbb{F}$-vector space $I^{\text{lex}} \cap S_i$ is spanned by the first $\dim_{\mathbb{F}} S_i - \dim_{\mathbb{F}}(S/I)_i$ largest monomials of degree i in S. Note that the correspondence between I and I^{lex} is far from being one to one, since I^{lex} only depends on the Hilbert function of I. We conclude this section with two fundamental results on the graded Betti numbers of lex ideals.

Lemma 5.3. [Big93; Hul93; Par96] *For any homogeneous ideal $I \subseteq S$ it holds that*

$$\beta^S_{i,i+j}(S/I) \leq \beta^S_{i,i+j}(S/I^{\text{lex}}),$$

for all $i, j \geq 0$.

Lemma 5.3, due to Bigatti and Hulett (for the case $\text{char}(\mathbb{F}) = 0$) and to Pardue ($\text{char}(\mathbb{F}) = p$), states that among all graded rings with the same Hilbert functions, the quotient with respect to the lex ideal maximizes all graded Betti number simultaneously. Another peculiar property of lex ideals is that their graded Betti numbers are determined just by the combinatorics of their minimal generating set $G(I^{\text{lex}})$. For a monomial $u \in S$ denote by $\max(u) := \max\{i \ : \ x_i | u\}$.

Lemma 5.4. [EK90] *Let $I^{\text{lex}} \subseteq S$ be a lexicographic ideal. Then*

$$\beta^S_{i,i+j}(S/I^{\text{lex}}) = \sum_{u \in G(I^{\text{lex}}) \cap S_{j+1}} \binom{\max(u) - 1}{i - 1},$$

for all $i \geq 1$, $j \geq 0$.

For the case of Stanley-Reisner rings the graded Betti numbers can be computed by means of the simplcial homology of induced subcomplexes. Indeed an unpublished result by Hochster establishes the following brilliant formula.

Lemma 5.5 (Hochster's formula)**.**

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) = \sum_{\substack{W \subseteq V(\Delta) \\ |W| = i+j}} \dim_{\mathbb{F}} \widetilde{H}_{j-1}(\Delta_W; \mathbb{F}).$$

Hochster formulated also a similar result for the dimension of local cohomology modules supported on $\mathfrak{m}$, which was used to derive Theorem 1.31. To conclude this section we mention a result of Migliore and Nagel which uses lex ideals to obtain sharp upper bounds for the Betti numbers of polytopal spheres.

Theorem 5.6. [MN03] *Let ∂P be a polytopal $(d-1)$-sphere on n vertices. Let L be the lex ideal of $R' := \mathbb{F}[x_1, \dots, x_{n-d-1}]$ with $\dim_{\mathbb{F}}(R'/L)_i = g_i(\partial P)$ for every $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$. Then*

$$\beta_{i,i+j}(\mathbb{F}[\partial P]) \leq \begin{cases} \beta^{R'}_{i,i+j}(R'/L) & j < \frac{d}{2} \\ \beta^{R'}_{i,i+j}(R'/L) + \beta^{R'}_{n-d-i,n-i-j}(R'/L) & j = \frac{d}{2} \\ \beta^{R'}_{n-d-i,n-i-j}(R'/L) & j > \frac{d}{2} \end{cases}. \tag{5.3}$$

5.2 General balanced simplicial complexes

In the following, we consider arbitrary balanced simplicial complexes without assuming any further algebraic or combinatorial properties. Our aim is to prove explicit upper bounds for the graded Betti numbers of the Stanley-Reisner rings of those simplicial complexes. This will be achieved by exhibiting (non-balanced) simplicial complexes (one for each strand in the linear resolution), whose graded Betti numbers are larger than those of all balanced complexes on a fixed vertex partition.

We first need to introduce some notation. Recall that the *clique complex* of a graph $G = (V, E)$ on vertex set V and edge set E is the simplicial complex $\Delta(G)$ on vertex set V, whose faces correspond to cliques of G, i.e.,

$$\Delta(G) := \{F \subseteq V \ : \ \{i,j\} \in E \text{ for all } \{i,j\} \subseteq F \text{ with } i \neq j\}.$$

Let Δ be a $(d-1)$-dimensional balanced simplicial complex with vertex partition $V(\Delta) = \dot\bigcup_{i=1}^d V_i$. Let $n_i := |V_i|$ denote the sizes of the color classes of $V(\Delta)$. Throughout this section, we denote by $K_{n_1,\dots,n_d}$ the complete d-partite graph on vertex set $\dot\bigcup_{i=1}^d V_i$. Note that the 1-skeleton of Δ, considered as a graph, is clearly a subgraph of $K_{n_1,\dots,n_d}$ and we have by definition that $\Delta \subseteq \Delta(K_{n_1,\dots,n_d})$.

We can now state our first bound, though not yet in an explicit form.

Theorem 5.7. *Let Δ be a $(d-1)$-dimensional balanced simplicial complex on $V = \dot\bigcup\limits_{i=1}^{d} V_i$ with $n_i := |V_i|$. Then*

$$\beta_{i,i+j}\left(\mathbb{F}\left[\Delta\right]\right) \leq \beta_{i,i+j}\left(\mathbb{F}\left[\mathrm{Skel}_{j-1}\left(\Delta(K_{n_1,\dots,n_d})\right)\right]\right)$$

for every $i, j \geq 0$.

Proof. The proof relies on Hochster's formula. We fix $j \geq 0$. To simplify notation we set $\Sigma = \mathrm{Skel}_{j-1}(\Delta(K_{n_1,\dots,n_d}))$. Given a simplicial complex Γ, we denote by $(C_\bullet(\Gamma), \partial_j^\Gamma)$ the chain complex which computes its simplicial homology over $\mathbb{F}$.
Let $W \subseteq V$. As $\dim \Sigma = j-1$, we have $\dim\left(\Sigma_W\right) \leq j-1$ and hence $C_j\left(\Sigma_W\right) = 0$. This implies

$$\widetilde{H}_{j-1}\left(\Sigma_W; \mathbb{F}\right) = \ker \partial_{j-1}^{\Sigma_W}. \tag{5.4}$$

As $\Delta(K_{n_1,\dots,n_d})$ is the "maximal" balanced simplicial complex with vertex partition $\dot\bigcup_{i=1}^d V_i$, it follows that $\mathrm{Skel}_{j-1}(\Delta) \subseteq \Sigma$ and thus $C_{j-1}\left(\Delta_W\right) \subseteq C_{j-1}\left(\Sigma_W\right)$. In particular, we conclude

$$\ker \partial_{j-1}^{\Delta_W} \subseteq \ker \partial_{j-1}^{\Sigma_W}$$

and, using (5.4), we obtain

$$\dim_{\mathbb{F}} \widetilde{H}_{j-1}(\Delta_W;\mathbb{F}) \leq \dim_{\mathbb{F}} \widetilde{H}_{j-1}(\Sigma_W;\mathbb{F}).$$

The claim follows from Hochster's formula (Lemma 5.5). ☐

We now provide a specific example of the bounds in Theorem 5.7.

Example 5.8. The graded Betti numbers of any 3-dimensional balanced simplicial complex on 12 vertices with 3 vertices in each color class can be bounded by the graded Betti numbers of the skeleta of $\Gamma := \Delta(K_{3,3,3,3})$. More precisely, we can bound $\beta_{i,i+j}(\mathbb{F}[\Delta])$ by the corresponding Betti number of the $(j-1)$-skeleton of Γ. We record those numbers in the following table, in which the entry (i, j) is $\beta_{i,i+j}(\mathbb{F}[\mathrm{Skel}_{j-1}(\Gamma)])$.

$j \setminus i$	0	1	2	3	4	5	6	7	8	9	10	11
1	0	66	440	1485	3168	4620	4752	3465	1760	594	120	11
2	0	108	945	3312	6720	8856	7875	4720	1836	420	43	0
3	0	81	648	2376	4752	5733	4352	2052	552	65	0	0
4	0	0	0	0	81	216	216	96	16	0	0	0

TABLE 5.1: The numbers $\beta_{i,i+j}(\mathbb{F}[\mathrm{Skel}_{j-1}(\Gamma)])$, for $\Gamma = \Delta(K_{3,3,3,3})$.

Remark 5.9. Observe that the $(j-1)$-skeleton of the clique complex $\Delta(K_{n_1,\dots,n_d})$ is balanced if and only if $j = d$ (or, less interestingly, if $j = 1$). It follows that the upper bounds for the graded Betti numbers of a $(d-1)$-dimensional balanced simplicial complex, given in Theorem 5.7, are attained for the d-th (and trivially, the 0-th) row of the Betti table. However, they are not necessarily sharp for the other rows of the Betti table and we do not expect them to be so.

In order to turn the upper bounds from Theorem 5.7 into explicit ones, we devote the rest of this section to the computation of the graded Betti numbers of the skeleta of $\Delta(K_{n_1,\dots,n_d})$. We first consider $\Delta(K_{n_1,\dots,n_d})$. As a preparation we determine the homology of induced subcomplexes of $\Delta(K_{n_1,\dots,n_d})$.

Lemma 5.10. *Let $\Gamma = \Delta(K_{n_1,\dots,n_d})$ with vertex partition $V := \dot{\bigcup}_{i=1}^{d} V_i$. For $W \subseteq V$, set $W_i := W \cap V_i$, for $1 \leq i \leq d$ and $\{i_1, \dots, i_k\} := \{i \,:\, W_i \neq \emptyset\}$. Then*

$$\widetilde{H}_{j-1}(\Gamma_W;\mathbb{F}) = \begin{cases} \mathbb{F}^{|W_{i_1}|-1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}^{|W_{i_k}|-1}, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases}.$$

In particular, $\widetilde{H}_{j-1}(\Gamma_W;\mathbb{F}) \neq 0$ if and only if $k = j$ and $|W_{i_\ell}| \geq 2$ for $1 \leq \ell \leq k$.

Proof. Denoting by $\langle V_i \rangle$ the simplicial complex consisting of the isolated vertices in V_i, we can write Γ as the join of those $\langle V_i \rangle$:

$$\Gamma = \langle V_1 \rangle * \cdots * \langle V_d \rangle. \tag{5.5}$$

In particular, we have

$$\Gamma_W = \langle W_{i_1} \rangle * \cdots * \langle W_{i_k} \rangle.$$

Using the Künneth formula in Theorem 1.19 and the fact that

$$\widetilde{H}_j\left(\langle W_i\rangle;\mathbb{F}\right)=\begin{cases}\mathbb{F}^{|W_i|-1} & \text{, if } j=0\\ 0 & \text{, if } j\neq 0,\end{cases}$$

we deduce the desired formula for the homology. The "In particular"-part follows directly from this formula. □

Remark 5.11. Since Cohen-Macaulayness is preserved under taking joins and since every 0-dimensional simplicial complex is Cohen-Macaulay, it follows directly from (5.5) that the clique complex $\Delta(K_{n_1,\ldots,n_d})$ is a Cohen-Macaulay complex. Accordingly, the same is true for the skeleta of $\Delta(K_{n_1,\ldots,n_d})$.

Lemma 5.10 enables us to compute the graded Betti numbers of $\Delta(K_{n_1,\ldots,n_d})$.

Lemma 5.12. *Let $d,n_1,\ldots,n_d$ be positive integers. Then*

$$\beta_{i,i+j}\left(\mathbb{F}\left[\Delta(K_{n_1,\ldots,n_d})\right]\right)=\sum_{\substack{I\subseteq[d]\\ I=\{i_1,\ldots,i_j\}}}\left(\sum_{\substack{c_1+\cdots+c_j=i\\ c_\ell\geq 1,\forall\ell\in[1,j]}}\left(\prod_{\ell=1}^{j}c_\ell\cdot\binom{n_{i_\ell}}{c_\ell-1}\right)\right)\tag{5.6}$$

for $i,j\geq 0$. In particular, if $n_1=\cdots=n_d=k$, then

$$\beta_{i,i+j}\left(\mathbb{F}\left[\Delta(K_{k,\ldots,k})\right]\right)=\binom{d}{j}\left(\sum_{\substack{c_1+\cdots+c_j=i\\ c_\ell\geq 1,\forall\ell\in[1,j]}}\left(\prod_{\ell=1}^{j}c_\ell\cdot\binom{k}{c_\ell-1}\right)\right)$$

for $i,j\geq 0$.

Proof. We prove the statement by a direct application of Hochster's formula. Fix $i,j\geq 0$. By Lemma 5.10 and Lemma 5.5, to compute $\beta_{i,i+j}(\Delta(K_{n_1,\ldots,n_d}))$, we need to count subsets $W\subseteq\bigcup_{\ell=1}^{d}V_i$ such that $|\{\ell\ :\ W\cap V_\ell\neq\emptyset\}|=j$ and $|W\cap V_\ell|\neq 1$ for $1\leq\ell\leq d$. To construct such a set, we proceed as follows:

- We first choose $i_1<\cdots<i_j$ such that $W\cap V_{i_\ell}\neq\emptyset$ for $1\leq\ell\leq j$.
- Next, for each i_ℓ we pick an integer $c_t\geq 2$, with the property that $c_1+\cdots+c_j=i+j$.
- Finally, there are $\binom{n_{i_\ell}}{c_\ell}$ ways to choose c_ℓ vertices among the n_{i_ℓ} vertices of V_{i_ℓ}.

By Lemma 5.10 the dimension of the $(j-1)$-st homology of such a subset W equals $\prod_{\ell=1}^{j}(c_\ell-1)$. Combining the previous argument, we deduce the required formula (5.6). The second statement now is immediate. □

We illustrate (5.6) with an example.

Example 5.13. Consider the clique complex $\Delta(K_{3,3,2})$ of $K_{3,3,2}$. To compute $\beta_{3,5}(\mathbb{F}[\Delta(K_{3,3,2})])$, we need to consider the 2-element subsets of $[3]$.

For the set $\{1,2\}$ the inner sum in (5.6) equals

$$\sum_{\substack{c_1+c_2=3\\ c_1,c_2\geq 1}}c_1\cdot c_2\cdot\binom{3}{c_1-1}\cdot\binom{3}{c_2-1}=12,$$

since the sum has two summands (corresponding to $(c_1, c_2) \in \{(1,2),(2,1)\}$), each contributing with 6.
Similarly, for $\{1,3\}$ and $\{2,3\}$, we obtain 2 for the value of the inner sum. In total, this yields:

$$\beta_{3,5}(\mathbb{F}[\Delta]) = 12 + 2 + 2 = 16.$$

We now turn our attention to the computation of the graded Betti numbers of the skeleta of $\Delta(K_{n_1,\ldots,n_d})$. The following result, which is a special case of [RV15, Theorem 3.1] by Roksvold and Verdure, is crucial for this aim.

Lemma 5.14. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex with $f_0(\Delta) = n$. Then the number $\beta_{i,i+j}(\mathbb{F}[\mathrm{Skel}_{d-2}(\Delta)])$ equals*

$$\begin{cases} \beta_{i,i+j}(\mathbb{F}[\Delta]), & \text{if } j < d-1 \\ \beta_{i,i+d-1}(\mathbb{F}[\Delta]) - \beta_{i-1,i+d-1}(\mathbb{F}[\Delta]) + \binom{n-d}{i-1} f_{d-1}(\Delta), & \text{if } j = d-1 \\ 0, & \text{if } j \geq d \end{cases}$$

for $0 \leq i \leq n-d+1$.

Applying Lemma 5.14 iteratively, we obtain the following recursive formula for the graded Betti numbers of general skeleta of a Cohen-Macaulay complex:

Corollary 5.15. *Let s be a positive integer and let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex with $f_0(\Delta) = n$. Then the number $\beta_{i,i+j}(\mathbb{F}[\mathrm{Skel}_{d-s-1}(\Delta)])$ equals*

$$\begin{cases} \beta_{i,i+j}(\mathbb{F}[\Delta]), & \text{if } j < d-s \\ \sum_{k=0}^{s}(-1)^k \beta_{i-k,i+d-s}(\mathbb{F}[\Delta]) + \sum_{t=0}^{s-1}(-1)^{t-s+1}\binom{n-d+t}{i-s+t} f_{d-t-1}(\Delta), & \text{if } j = d-s \\ 0, & \text{if } j \geq d-s+1 \end{cases}$$

for $0 \leq i \leq n-d+s$.

Since the clique complex $\Delta(K_{n_1,\ldots,n_d})$ is Cohen-Macaulay (see Remark 5.11), we can use Corollary 5.15 to compute the graded Betti numbers of its skeleta. Combining this with Theorem 5.7, we obtain the following bounds for the graded Betti numbers of an arbitrary balanced simplicial complex.

Corollary 5.16. *Let Δ be a $(d-1)$-dimensional balanced simplicial complex on vertex set $V = \bigcupdot_{i=1}^{d} V_i$, with $n := |V|$ and $n_i := |V_i|$. Let $\Gamma = \Delta(K_{n_1,\ldots,n_d})$. Then*

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) \leq \sum_{k=0}^{d-j}(-1)^k \beta_{i-k,i+j}(\mathbb{F}[\Gamma]) + \sum_{t=0}^{d-j-1}(-1)^{t-d+j+1}\binom{n-d+t}{i-d+j+t} f_{d-t-1}(\Gamma).$$

Note that the graded Betti numbers of $\Gamma := \Delta(K_{n_1,\ldots,n_d})$ are given in Lemma 5.12 and that the f-vector of Γ is given by

$$f_i(\Gamma) = \sum_{I \subseteq [d], |I| = i+1} \prod_{\ell \in I} n_\ell$$

for $0 \leq i \leq d-1$. Therefore, the previous corollary really provides explicit bounds for the graded Betti numbers of a balanced simplicial complex.

5.3 A first bound in the Cohen-Macaulay case

As in the previous sections we let $S = \mathbb{F}[x_1, \dots, x_n]$ denote the polynomial ring in n variables over an arbitrary field $\mathbb{F}$. The ultimate aim of this section is to show upper bounds for the graded Betti numbers of the Stanley-Reisner rings of balanced Cohen-Macaulay complexes. On the way, more generally, we will prove upper bounds for the graded Betti numbers of Artinian quotients S/I, where $I \subseteq S$ is a homogeneous ideal having *many* generators in degree 2.

5.3.1 Ideals with many generators in degree 2

Throughout this section, we let $I \subsetneq S$ be a homogeneous ideal that has no generators in degree 1, i.e., $I \subseteq \mathfrak{m}^2$.

First assume that S/I is of dimension 0. It is well known and essentially follows from Lemma 5.3 by passing to the lex ideal I^{lex}, that we can bound $\beta_{i,i+j}(S/I)$ by the corresponding Betti number $\beta_{i,i+j}(S/\mathfrak{m}^{j+1})$ of the quotient of S with the $(j+1)$-st power of the maximal homogeneous ideal $\mathfrak{m} \subseteq S$. Lemma 5.4 then yields

$$\beta_{i,i+j}(S/I) \leq \binom{i-1+j}{j}\binom{n+j}{i+j}$$

for all $i \geq 1$, $j \geq 0$. Moreover, if S/I is Cohen-Macaulay of dimension d, then, by modding out a linear system of parameters $\Theta \subseteq S$ (which is a regular sequence by assumption) and using Lemma 5.2, we can reduce to the 0-dimensional case, which yields the well known upper bound (see e.g., [Mur15, Lemma 3.4 (i)]):

$$\beta_{i,i+j}(S/I) \leq \binom{i-1+j}{j}\binom{n-d+j}{i+j},$$

for all $i \geq 1$, $j \geq 0$. In particular, those bounds apply to Stanley-Reisner rings of Cohen-Macaulay complexes. Moreover, if equality holds in the j-th strand, then I has $(j+1)$-linear resolution (see e.g., [HH11] for the precise definition).

In the following, assume that S/I is Artinian and that there exists a positive integer b such that

$$\dim_{\mathbb{F}}(S/I)_2 \leq \binom{n+1}{2} - b.$$

In other words, I has at least b generators in degree 2. Our goal is to prove upper bounds for $\beta_{i,i+j}(S/I)$ in this setting. This will be achieved using similar arguments as the ones we just recalled that are used in the general setting. First, we need some preparations.

As, by assumption, I does not contain polynomials of degree 1, neither does its lex ideal $I^{\mathrm{lex}} \subseteq S$. In particular, we have

$$|G(I^{\mathrm{lex}}) \cap S_2| \geq b$$

and I^{lex} contains at least the b *largest* monomials of degree 2 in lexicographic order. The next lemma describes this set of monomials explicitly.

Lemma 5.17. *Let $n \in \mathbb{N}$ be a positive integer and let $b < \binom{n+1}{2}$. Let $x_p x_q$ be the b-th largest monomial in the lexicographic order of degree 2 monomials in variables*

$x_1, \dots, x_n$ and assume $p \le q$. Then:

$$p = n - \left\lfloor \frac{\sqrt{-8b + 4n(n+1) + 1}}{2} - \frac{1}{2} \right\rfloor,$$

and

$$q = b + \frac{(p-1)(p-2n)}{2}.$$

Proof. Let M be the $n \times n$ upper triangular matrix obtained by listing the degree 2 monomials in variables $x_1, \dots, x_n$ in decreasing lexicographic order from left to right and top to bottom:

$$M = \begin{bmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n \\ 0 & x_2^2 & \dots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n^2 \end{bmatrix}.$$

From this ordering, it is easily seen, that, if x_px_q (with $p < q$) is the b-th largest degree 2 monomial in lexicographic order, then

$$n - p = \max\{s \in \mathbb{N} \ : \ \sum_{\ell=1}^{s} \ell \le \binom{n+1}{2} - b\}.$$

As $s = -\frac{1}{2} + \frac{\sqrt{4n(n+1)+1-8b}}{2}$ is the unique non-negative solution to the equation

$$(s+1)s/2 = (n+1)n/2 - b,$$

we conclude that

$$p = n - \left\lfloor -\frac{1}{2} + \frac{\sqrt{4n(n+1)+1-8b}}{2} \right\rfloor.$$

Looking at the matrix M, we deduce that the index q, (i.e., the column index of x_px_q in M) is given by

$$\begin{aligned} q =& b - \sum_{\ell=1}^{p-1}(n+1-\ell) + (p-1) \\ =& b - (p-1)(n+1) + \frac{p(p-1)}{2} + (p-1) \\ =& b + \frac{(p-1)(-2-2n+p+2)}{2} = b + \frac{(p-1)(p-2n)}{2}. \end{aligned}$$

The claim follows. □

Intuitively, if a lex ideal $J \subseteq S$ has *many* generators in degree 2, then there can only exist relatively few generators of higher degree. More precisely, the next lemma provides a necessary condition for a monomial u to lie in $G(J)_j$ for $j > 2$ and thus enables us to bound the number of generators of J of degree j.

Lemma 5.18. *Let $j > 2$ be an integer and let $J \subseteq S$ be a lex ideal. Let x_px_q be the lexicographically smallest monomial of degree 2 that is contained in J. If $u \in G(J)_j$ is a minimal generator of J of degree j, then $u <_{\mathrm{lex}} x_px_qx_n^{j-2}$. In other words:*

$$G(J)_j \subseteq \mathrm{Mon}_j(S)_{<_{\mathrm{lex}} x_px_qx_n^{j-2}}.$$

Proof. To simplify notation we set $w = x_p x_q x_n^{j-2}$. First note that any monomial of degree j that is divisible by $x_p x_q \in G(J)$ cannot be a minimal generator of J. Let u be a monomial of degree j with $u >_{\text{lex}} w$, that is not divisible by $x_p x_q$. Then, there exists $\ell < p$ such that x_ℓ divides u or u is divisible by x_p and there exists $p \leq r < q-1$ such that $x_p x_r$ divides u. In the first case, let x_r such that $x_\ell x_r$ divides u. Then, $x_\ell x_r >_{\text{lex}} x_p x_q$ and hence $x_\ell x_r \in J$, since J is a lex ideal. This implies $u \notin G(J)$. Similarly, in the second case, we have $x_p x_r >_{\text{lex}} x_p x_q \in G(J)$ and hence $u \notin G(J)$. The claim follows. □

Recall that a homogeneous ideal $I \subseteq S$, which is generated in degree d, is called *Gotzmann ideal* if the number of generators of $\mathfrak{m}I$ is smallest possible. More generally, a graded ideal $I \subseteq S$ is called *Gotzmann ideal* if all components $I_{\langle j \rangle}$ are Gotzmann ideals. Here, $I_{\langle j \rangle}$ denotes the ideal generated by all the elements in I of degree j. By Gotzmann's persistence theorem [Got78], a graded ideal $I \subseteq S$ is Gotzmann if and only if I and $(I^{\text{lex}})_{\langle d \rangle}$ have the same Hilbert function. Moreover, as shown in [HH99, Corollary 1.4], this is equivalent to S/I and S/I^{lex} having the same graded Betti numbers, i.e.,

$$\beta_{i,i+j}(S/I) = \beta_{i,i+j}(S/I^{\text{lex}}) \tag{5.7}$$

for all $i, j \geq 0$. We state an easy lemma, which well be helpful to prove the main result of this section.

Lemma 5.19. *Let $j \geq d$ be positive integers and let $J \subseteq S$ be a Gotzmann ideal that is generated in degree d. Let $I = J + \mathfrak{m}^{j+1}$. Then*

$$\beta_{i,i+\ell}(S/I) = \beta_{i,i+\ell}(S/I^{\text{lex}}) \tag{5.8}$$

for all $i, \ell \geq 0$.

Proof. We first note that, as J is Gotzmann, so are its graded components $I_{\langle j \rangle}$. Moreover, as any power of $\mathfrak{m}$ is Gotzmann, it follows from the definition of a Gotzmann ideal that I has to be Gotzmann as well. The claim now follows from [HH99, Corollary 1.4]. □

We can now state the main result of this section.

Theorem 5.20. *Let $I \subseteq S$ be a homogeneous ideal, that does not contain linear forms. Let $\dim_{\mathbb{F}}(S/I)_2 \leq \binom{n+1}{2} - b$ for some positive integer b. Let $x_p x_q$, where $p \leq q$, be the b-th largest monomial of degree 2 in lexicographic order on S. Then*

$$\beta_{i,i+j}(S/I) \leq \sum_{\ell=p+1}^{n} \binom{\ell - p + j - 1}{j} \binom{\ell - 1}{i - 1} + \sum_{\ell=q+1}^{n} \binom{\ell - q + j - 2}{j - 1} \binom{\ell - 1}{i - 1}, \tag{5.9}$$

for any $i \geq 0$ and $j \geq 2$. Moreover if $I = J + \mathfrak{m}^{j+1}$, where $J \subseteq S$ is a Gotzmann ideal that is generated by b elements of degree 2, then equality is attained for a fixed $j \geq 2$ and all $i \geq 0$.

Proof. We fix $j \geq 2$ and we set $w := x_p x_q x_n^{j-1}$. By Lemma 5.3 we can use the graded Betti numbers of the lex ideal $I^{\text{lex}} \subseteq S$ of I to bound the ones of I. Using Lemma 5.4

we infer

$$\begin{aligned}
\beta_{i,i+j}(S/I) &\leq \beta_{i,i+j}(S/I^{\mathrm{lex}}) \\
&= \sum_{u \in G(I^{\mathrm{lex}})_{j+1}} \binom{\max(u)-1}{i-1} \\
&\overset{(\text{Lemma 5.18})}{\leq} \sum_{u \in \mathrm{Mon}_{j+1}(S)_{<w}} \binom{\max(u)-1}{i-1} \\
&= \sum_{\substack{u \in \mathrm{Mon}_{j+1}(S)_{<w} \\ x_p \mid u}} \binom{\max(u)-1}{i-1} + \sum_{\substack{u \in \mathrm{Mon}_{j+1}(S)_{<w} \\ x_p \nmid u}} \binom{\max(u)-1}{i-1}.
\end{aligned}$$

Let u be a monomial of degree $j+1$, such that $u <_{\mathrm{lex}} w$. If $x_p|u$, then $\max(u) \geq q+1$ and u is of the form $x_p x_{\max(u)} \cdot v$, where v is a monomial in $\mathbb{F}[x_{q+1}, \ldots, x_{\max(u)}]$ of degree $j-1$. In particular, there are $\binom{(\ell-q)+(j-1)-1}{j-1}$ many such monomials with $\max(u) = \ell$. Similarly, if u is not divisible by x_p, then $\max(u) \geq p+1$ and u is of the form $x_{\max(u)} \cdot v$, where v is a monomial of degree j in $\mathbb{F}[x_{p+1}, \ldots, x_{\max(u)}]$. There are $\binom{(\ell-p)+j-1}{j}$ many such monomials with $\max(u) = \ell$. The desired inequality follows.

For the equality case first note that if $I = J + \mathfrak{m}^{j+1}$, where J is a Gotzmann ideal generated in degree d, then it follows from Lemma 5.19 that $\beta_{i,i+j}(S/I) = \beta_{i,i+j}(S/I^{\mathrm{lex}})$ for all i. Moreover, as $I^{\mathrm{lex}} = \mathrm{Lex}(b) + \mathfrak{m}^{j+1}$, where $\mathrm{Lex}(b)$ denotes the lex ideal generated by the b lexicographically largest monomials of degree 2, the lex ideal I^{lex} attains equality in (5.9).

□

Remark 5.21. It is worth remarking that if an ideal I attains equality in (5.9) for a fixed j, then the ideal J (where $I = J + \mathfrak{m}^{j+1}$ as above) is not necessarily a monomial ideal. E.g., for $n = 2$ and $b = 2$ the ideals

$$(x_1^2, x_1x_2) + (x_1, x_2)^3 \quad \text{and} \quad (x_1^2 + x_1x_2, x_2^2 + x_1x_2) + (x_1, x_2)^3$$

both maximize $\beta_{i,i+2}$ for any i. The maximal Betti numbers in this case are $\beta_{1,3} = \beta_{2,4} = 1$.

5.3.2 Application: Balanced Cohen-Macaulay complexes

The aim of this section is to use the results from the previous section in order to derive upper bounds for the graded Betti numbers of balanced Cohen-Macaulay complexes.

In the following, let Δ be a balanced Cohen-Macaulay simplicial complex and let $\Theta \subseteq \mathbb{F}[\Delta]$ be a linear system of parameters for $\mathbb{F}[\Delta]$. In order to apply Theorem 5.20 we need to bound the Hilbert function of the Artinian reduction $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$ in degree 2 from above. As Δ is Cohen-Macaulay, it follows from Lemma 1.25 that

$$\dim_{\mathbb{F}} \left(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]\right)_2 = h_2(\Delta).$$

We can therefore employ an upper bound for $h_2(\Delta)$.

Lemma 5.22. *Let Δ be a $(d-1)$-dimensional balanced simplicial complex with vertex partition $V(\Delta) = \dot{\bigcup}_{i=1}^{d} V_i$. Let $n := |V|$ and $n_i := |V_i|$. Then*

$$h_2(\Delta) \leq \binom{n-d+1}{2} - \sum_{i=1}^{d} \binom{n_i}{2}. \tag{5.10}$$

Proof. As Δ is balanced, it does not have monochromatic edges, i.e., we have $\{v, w\} \notin \Delta$, if v and w belong to the same color class V_i $(1 \leq i \leq d)$. As there are $\binom{n_i}{2}$ monochromatic non-edges of color i, this gives the following upper bound for $f_1(\Delta)$:

$$f_1(\Delta) \leq \binom{n}{2} - \sum_{i=1}^{d} \binom{n_i}{2}.$$

The claim now directly follows from the relation

$$h_2(\Delta) = \binom{d}{2} - (d-1) f_0(\Delta) + f_1(\Delta).$$

□

A direct application of Theorem 5.20 combined with Lemma 5.22 finally yields:

Theorem 5.23. *Let Δ be a $(d-1)$-dimensional balanced Cohen-Macaulay complex with vertex partition $V = \dot{\bigcup}_{i=1}^{d} V_i$. Let $n := |V|$, $n_i := |V_i|$ and $b := \sum_{i=1}^{d} \binom{n_i}{2}$. Let $x_p x_q$ be the b-th largest degree 2 monomial of $\mathbb{F}[x_1, \ldots, x_{n-d}]$ in lexicographic order with $p \leq q$. Then*

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) \leq \sum_{\ell=p+1}^{n-d} \binom{\ell-p+j-1}{j}\binom{\ell-1}{i-1} + \sum_{\ell=q+1}^{n-d} \binom{\ell-q+j-2}{j-1}\binom{\ell-1}{i-1},$$

for any $i \geq 0$ and $2 \leq j \leq d$.

The above statement is trivially true also for $j > d$. However, as the Castelnuovo-Mumford regularity of $\mathbb{F}[\Delta]$ is at most d, we know that $\beta_{i,i+j}(\mathbb{F}[\Delta]) = 0$ for any $i \geq 0$ and $j > d$.

Proof. Let $S = \mathbb{F}[x_1, \ldots, x_n]$. Let Θ be an l.s.o.p. for $\mathbb{F}[\Delta]$. It follows from Lemma 5.2 that

$$\beta^S_{i,i+j}(\mathbb{F}[\Delta]) = \beta^{S/\Theta S}_{i,i+j}(S/(I_\Delta + (\Theta))).$$

Moreover, $S/\Theta S \cong \mathbb{F}[x_1, \ldots, x_{n-d}] := R$ as rings and there exists a homogeneous ideal $J \subseteq R$ with $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta] \cong R/J$ and $\beta^R_{i,i+j}(R/J) = \beta^{S/\Theta S}_{i,i+j}(S/(I_\Delta + (\Theta)))$. In particular, as Δ is Cohen-Macaulay, $\dim_{\mathbb{F}}(R/J)_2 = h_2(\Delta)$ satisfies the bound from Lemma 5.22. As $h_1(\Delta) = \dim_{\mathbb{F}}(R/J)_1$, the ideal J does not contain any linear form and the result now follows from Theorem 5.20. □

Remark 5.24. Whereas we have seen that the bounds in Theorem 5.20 are tight, the ones in Theorem 5.23 are not. For example, consider the case that $n_1 = n_2 = 2$ and $d = 2$. In this situation, we have $b := \sum_{i=1}^{d} \binom{n_i}{2} = 2$ and $x_1 x_2$ is the second largest degree 2 monomial in the lexicographic order. Theorem 5.23 gives $\beta_{1,3} \leq 1$. However, by Hochster's formula, if Δ is a 1-dimensional simplicial complex with $\beta_{1,3}(\mathbb{F}[\Delta]) = 1$, then Δ must contain an induced 3-cycle. But this means that Δ cannot be balanced.

Example 5.25. Let Δ be a 3-dimensional balanced Cohen-Macaulay complex with 3 vertices in each color class, i.e., $n_i = 3$ for $1 \leq i \leq 4$. We have $b := \sum_{i=1}^{4} \binom{3}{2} = 12$ and x_2x_5 is the 12-th largest monomial of degree 2 in variables $x_1, \ldots, x_8$. The bounds from Theorem 5.23 are recorded in the following table: We set $S = \mathbb{F}[x_1, \ldots, x_8]$ and

$j \setminus i$	0	1	2	3	4	5	6	7	8
2	0	62	360	915	1317	1156	617	185	24
3	0	136	821	2155	3184	2855	1551	472	62
4	0	267	1653	4432	6665	6065	3336	1026	136

TABLE 5.2: The bound given in Theorem 5.23 for a 3-dimensional balanced Cohen-Macaulay complex with 3 vertices in each color class.

we let $I \subseteq S$ be the lex ideal generated by the 12 largest monomials of degree 2 in variables $x_1, \ldots, x_8$. It follows from Theorem 5.20 that $\beta_{i,i+j}(S/(I+\mathfrak{m}^{j+1}))$ equals the entry of the above table in the row, labeled i and the column, labeled j. Moreover, it is shown in the proof of Theorem 5.20 that $\beta_{i,i+\ell}(S/(I+\mathfrak{m}^{j+1})) = 0$ if $\ell \notin \{1, j\}$. One can easily compute that for any j the first row of the Betti table of $S/(I+\mathfrak{m}^{j+1})$ is given by

$j \setminus i$	0	1	2	3	4	5	6	7	8
1	0	12	38	66	75	57	28	8	1

Finally, we compare the bounds from the upper table with the numbers $\beta_{i,i+j}(S/\mathfrak{m}^j)$, for general 3-dimensional Cohen-Macaulay complexes on 12 vertices. Those are displayed in the next table:

$j \setminus i$	0	1	2	3	4	5	6	7	8
2	0	120	630	1512	2100	1800	945	280	36
3	0	330	1848	4620	6600	5775	3080	924	120
4	0	792	4620	11880	17325	15400	8316	2520	330

TABLE 5.3: The bound for the numbers $\beta_{i,i+j}$ for a 3-dimensional Cohen-Macaulay complex on 12 vertices.

We point out that while Theorem 5.23 provides bounds for $\beta_{i,i+j}(\mathbb{F}[\Delta])$ for all i and all $j \geq 2$, it does not give bounds for the graded Betti numbers of the linear strand (i.e., for $j = 1$). This seems a natural drawback of our approach, since our key ingredient is the concentration of monomials of degree 2 in the lex ideal of $I_\Delta + (\Theta)$ (cf., (5.10)). However, it follows from the next lemma, that there is no better bound in terms of the total number of vertices n and the dimension $d-1$ than in the standard (non-balanced) Cohen-Macaulay case. More precisely, for any n and any d we construct a balanced Cohen-Macaulay complex whose graded Betti numbers equal $\beta_{i,i+j}(S/\mathfrak{m}^j)$ for $j = 1$ and for every $i > 0$, where $S = \mathbb{F}[x_1, \ldots, x_{n-d}]$.

Lemma 5.26. *Let n and d be positive integers. Let Γ_{n-d+1} denote the simplicial complex consisting of the isolated vertices $1, 2, \ldots, n-d+1$ and let Δ_{d-2} be the $(d-2)$-simplex with vertices $\{n-d+2, \ldots, n\}$. Then $\Delta_{d-2} * \Gamma_{n-d+1}$ is a balanced $(d-1)$-dimensional Cohen-Macaulay complex. Moreover*

$$\beta_{i,i+1}(\mathbb{F}[\Delta_{d-2} * \Gamma_{n-d+1}]) = i\binom{n-d+1}{i+1} \quad \textit{for all } i.$$

Proof. We set $\Delta = \Delta_{d-2} * \Gamma_{n-d+1}$. As Δ is the join of a $(d-2)$-dimensional and a 0-dimensional Cohen-Macaulay complex, it is Cohen-Macaulay of dimension $d-1$. Moreover, coloring the vertices of Δ_{d-2} with the colors $1, \dots, d-1$ and assigning color d to all vertices of Γ_{n-d+1} gives a proper d-coloring of Δ, i.e., Δ is balanced.

By Hochster's formula (Lemma 5.5), the graded Betti numbers $\beta_{i,i+1}(\mathbb{F}[\Delta])$ are given by

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) = \sum_{W \subseteq [n]: |W| = i+1} \dim_{\mathbb{F}} \widetilde{H}_0(\Delta_W; \mathbb{F}). \tag{5.11}$$

As $\Delta_W = (\Delta_{d-2})_W * (\Gamma_{n-d+1})_W$, the induced complex Δ_W is connected whenever $W \cap \{n-d+2, \dots, n\} \neq \emptyset$. Hence the only non-trivial contributions to (5.11) come from $(i+1)$-element subsets of $[n-d+1]$. For such a subset W, the complex Δ_W consists of i connected components and since there are $\binom{n-d+1}{i+1}$ many such sets, the claim follows. $\square$

Though we have just seen that Betti numbers (in the linear strand) of balanced Cohen-Macaulay complexes can be as big as the ones for general Cohen-Macaulay complexes, it should also be noted that the simplicial complex $\Delta_{d-2} * \Gamma_{n-d+1}$ is special, in the sense that all but one "big" color classes are singletons. It is therefore natural to ask, if there are better bounds than those for the general Cohen-Macaulay situation, that take into account the size of the color classes.

5.4 A second bound in the Cohen-Macaulay case via lex-plus-squares ideals

The aim of this section is to provide further upper bounds for the graded Betti numbers of balanced Cohen-Macaulay complexes. On the one hand, those bounds will be a further improvement of the ones from Theorem 5.23. On the other hand, however, they are slightly more complicated to state. Our approach is similar to the one used in Theorem 5.23 with *lex-plus-squares* ideals as an additional ingredient. More precisely, we will prove upper bounds for the graded Betti numbers of Artinian quotients S/I, where $I \subseteq S$ is a homogeneous ideal having *many* generators in degree 2, including the squares of the variables $x_1^2, \dots, x_n^2$. The desired bound for balanced Cohen-Macaulay complexes is then merely an easy application of those more general results.

5.4.1 Ideals containing the squares $x_1^2, \dots, x_n^2$ with many degree 2 generators

We recall some necessary definitions and results. As in the previous sections, we let $S = \mathbb{F}[x_1, \dots, x_n]$. We further let $P := (x_1^2, \dots, x_n^2) \subseteq S$. A monomial ideal $L \subseteq S$ is called *squarefree lex ideal* if for every squarefree monomial $u \in L$ and every monomial $v \in S$ with $\deg(u) = \deg(v)$ and $v >_{\text{lex}} u$ it follows that $v \in L$. For homogeneous ideals containing the squares of the variables the following analog of Lemma 5.3 was shown by Mermin, Peeva and Stillman [MPS08] in characteristic 0 and by Mermin and Murai [MM11] in arbitrary characteristic:

Theorem 5.27. *Let $I \subseteq S = \mathbb{F}[x_1, \dots, x_n]$ be a homogeneous ideal containing P. Let $I^{\text{sqlex}} \subseteq S$ be the squarefree lex ideal such that I and $I^{\text{sqlex}} + P$ have the same Hilbert function. Then*

$$\beta^S_{i,i+j}(S/I) \leq \beta^S_{i,i+j}(S/(I^{\text{sqlex}} + P)), \tag{5.12}$$

for all $i, j \geq 0$.

The existence of a squarefree lex ideal I^{sqlex} as in the previous theorem is a straightforward consequence of the Clements-Lindström Theorem [CL69]. Moreover, Theorem 5.27 provides an instance for which the so-called *lex-plus-powers Conjecture* is known to be true (see [ER02], [Fra04], [FR07] for more details on this topic).

An ideal of the form $I^{\text{sqlex}} + P$ is called *lex-plus-squares* ideal. It was shown in [MPS08, Theorem 2.1 and Lemma 3.1(2)] that the graded Betti numbers of ideals of the form $I + P \subseteq S$, where $I \subseteq S$ is a squarefree monomial ideal can be computed via the Betti numbers of *smaller* squarefree monomial ideals, via iterated mapping cones. In the next result, we use $\binom{[n]}{k}$ to denote the set of k-element subsets of $[n]$.

Proposition 5.28. *Let $I \subseteq S$ be a squarefree monomial ideal. Then*

i.

$$\beta^S_{i,i+j}(S/(I+P)) = \sum_{k=0}^{j} \left(\sum_{F \in \binom{[n]}{k}} \beta^S_{i-k,i+j-2k}(S/(I : x_F)) \right),$$

where $x_F = \prod_{f \in F} x_f$.

ii. If I is squarefree lex, then the ideal $(I^{\text{sqlex}} : x_F)$ is a squarefree lex ideal in $S_F = S/(x_f \ : \ f \in F)$ for any $F \in \binom{[n]}{k}$.

We have the following analog of Lemma 5.17 in the squarefree setting.

Lemma 5.29. *Let $n \in \mathbb{N}$ be a positive integer and let $b < \binom{n}{2}$. Let $x_p x_q$ be the b-th largest monomial in the lexicographic order of degree 2 squarefree monomials in variables $x_1, \ldots, x_n$ and assume $p < q$. Then:*

$$p = n - 1 + \left\lfloor \frac{1}{2} - \frac{\sqrt{4n(n-1) - 8b + 1}}{2} \right\rfloor,$$

and

$$q = b + \binom{p+1}{2} - (p-1)n.$$

Proof. As in the proof of Lemma 5.17 it is easy to see that, if $x_p x_q$ (with $p < q$) is the b-th largest squarefree degree 2 monomial, then

$$n - p = \max\{s \in \mathbb{N} \ : \ \sum_{\ell=1}^{s} \ell \leq \binom{n}{2} - b\} + 1.$$

Since $s = -\frac{1}{2} + \frac{\sqrt{4n(n-1)-8b+1}}{2}$ is the unique non-negative solution to the equation

$$(s+1)s/2 = n(n-1)/2 - b,$$

we infer that $p = n - 1 + \left\lfloor \frac{1}{2} - \frac{\sqrt{4n(n-1)-8b+1}}{2} \right\rfloor$. As $q = b - \sum_{\ell=1}^{p-1}(n-\ell) + p$, the claim follows from a straightforward computation. □

For squarefree lex ideals (or more generally squarefree stable ideals) the following analog of the Eliahou-Kervaire formula Lemma 5.4 is well known:

Lemma 5.30. [HH11, Corollary 7.4.2] *Let $I \subseteq S$ be a squarefree lex ideal. Then:*

$$\beta^S_{i,i+j}(S/I) = \sum_{u \in G(I)_{j+1}} \binom{\max(u) - j - 1}{i-1}, \tag{5.13}$$

for every $i \geq 1$, $j \geq 0$.

We can now formulate the main result of this section:

Theorem 5.31. *Let $I \subseteq S$ be a homogeneous ideal not containing any linear form. Let $\dim_{\mathbb{F}}(S/(I+P))_2 \leq \binom{n}{2} - b$ for some positive integer b. Let $x_p x_q$, where $p < q$, be the b-th largest squarefree monomial in S of degree 2 in lexicographic order. Then:*

$$\begin{aligned}\beta_{i,i+j}(S/(I+P)) \leq \sum_{k=0}^{j-1} \Bigg[&\binom{n-p}{k} \sum_{\ell=p+j-k+1}^{n-k} \binom{\ell-p-1}{j-k}\binom{\ell-j+k-1}{i-k-1} \\ &+\binom{n-q}{k} \sum_{\ell=q+j-k}^{n-k} \binom{\ell-q-1}{j-k-1}\binom{\ell-j+k-1}{i-k-1} \\ &+\binom{n-q}{k-1} \sum_{\ell=q+j-k}^{n-k} \binom{\ell-q}{j-k}\binom{\ell-j+k-1}{i-k-1}\Bigg] \\ &+\binom{n-j}{i-j}\left(\binom{n-p}{j}+\binom{n-q}{j-1}\right)\end{aligned}$$

for all $i > 0$, $j \geq 2$.

Proof. By Theorem 5.27 we have $\beta_{i,i+j}(S/(I+P)) \leq \beta_{i,i+j}(S/(L+P))$, where $L \subseteq S$ is the squarefree lex ideal such that $L+P$ and $I+P$ have the same Hilbert function. By assumption, L does not contain variables and $\dim_{\mathbb{F}} L_2 \geq b$. Hence, L contains all squarefree degree 2 monomials that are lexicographically larger or equal to $x_p x_q$. We can further compute $\beta_{i,i+j}(S/(L+P))$ using Proposition 5.28. For this, we need to analyze the ideals $(L : x_F)$, where $F \in \binom{[n]}{k}$. We distinguish four cases (having several subcases):

Case 1: Assume that $F = \{f\}$ for $1 \leq f < p$. In particular, we have $p > 1$. Since L is squarefree lex and $x_p x_q \in L$, it holds that $x_f x_\ell \in L$ for all $\ell \in [n] \setminus \{f\}$. This implies $(x_i : i \in [n] \setminus \{f\}) \subseteq (L : x_F)$. As, by Proposition 5.28 *ii* $(L : x_F)$ can be considered as an ideal in S_F and hence no minimal generator is divisible by x_f, we infer that $(L : x_F) = (x_i : i \in [n] \setminus \{f\})$. As $(L : x_F)$ and $(x_1, \ldots, x_{n-1})$ have the same graded Betti numbers, it follows from Lemma 5.30 that F only contributes to $\beta_{i,i+j}(S/(L+P))$ if $j = 1$, a case which we do not consider.

Case 2: Assume that there exist $1 \leq s < t \leq n$ such that $\{s,t\} \subseteq F$ and $x_s x_t \geq_{\text{lex}} x_p x_q$. As L is squarefree lex and $x_p x_q \in L$, we infer that $x_s x_t \in L$ and hence $1 \in (L : x_F)$, i.e., $(L : x_F) = S$. In particular, such F never contributes to $\beta_{i,i+j}(S/(L+P))$.

Case 3: Suppose that there do not exist $s,t \in F$ $(s \neq t)$ with $x_s x_t \geq_{\text{lex}} x_p x_q$. We then have to consider the following two subcases:

Case 3.1: $f > p$ for all $f \in F$.

Case 3.2: $p \in F$ and $f > q$ for all $f \in F \setminus \{p\}$.

Case 3.1 (a): Assume in addition that there exists $f \in F$ with $p < f \leq q$. As $x_p x_q \in L$, $x_\ell x_f \geq_{\text{lex}} x_p x_q$ for $1 \leq \ell \leq p$ and since L is squarefree lex, we infer that $(x_1, \ldots, x_p) \subseteq (L : x_F)$. Moreover, by Proposition 5.28 *ii* $(L : x_F)$ is squarefree lex as an ideal in S_F. If we reorder (and relabel) the variables $x_1, \ldots, x_n$ by first ordering $\{x_i : i \notin F\}$ from largest to smallest by increasing indices and then adding

$\{x_f \ : \ f \in F\}$ in any order, the ideal $(L : x_F)$ will be a squarefree lex ideal in S with respect to this ordering of the variables. If $j \neq k$, then, using Lemma 5.30, we conclude

$$\begin{aligned}\beta_{i-k,i+j-2k}(S/(L:x_F)) &= \sum_{\ell=p+j-k+1}^{n-k}\left(\sum_{u\in G(L:x_F)_{j-k+1}}\binom{\ell-(j-k)-1}{i-k-1}\right)\\ &\leq \sum_{\ell=p+j-k+1}^{n-k}\binom{\ell-p-1}{j-k}\binom{\ell-j+k-1}{i-k-1},\end{aligned}$$

where the last inequality follows from the fact that $G(L : x_F)_{j-k+1} \subseteq G((x_{p+1},\ldots, x_{n-k})^{j-k+1})$. For $j = k$, we note that (after relabeling) we have $G(L : x_F)_1 \subseteq (x_1,\ldots,x_{n-k})$, from which it follows that F contributes to $\beta_{i,i+j}(S/(L+P))$ with at most

$$\sum_{\ell=1}^{n-j}\binom{\ell-1}{i-j-1}=\binom{n-j}{i-j}.$$

Case 3.1 (b): Now suppose that $f > q$ for all $f \in F$. As $F \neq \emptyset$, such f exists. If $p > 1$, then, as L is squarefree lex and $x_px_q \in L$, we have $x_\ell x_f \in L$ for all $1 \leq \ell \leq p-1$. It follows that $x_F \cdot x_\ell = x_{F\setminus\{f\}} \cdot (x_\ell \cdot x_f) \in L$ for $1 \leq \ell \leq p-1$, which implies $(x_1,\ldots,x_{p-1}) \subseteq (L : x_F)$. Moreover, for any p, as $x_px_q \in L$, we also have $x_px_\ell \in (L : x_F)$ for $p+1 \leq \ell \leq q$. Similar as in Case 3.1 (a) we can assume that, after reordering (and relabeling) the variables, $(L : x_F)$ is a squarefree lex ideal in S. As the order of $x_1,\ldots,x_q$ is not affected by this reordering, the previous discussion implies

$$\begin{aligned}G(L:x_F)_{j-k+1} \subseteq &\{u\in \mathrm{Mon}_{j-k+1}(x_{p+1},\ldots,x_{n-k}) \ : \ u \text{ squarefree}\}\cup\\ &\{x_pu \ : u\in \mathrm{Mon}_{j-k}(x_{q+1},\ldots,x_{n-k}),\ u \text{ squarefree}\}\end{aligned}$$

if $j \neq k$. Using Lemma 5.30 we thus obtain

$$\begin{aligned}\beta_{i-k,i+j-2k}(S/(L:x_F)) \leq &\sum_{\ell=p+1+j-k}^{n-k}\binom{\ell-1-p}{j-k}\binom{\ell-j+k-1}{i-k-1}\\ &+\sum_{\ell=q+j-k}^{n-k}\binom{\ell-1-q}{j-k-1}\binom{\ell-j+k-1}{i-k-1}\end{aligned}$$

if $j \neq k$. For $j = k$, a similar computation as in Case 3.1 (a) shows that F contributes to $\beta_{i,i+j}(S/(L+P))$ with at most $\binom{n-j}{i-j}$.

Case 3.2: Consider $F \in \binom{[n]}{k}$ such that $p \in F$ and $f > q$ for all $f \in F \setminus \{p\}$. As $x_px_q \in L$ and as L is squarefree lex, it follows that $(x_1,\ldots,x_{p-1},x_{p+1},\ldots,x_q) \subseteq (L : x_F)$. As in Case 3.1, we can assume that after a suitable reordering (and relabeling) of the variables $(L : x_F)$ is a squarefree lex ideal in S. (Note that after relabeling $(L : x_F)$ contains $x_1,\ldots,x_{q-1}$.) We infer that

$$G(I:x_F)_{j-k+1} \subseteq \{u\in \mathrm{Mon}_{j-k+1}(x_q,\ldots,x_{n-k}) \ : \ u \text{ squarefree}\},$$

if $j \neq k$ and it hence follows from Lemma 5.30 that

$$\beta_{i-k,i+j-2k}(S/(L:x_F)) \leq \sum_{\ell=q+j-k}^{n-k}\binom{\ell-q}{j-k}\binom{\ell-j+k-1}{i-k}$$

if $\neq k$. For $j = k$, it follows from the same arguments as in Case 3.1 (a) that the set F contributes to $\beta_{i,i+j}(S/(L+P))$ with at most $\binom{n-j}{i-j}$.

Case 4: If $F = \emptyset$, then clearly $(L : x_F) = L$. As $x_p x_q \in L$, we obtain that

$$\begin{aligned} G(L)_{j+1} \subseteq &\{u \in \operatorname{Mon}_{j+1}(x_{p+1}, \dots, x_n) \ : \ u \text{ squarefree}\} \cup \\ &\{x_p u \ : u \in \operatorname{Mon}_j(x_{q+1}, \dots, x_n), \ u \text{ squarefree}\} \end{aligned}$$

for $j \geq 2$. The same computation as in Case 3.1 (b) now yields that

$$\beta_{i,i+j}(S/(L : x_F)) \leq \sum_{\ell=p+1+j}^{n} \binom{\ell-1-p}{j}\binom{\ell-j-1}{i-1} + \sum_{\ell=q+j}^{n} \binom{\ell-1-q}{j-1}\binom{\ell-j-1}{i-1}.$$

Combining Case 1–4, we finally obtain for $i > 0$ and $j > 1$:

$$\begin{aligned}
&\beta_{i,i+j}(S/(I+P)) \leq \beta_{i,i+j}(S/(L+P)) \\
&= \underbrace{\binom{n-j}{i-j}\left(\binom{n-p}{j} - \binom{n-q}{j}\right)}_{\text{Case 3.1(a), } j=k} + \underbrace{\binom{n-j}{i-j}\binom{n-q}{j}}_{\text{Case 3.1(b), } j=k} + \underbrace{\binom{n-j}{i-j}\binom{n-q}{j-1}}_{\text{Case 3.2, } j=k} \\
&+ \sum_{k=1}^{j-1} \Bigg[\underbrace{\left(\binom{n-p}{k} - \binom{n-q}{k}\right) \sum_{\ell=p+j-k+1}^{n-k} \binom{\ell-p-1}{j-k}\binom{\ell-j+k-1}{i-k-1}}_{\text{Case 3.1(a)}} \\
&+ \underbrace{\binom{n-q}{k}\Bigg(\sum_{\ell=p+j-k+1}^{n-k} \binom{\ell-1-p}{j-k}\binom{\ell-j+k-1}{i-k-1}}_{\text{Case 3.1(b)}} \\
&+ \underbrace{\sum_{\ell=q+j-k}^{n-k} \binom{\ell-q-1}{j-k-1}\binom{\ell-j+k-1}{i-k-1}\Bigg)}_{\text{Case 3.1(b)}} \\
&+ \underbrace{\binom{n-q}{k-1} \sum_{\ell=q+j-k}^{n-k} \binom{\ell-q}{j-k}\binom{\ell-j+k-1}{i-k-1}}_{\text{Cases 3.2}} \Bigg] \\
&= \sum_{k=0}^{j-1} \Bigg[\binom{n-p}{k} \sum_{\ell=p+j-k+1}^{n-k} \binom{\ell-p-1}{j-k}\binom{\ell-j+k-1}{i-k-1} \\
&+ \binom{n-q}{k} \sum_{\ell=q+j-k}^{n-k} \binom{\ell-q-1}{j-k-1}\binom{\ell-j+k-1}{i-k-1} \\
&+ \binom{n-q}{k-1} \sum_{\ell=q+j-k}^{n-k} \binom{\ell-q}{j-k}\binom{\ell-j+k-1}{i-k-1} \Bigg] \\
&+ \binom{n-j}{i-j}\left(\binom{n-p}{j} - \binom{n-q}{j} + \binom{n-q+1}{j}\right).
\end{aligned}$$

This completes the proof. □

There might be several ways to simplify the bound of Theorem 5.31 by losing tightness. However, we decided to state it in the best possible form.

5.4.2 Application: Balanced Cohen-Macaulay complexes revisited

The aim of this section is to use Theorem 5.31 in order to get bounds for the graded Betti numbers of balanced Cohen-Macaulay complexes. Our starting point is the result of Stanley presented in Proposition 2.6, which shows for a balanced d-dimensional Cohen-Macaulay simplicial complex Δ the existence of a special l.s.o.p. $\Theta = (\theta_1, \ldots, \theta_d)$, which we call colored l.s.o.p.. This linear forms are simply sums of variables colored using the same color. As a consequence, see Proposition 2.6, we have that $x_v^2 \in I_\Delta + (\theta_1, \ldots, \theta_d)$ for all $v \in V(\Delta)$.

An almost immediate application of Theorem 5.31, combined with Proposition 2.6 *ii* yields the desired bound for the graded Betti numbers of a balanced Cohen-Macaulay complex:

Theorem 5.32. *Let Δ be a $(d-1)$-dimensional balanced Cohen-Macaulay complex with vertex partition $V = \dot{\bigcup}_{i=1}^{d} V_i$. Let $n := |V|$, $n_i := |V_i|$ and $b := \sum_{i=1}^{d} \binom{n_i - 1}{2}$. Let $x_p x_q$ be the b-th largest squarefree degree 2 monomial of $\mathbb{F}[x_1, \ldots, x_{n-d}]$ in lexicographic order with $p \leq q$. Then*

$$\begin{aligned}\beta_{i,i+j}(\mathbb{F}[\Delta]) \leq \sum_{k=0}^{j-1} \Bigg[& \binom{n-d-p}{k} \sum_{\ell=p+j-k+1}^{n-d-k} \binom{\ell-p-1}{j-k} \binom{\ell-j+k-1}{i-k-1} \\ & + \binom{n-d-q}{k} \sum_{\ell=q+j-k}^{n-d-k} \binom{\ell-q-1}{j-k-1} \binom{\ell-j+k-1}{i-k-1} \\ & + \binom{n-d-q}{k-1} \sum_{\ell=q+j-k}^{n-d-k} \binom{\ell-q}{j-k} \binom{\ell-j+k-1}{i-k-1} \Bigg] \\ & + \binom{n-d-j}{i-j} \left(\binom{n-d-p}{j} + \binom{n-d-q}{j-1} \right)\end{aligned}$$

for all $i > 0$, $j > 1$.

Proof. The proof follows exactly along the same arguments as the one of Theorem 5.23, using the colored l.s.o.p. of $\mathbb{F}[\Delta]$. By Proposition 2.6 it then holds that the ideal $(\Theta) + I_\Delta$ contains the squares of the variables. It remains to observe that under the isomorphism $\mathbb{F}[x_1, \ldots, x_n]/(\Theta) \cong R$, the ideal $P = (x_1^2, \ldots, x_n^2) \subseteq \mathbb{F}[x_1, \ldots, x_n]$ is mapped to a homogeneous ideal containing $(x_1^2, \ldots, x_{n-d}^2)$ and thus $\mathbb{F}[\Delta]/\Theta \cong R/(I+P)$ for a homogeneous ideal $I \subseteq R$ (not containing linear forms). We further observe that

$$\dim_{\mathbb{F}}(R/(I+P))_2 = h_2(\Delta) \leq \binom{n-d+1}{2} - \sum_{i=1}^{d} \binom{n_i}{2} = \binom{n-d}{2} - \sum_{i=1}^{d} \binom{n_i - 1}{2}.$$

The claim now follows from Theorem 5.31. □

Example 5.33. As in Example 5.25, we consider 3-dimensional balanced Cohen-Macaulay complexes with 3 vertices in each color class, i.e., $n_i = 3$ for $1 \leq i \leq 4$. We have $b := \sum_{i=1}^{4} \binom{3}{2} - 8 = 4$ and $x_1 x_5$ is the 4-th largest monomial of degree 2 in variables $x_1, \ldots, x_8$. The bounds from Theorem 5.32 are recorded in the following table: Comparing those bounds with the ones from Table 5.2, we see that the lex-plus-squares approach gives better bounds for all entries of the Betti table. The improvement is more significant in the lower rows of the Betti tables.

$j \setminus i$	0	1	2	3	4	5	6	7	8
2	0	38	292	827	1249	1125	609	184	24
3	0	36	267	885	1529	1510	877	280	38
4	0	21	161	533	1024	1145	727	249	36

TABLE 5.4: The bound given in Theorem 5.32 for a 3-dimensional balanced Cohen-Macaulay complex with 3 vertices in each color class.

Remark 5.34. Consider again a 3-dimensional balanced Cohen-Macaulay complex Δ on 12 vertices, but with a different color partition, namely $n_1 = 1$, $n_2 = 3$, and $n_3 = n_4 = 4$. Then since every facet must contain the unique vertex of color 1, Δ is a cone, hence contractible. Theorem 5.32 yields $\beta_{8,12}(\mathbb{F}[\Delta]) = \dim_{\mathbb{F}} \widetilde{H}_3(\Delta;\mathbb{F}) \leq 35$. This shows that the bound is not necessarily tight.

5.5 The linear strand for balanced pseudomanifolds

The aim of this section is to study the linear strand of the minimal graded free resolution of the Stanley-Reisner ring of a balanced normal pseudomanifold. In particular, we will provide upper bounds for the graded Betti numbers in the linear strand. Previously, such bounds have been shown for general (not necessarily balanced) pseudomanifolds by Murai [Mur15, Lemma 5.6 *ii*] and it follows from a result by Hibi and Terai [TH97, Corollary 2.3.2] that they are tight for stacked spheres (see Definition 1.41). The mentioned results of Murai [Mur15, Lemma 5.6 (ii)] and Hibi and Terai [TH97, Corollary 2.3.2] can be summarized as follows:

Lemma 5.35. *Let $d \geq 3$. Let Δ be a $(d-1)$-dimensional normal pseudomanifold with n vertices. Then:*

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) \leq i\binom{n-d}{i+1} \qquad \textit{for all } i \geq 0.$$

Moreover, those bounds are attained if Δ is a stacked sphere.

We remark that, in [TH97], the authors provide explicit formulas not only for the Betti numbers of the linear strand but for *all* graded Betti numbers of a stacked sphere. In particular, it is shown that these numbers only depend on the number of vertices n and the dimension $d-1$.

In order to prove a balanced analog of the first statement of Lemma 5.35, the following result due to Fogelsanger [Fog88] will be crucial (see also [NS09a, Section 5]).

Lemma 5.36. *Let $d \geq 3$. Let Δ be a $(d-1)$-dimensional normal pseudomanifold. Then there exist linear forms $\theta_1, \ldots, \theta_{d+1}$ such that the multiplication map*

$$\times\theta_i : \left(\mathbb{F}[\Delta]/(\theta_1, \ldots, \theta_{i-1})\mathbb{F}[\Delta]\right)_1 \longrightarrow \left(\mathbb{F}[\Delta]/(\theta_1, \ldots, \theta_{i-1})\mathbb{F}[\Delta]\right)_2$$

is injective for all $1 \leq i \leq d+1$.

Intuitively, the previous result compensates the lack of a regular sequence for normal pseudomanifolds in small degrees, since those need not to be Cohen-Macaulay.

Recall that a key step for the proofs of Theorem 5.20 and Theorem 5.31 was to find upper bounds for the number of generators of the lex ideal and the lex-plus-squares ideal, respectively, of degree ≥ 3. For the proof of our main result in this section we will use a similar strategy, but since we are interested in the linear strand of the minimal free resolution, we rather need to bound the number of degree 2 generators in a certain lex-ideal. This can be accomplished via the Lower Bound Theorem for balanced normal pseudomanifolds presented in Theorem 2.20, which, for every $(d-1)$-dimensional balanced normal pseudomanifold Δ with $d \geq 3$, establishes the inequality

$$h_2(\Delta) \geq \frac{d-1}{2} h_1(\Delta). \tag{5.14}$$

We can now state the main result of this section.

Theorem 5.37. *Let $d \geq 3$ and let Δ be a $(d-1)$-dimensional balanced normal pseudomanifold on n vertices. Let $b := \frac{(n-d)(n-2d+2)}{2}$ and let $x_p x_q$ (where $p \leq q$) be the b-th largest degree 2 monomial of $\mathbb{F}[x_1, \ldots, x_{n-d-1}]$ in lexicographic order. Then*

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) \leq (p-1)\binom{n-d-1}{i} - \binom{p}{i+1} + \binom{q}{i}. \tag{5.15}$$

Proof. Let $R' := \mathbb{F}[x_1, \ldots, x_{n-d-1}]$ and let $\Theta = \{\theta_1, \ldots, \theta_{d+1}\}$ be linear forms given by Lemma 5.36. Then, as in the proof of Theorem 5.23, we let $J \subseteq R$ be the homogeneous ideal with $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta] \cong R/J$ and we let $J^{\text{lex}} \subseteq R$ be the lex ideal of J. Using Lemma 5.36, Lemma 5.2 and Lemma 5.3 we conclude

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) \leq \beta_{i,i+1}^{S/\Theta S}(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]) = \beta_{i,i+1}^{R}(R/J) \leq \beta_{i,i+1}^{R}(R/J^{\text{lex}}).$$

To prove inequality (5.15) we will compute upper bounds for $\beta_{i,i+1}^{R}(R/J^{\text{lex}})$ using Lemma 5.4. For those we need an upper bound for the number of generators of degree 2 in J^{lex}. More precisely, we will prove the following claim:
Claim: $\dim_{\mathbb{F}}(J^{\text{lex}})_2 \leq b$.
By the definition of the ideals J and J^{lex} we have

$$\begin{aligned}
\dim_{\mathbb{F}}(R/J^{\text{lex}})_2 &= \dim_{\mathbb{F}}(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_2 \\
&= h_2(\Delta) - h_1(\Delta) \\
&\geq \frac{d-1}{2} h_1(\Delta) - h_1(\Delta) \\
&= \frac{d-3}{2}(n-d).
\end{aligned}$$

Here, the second equality follows from the injectivity of the multiplication maps in Lemma 5.36 and the inequality holds by (5.14). We conclude

$$\dim_{\mathbb{F}}(J^{\text{lex}})_2 \leq \binom{n-d}{2} - \frac{d-3}{2}(n-d) = \frac{(n-d)(n-2d+2)}{2} = b,$$

which shows the claim.
Since $\dim_{\mathbb{F}}(R/J^{\text{lex}})_1 = n-d-1 = \dim_{\mathbb{F}}(R)_1$, the ideal J^{lex} does not contain variables. Using the just proven claim we conclude that $G(J^{\text{lex}})_2$ contains at most the b lexicographically largest degree 2 monomials of R, i.e.,

$$G(J^{\text{lex}})_2 \subseteq \{u \in \text{Mon}_2(R) \ : \ u \geq_{\text{lex}} x_p x_q\}.$$

To simplify notation, we set $M := \{u \in \mathrm{Mon}_2(R) \ : \ u \geq_{\mathrm{lex}} x_p x_q\}$. Using Lemma 5.4, we infer:

$$\begin{aligned}
\beta^R_{i,i+1}(R/J^{\mathrm{lex}}) \leq & \sum_{u\in M} \binom{\max(u)-1}{i-1} \\
= & \sum_{\ell=1}^{p} \sum_{\substack{u\in M\\ \max(u)=\ell}} \binom{\ell-1}{i-1} + \sum_{\ell=p+1}^{q} \sum_{\substack{u\in M\\ \max(u)=\ell}} \binom{\ell-1}{i-1} + \\
& + \sum_{\ell=q+1}^{n-d-1} \sum_{\substack{u\in M\\ \max(u)=\ell}} \binom{\ell-1}{i-1} \\
= & \sum_{\ell=1}^{p} \ell \binom{\ell-1}{i-1} + p \sum_{\ell=p+1}^{q} \binom{\ell-1}{i-1} + (p-1) \sum_{\ell=q+1}^{n-d-1} \binom{\ell-1}{i-1} \\
= & i\binom{p+1}{i+1} + (p-1)\binom{n-d-1}{i} - p\binom{p}{i} + \binom{q}{i} \\
= & (p-1)\binom{n-d-1}{i} - \binom{p}{i+1} + \binom{q}{i}
\end{aligned}$$

for all $i \geq 0$. This finishes the proof. □

Note that, unlike the bounds from Theorem 5.20 and Theorem 5.32, the bounds from Theorem 5.37 do not depend on the sizes of the color classes.

Example 5.38. Let Δ be a 3-dimensional balanced pseudomanifold on 12 vertices, with an arbitrary partition of the vertices into color classes. We have $b = \frac{(n-d)(n-2d+2)}{2} = 24$ and x_5x_6 is the 24-th largest degree 2 monomial in variables $x_1, \ldots, x_7$. The bounds for $\beta_{i,i+1}(\mathbb{F}[\Delta])$ provided by Theorem 5.37 are recorded in Example 5.38. One should compare those with the bounds provided by Lemma 5.35

$j \setminus i$	0	1	2	3	4	5	6	7	8
1	0	24	89	155	154	90	29	4	0

Table 5.5: The bound given in Theorem 5.37 for a 3-dimensional balanced normal pseudomanifold on 12 vertices.

for arbitrary (not necessarily balanced) pseudomanifolds: While the bounds in Ex-

$j \setminus i$	0	1	2	3	4	5	6	7	8
1	0	28	112	210	224	140	48	7	0

Table 5.6: The bound for an arbitrary 3-dimensional normal pseudomanifold on 12 vertices.

ample 5.38 are realized by any stacked 3-sphere on 12 vertices, we do not know if the ones for the balanced case, shown in Example 5.38, are attained. In the next section we will see that they are not attained by the balanced analog of stacked spheres.

Remark 5.39. In view of Theorem 5.37 a natural question that arises is if one can also bound the entries of the j-th row of the Betti table of a balanced pseudomanifold for $j \geq 2$. In order for our approach to work, this would require the multiplication maps from Lemma 5.36 to be injective also for higher degrees; a property that is closely related to Lefschetz properties.

5.6 Betti numbers of stacked cross-polytopal spheres

The aim of this section is to compute explicitly the graded Betti numbers of cross-polytopal stacked spheres, which have been defined in Definition 2.19. These spheres can be considered as the balanced analog of stacked spheres, in the sense that both minimize the h-vector among the class of balanced normal pseudomanifolds respectively all normal pseudomanifolds (see Theorem 1.43 and Theorem 2.20). For stacked spheres, explicit formulas for their graded Betti numbers were provided by Hibi and Terai [TH97] and it was shown that they only depend on the number of vertices and the dimension but not on the combinatorial type of the stacked sphere (see also Lemma 5.35).

Observe that $\mathcal{ST}^{\times}(2d, d) = \{\partial\mathcal{C}_d\}$. In analogy with the non-balanced setting, for $k \geq 4$, there exist stacked cross-polytopal spheres in $\mathcal{ST}^{\times}(kd, d)$ of different combinatorial types, as depicted in Figure 2.3. Nevertheless, we observed in Chapter 2 that the f-vector of a stacked cross-polytopal only depends on n and d. In this section, we will show the same behavior for their graded Betti numbers.

As a warm-up, we compute the Betti numbers of the boundary complex of the cross-polytope.

Lemma 5.40. *Let $d \geq 1$. Then $\beta_{i,i+j}(\mathbb{F}[\partial\mathcal{C}_d]) = 0$ for all $i \geq 0$ and $j \neq i$. Moreover,*

$$\beta_{i,2i}(\mathbb{F}[\partial\mathcal{C}_d]) = \binom{d}{i}$$

for all i.

Proof. Being generated by d pairwise coprime monomials, the Stanley-Reisner ideal of $\partial\mathcal{C}_d$ is a complete intersection, and hence it is minimally resolved by the Koszul complex. □

The following immediate lemma will be very useful, in order to derive a recursive formula for the graded Betti numbers of stacked cross-polytopal spheres.

Lemma 5.41. *Let $d \geq 3$. Let $\Delta \in \mathcal{ST}^{\times}(kd, d)$ be a stacked cross-polytopal sphere on vertex set V and let F be a facet of Δ. Then for any $W \subseteq V$,*

$$\widetilde{H}_j(\Delta_W; \mathbb{F}) = \widetilde{H}_j((\Delta \setminus \{F\})_W; \mathbb{F}) \quad \text{for all } 0 \leq j \leq d-3.$$

Proof. The statement is immediate since Δ and $\Delta \setminus \{F\}$ share the same skeleta up to dimension $d-2$. □

Consider $\Delta \in \mathcal{ST}^{\times}(kd, d)$ and let $\Diamond_1, \ldots, \Diamond_{k-1}$ denote the copies of $\partial\mathcal{C}_d$ from which Δ was constructed. We call a facet $F \in \Delta \cap \Diamond_i$ *extremal* if $V(\Diamond_i) \setminus F \notin \Delta$, and the facet $V(\Diamond_i) \setminus F$ is called the *opposite* of F. Intuitively a facet F of Δ is extremal if removing all the vertices in F from Δ yields a complex $\Gamma \setminus \{G\}$, where $\Gamma \in \mathcal{ST}^{\times}((k-1)d, d)$ and G is the opposite of F.

We have the following recursive formulas for Betti numbers of stacked cross-polytopal spheres.

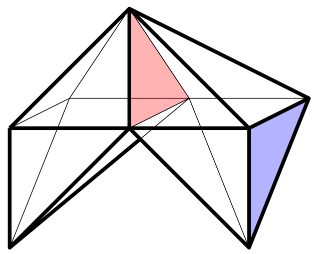

FIGURE 5.1: An extremal facet (blue) and its opposite (red).

Theorem 5.42. *Let $n \geq 3d$ and $\Delta \in \mathcal{ST}^{\times}(n,d)$. Then*

$$\beta_{i,i+j}(\mathbb{F}[\Delta]) = \begin{cases} A_{i,1}(\Gamma) + d\binom{n-2d}{i-1} + \sum_{\ell=1}^{\min\{i,d\}} \binom{d}{\ell}\binom{n-2d}{i+1-\ell}, & \text{if } j = 1 \\ A_{i,j}(\Gamma) + \binom{d}{j}\binom{n-2d}{i-j}, & \text{if } 2 \leq j \leq d-2, \end{cases} \tag{5.16}$$

with $\Gamma \in \mathcal{ST}^{\times}(n-d,d)$ and $A_{i,j}(\Gamma) = \sum_{\ell=0}^{d}\binom{d}{\ell}\beta_{i-\ell,i-\ell+j}(\mathbb{F}[\Gamma])$. In particular, the graded Betti numbers of Δ only depend on n and d.

Remark 5.43. Note that for the case $j = 1$ the previous formula can be deduced from Corollary 3.4 in [CK10]. We report its proof anyway, as the idea is analogous to the case $j \geq 2$.

Proof. We will compute the graded Betti numbers using Hochster's formula. Let V be the vertex set of Δ and let F be an extremal facet of Δ with opposite G. Then, we can write $\Delta = (\Gamma \setminus \{G\}) \cup (\Diamond \setminus \{G\})$, where $\Gamma \in \mathcal{ST}^{\times}(n-d,d)$ and $\Diamond$ is the boundary complex of the d-dimensional cross-polytope on vertex set $F \cup G$. In particular, $(\Gamma \setminus \{G\}) \cap (\Diamond \setminus \{G\}) = \partial(G)$. We now distinguish two cases.
Case 1: $j = 1$. Let $W \subseteq V$. We have several cases:

(a) If $W \subseteq V(\Gamma)$, then $\Delta_W = (\Gamma \setminus \{G\})_W$. By Lemma 5.41, $(\Gamma \setminus \{G\})_W$ (thus Δ_W) and Γ_W have the same number of connected components and hence $\widetilde{H}_0(\Delta_W;\mathbb{F}) = \widetilde{H}_0(\Gamma_W;\mathbb{F})$.

(b) If $W \subseteq V(\Diamond)$, then it follows as in (b) that $\widetilde{H}_0(\Delta_W;\mathbb{F}) = \widetilde{H}_0(\Diamond_W;\mathbb{F})$.

(c) Assume that $W \cap (V(\Gamma) \setminus G) \neq \emptyset$ and $W \cap (V(\Diamond) \setminus G) \neq \emptyset$. Then, $\Delta_W = (\Gamma \setminus \{G\})_W \cup (\Diamond \setminus \{G\})_W$. If, in addition, $W \cap G = \emptyset$, then this union is disjoint and, using Lemma 5.41 we conclude that the number of connected components of Δ_W equals the sum of the number of connected components of Γ_W and $\Diamond_W$. Thus, as neither Γ_W nor $\Diamond_W$ is the empty complex,

$$\dim_{\mathbb{F}} \widetilde{H}_0(\Delta_W;\mathbb{F}) = \dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F}) + \dim_{\mathbb{F}} \widetilde{H}_0(\Diamond_W;\mathbb{F}) + 1.$$

If $W \cap G \neq \emptyset$, then the number of connected components of Δ_W is one less than the sum of the number of connected components of $(\Gamma \setminus \{G\})_W$ and $(\Diamond \setminus \{G\})_W$. In particular, using Lemma 5.41, we infer

$$\dim_{\mathbb{F}} \widetilde{H}_0(\Delta_W;\mathbb{F}) = \dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F}) + \dim_{\mathbb{F}} \widetilde{H}_0(\Diamond_W;\mathbb{F}).$$

Using Hochster's formula we obtain:

$$\begin{aligned}
\beta_{i,i+1}(\mathbb{F}[\Delta]) &= \sum_{W\subseteq V;\,|W|=i+1} \dim_{\mathbb{F}} \widetilde{H}_{i-1}(\Delta_W;\mathbb{F})\\
&= \sum_{\substack{W\subseteq V;\,|W|=i+1\\ W\cap G\neq\emptyset}} \Big(\dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F}) + \dim_{\mathbb{F}} \widetilde{H}_0(\Diamond_W;\mathbb{F})\Big)\\
&\quad + \sum_{\substack{W\subseteq V\setminus G;\,|W|=i+1\\ W\cap V(\Gamma)\neq\emptyset; W\cap V(\Diamond)\neq\emptyset}} \Big(\dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F}) + \dim_{\mathbb{F}} \widetilde{H}_0(\Diamond_W;\mathbb{F}) + 1\Big)\\
&\quad + \sum_{\substack{W\subseteq V(\Gamma)\setminus G\\ |W|=i+1}} \dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F}) + \sum_{\substack{W\subseteq V(\Diamond)\setminus G\\ |W|=i+1}} \dim_{\mathbb{F}} \widetilde{H}_0(\Diamond_W;\mathbb{F}).
\end{aligned}$$

For $W \subseteq V(\Gamma)$ (respectively $W \subseteq V(\Diamond)$) the term $\dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F})$ (respectively $\dim_{\mathbb{F}} \widetilde{H}_0(\Diamond_W;\mathbb{F})$) appears $\binom{d}{i+1-|W|}$ (respectively $\binom{n-2d}{i+1-|W|}$) times in the previous expression. Moreover, there are $\sum_{\ell=1}^{\min\{i,d\}} \binom{d}{\ell}\binom{n-2d}{i+1-\ell}$ $(i+1)$-subsets W of $V \setminus G$ with $W \cap V(\Gamma) \neq \emptyset$ and $W \cap V(\Diamond) \neq \emptyset$. This implies

$$\begin{aligned}
\beta_{i,i+1}(\mathbb{F}[\Delta]) &= \sum_{\ell=1}^{i+1} \binom{d}{i+1-\ell} \left(\sum_{W\subseteq V(\Gamma),\,|W|=\ell} \dim_{\mathbb{F}} \widetilde{H}_0(\Gamma_W;\mathbb{F}) \right)\\
&\quad + \sum_{\ell=1}^{2d} \binom{n-2d}{i+1-\ell} \left(\sum_{W\subseteq V(\Diamond),\,|W|=\ell} \dim_{\mathbb{F}} \widetilde{H}_0(\Diamond;\mathbb{F}) \right)\\
&\quad + \sum_{\ell=1}^{\min\{i,d\}} \binom{d}{\ell}\binom{n-2d}{i+1-\ell}\\
&= \sum_{\ell=i+1-d}^{i+1} \binom{d}{i+1-\ell} \beta_{\ell-1,\ell}(\mathbb{F}[\Gamma]) + \sum_{\ell=1}^{2d} \binom{n-2d}{i+1-\ell} \beta_{\ell-1,\ell}(\mathbb{F}[\Diamond])\\
&\quad + \sum_{\ell=1}^{\min\{i,d\}} \binom{d}{\ell}\binom{n-2d}{i+1-\ell},
\end{aligned}$$

where the last equality holds by Hochster's formula. The desired recursion for $\beta_{i,i+1}(\mathbb{F}[\Delta])$ now follows from a simple index shift.

Case 2: $2 \leq j \leq d-2$. Let $W \subseteq V$. We consider two cases.

(a) If $W \subseteq V(\Gamma)$, then it follows from Lemma 5.41 that

$$\widetilde{H}_j(\Delta_W;\mathbb{F}) = \widetilde{H}_j(\Gamma_W;\mathbb{F}) \text{ for } 0 \leq j \leq d-3.$$

(b) If $W \subseteq V(\Diamond)$, then it follows as in (a) that

$$\widetilde{H}_j(\Delta_W;\mathbb{F}) = \widetilde{H}_j(\Diamond_W;\mathbb{F}) \text{ for } 0 \leq j \leq d-3.$$

(c) Assume that $W \cap (V(\Gamma) \setminus G) \neq \emptyset$ and $W \cap (V(\Diamond) \setminus G) \neq \emptyset$. Then, $\Delta_W =$

$(\Gamma\setminus\{G\})_W\cup(\lozenge\setminus\{G\})_W$. Let $1\le j\le d-3$. We have the following Mayer-Vietoris exact sequence

$$\begin{aligned}\ldots\to\underbrace{\widetilde{H}_j(\partial(G)_W;\mathbb{F})}_{=0}&\to\widetilde{H}_j((\Gamma\setminus\{G\})_W;\mathbb{F})\oplus\widetilde{H}_j((\lozenge\setminus\{G\})_W;\mathbb{F})\\&\to\widetilde{H}_j(\Delta_W;\mathbb{F})\to\underbrace{\widetilde{H}_{j-1}(\partial(G)_W;\mathbb{F})}_{=0}\to\ldots,\end{aligned}\tag{5.17}$$

where we use that $(\Gamma\setminus\{G\})_W\cap(\lozenge\setminus\{G\})_W=(\partial(G))_W$, which has always trivial homology in dimension $\le d-3$. It follows from (5.17) combined with Lemma 5.41 that

$$\widetilde{H}_j(\Delta_W;\mathbb{F})\cong\widetilde{H}_j(\Gamma_W;\mathbb{F})\oplus\widetilde{H}_j(\lozenge_W;\mathbb{F})\quad\text{for } 1\le j\le d-3.$$

Using Hochster's formula we conclude

$$\begin{aligned}\beta_{i,i+j}(\mathbb{F}[\Delta])&=\sum_{W\subseteq V,\,|W|=i+1}\left(\dim_{\mathbb{F}}\widetilde{H}_{j-1}(\Gamma_W;\mathbb{F})+\dim_{\mathbb{F}}\widetilde{H}_{j-1}(\lozenge_W;\mathbb{F})\right)\\&=\sum_{\ell=i+j-d}^{i+j}\binom{d}{i+j-\ell}\left(\sum_{W\subseteq V(\Gamma),\,|W|=\ell}\dim_{\mathbb{F}}\widetilde{H}_{j-1}(\Gamma_W;\mathbb{F})\right)+\\&\quad+\sum_{\ell=1}^{2d}\binom{n-2d}{i+j-\ell}\left(\sum_{W\subseteq V(\lozenge),\,|W|=\ell}\dim_{\mathbb{F}}\widetilde{H}_{j-1}(\lozenge;\mathbb{F})\right)\\&=\sum_{\ell=i+j-d}^{i+j}\binom{d}{i+j-\ell}\beta_{\ell-j,\ell}(\mathbb{F}[\Gamma])+\sum_{\ell=1}^{2d}\binom{n-2d}{i+j-\ell}\beta_{\ell-j,\ell}(\mathbb{F}[\lozenge])\\&=\sum_{\ell=0}^{d}\binom{d}{\ell}\beta_{i-d+\ell,i-d+\ell+j}(\mathbb{F}[\Gamma])+\binom{n-2d}{i-j}\binom{d}{j},\end{aligned}$$

where the second equality follows, as in Case 1, by a simple counting argument and the last equality follows from Lemma 5.40.

The statement of the "In particular"-part follows directly by applying the recursion iteratively, and from $\mathcal{ST}^{\times}(2d,d)=\{\partial\mathcal{C}_d\}$. ☐

Remark 5.44. We remark that due to graded Poincaré duality the graded Betti numbers of any stacked cross-polytopal sphere $\Delta\in\mathcal{ST}^{\times}(n,d)$ exhibit the following symmetry:

$$\beta_{i,i+j}(\mathbb{F}[\Delta])=\beta_{n-d-i,n-i-j}(\mathbb{F}[\Delta]).\tag{5.18}$$

This in particular implies $\beta_{n-d,n}(\mathbb{F}[\Delta])=1$ and $\beta_{i,i+d}(\mathbb{F}[\Delta])=0$ for $0\le i<n-d$. Moreover, also $\beta_{i,i+d-1}(\mathbb{F}[\Delta])$ can be computed using the recursion from Theorem 5.42 (for the linear strand).

In order to derive explicit formulas for the graded Betti numbers of a stacked cross-polytopal sphere, we need to convert the recursive formula of Theorem 5.42 into a closed expression.

Theorem 5.45. *Let $d\ge 3$, $k\ge 2$ and let $\Delta\in\mathcal{ST}^{\times}(kd,d)$ be a stacked cross-polytopal sphere. Then, $\beta_{0,0}(\mathbb{F}[\Delta])=\beta_{(k-1)d,kd}(\mathbb{F}[\Delta])=1$ and for $i\ge 0$ the number*

$\beta_{i,i+j}(\mathbb{F}[\Delta])$ *equals:*

$$\begin{cases} (k-2)\binom{d(k-1)}{i+1} - (k-1)\binom{d(k-2)}{i+1} + d(k-1)\binom{d(k-2)}{i-1} & j=1 \\ (k-1)\binom{d}{j}\binom{d(k-2)}{i-j} & 2 \leq j \leq d-2\,. \\ (k-2)\binom{d(k-1)}{i-1} - (k-1)\binom{d(k-2)}{i-d-1} + d(k-1)\binom{d(k-2)}{i-d+1} & j=d-1 \end{cases}$$

Proof. We proof the claim by induction on k.
For $k=2$, the first line above equals d if $i=1$ and 0 otherwise. Similarly, the second line equals $\binom{d}{i}$ if $j=i$ and 0 otherwise. The claim for $k=2$ now follows from Lemma 5.40.
Let $k \geq 3$ and let $\Delta \in \mathcal{ST}^{\times}(kd, d)$. We first show the case $j=1$.
Using Theorem 5.42 and then the induction hypothesis, we conclude

$$\begin{aligned} \beta_{i,i+1}(\mathbb{F}[\Delta]) = &\sum_{\ell=0}^{\min\{i,d\}} \binom{d}{\ell}\beta_{i-\ell,i-\ell+1}(\mathbb{F}[\Gamma]) + d\binom{n-2d}{i-1} + \sum_{\ell=1}^{\min\{i,d\}}\binom{d}{\ell}\binom{n-2d}{i+1-\ell} \\ =&(k-3)\sum_{\ell=0}^{\min\{i,d\}}\binom{d}{\ell}\binom{d(k-2)}{(i+1)-\ell} - (k-2)\sum_{\ell=0}^{\min\{i,d\}}\binom{d}{\ell}\binom{d(k-3)}{(i+1)-\ell} \\ &+ d(k-2)\sum_{\ell=0}^{\min\{i,d\}}\binom{d}{\ell}\binom{d(k-3)}{(i-1)-\ell} + d\binom{d(k-2)}{i-1} \\ &+ \sum_{\ell=1}^{\min\{i,d\}}\binom{d}{\ell}\binom{d(k-2)}{(i+1)-\ell}, \end{aligned} \tag{5.19}$$

where $\Gamma \in \mathcal{ST} \times ((k-1)d, d)$. We now assume that $\min\{i,d\} = d$. We notice that in (5.19), we can shift the upper summation indices to $i+1$ in the first 2 sums and to $i-1$ in the third sum. Using Vandermonde identity we obtain

$$\begin{aligned} \beta_{i,i+1}(\mathbb{F}[\Delta]) =&(k-3)\binom{d(k-1)}{i+1} - (k-2)\binom{d(k-2)}{i+1} + d(k-2)\binom{d(k-2)}{i-1} \\ &+ d\binom{d(k-2)}{i-1} + \left(\binom{d(k-1)}{i+1} - \binom{d(k-2)}{i+1}\right) \\ =&(k-2)\binom{d(k-1)}{i+1} - (k-1)\binom{d(k-2)}{i+1} + d(k-1)\binom{d(k-2)}{i-1}. \end{aligned}$$

If $i < d$ (thus $\min\{i,d\} = i$), then the same computation as above with an additional summand of $-(k-3)\binom{d}{i+1}$, $(k-2)\binom{d}{i+1}$ and $-\binom{d}{i+1}$ for the first, second and fourth sum, respectively, shows the formula for the first line.

We now show the case $1 < j \leq d-2$:
Applying Theorem 5.42 and the induction hypothesis, we obtain

$$\begin{aligned}\beta_{i,i+j}(\mathbb{F}[\Delta]) &= \sum_{\ell=0}^{\min\{i,d\}} \binom{d}{\ell} \beta_{i-\ell,i-\ell+j}(\mathbb{F}[\Gamma]) + \binom{d}{j}\binom{d(k-2)}{i-j} \\ &= \sum_{\ell=0}^{\min\{i,d\}} \binom{d}{\ell}(k-2)\binom{d}{j}\binom{d(k-3)}{i-j-\ell} + \binom{d}{j}\binom{d(k-2)}{i-j} \\ &=(k-2)\binom{d}{j}\sum_{\ell=0}^{\min\{i-j,d\}} \binom{d}{\ell}\binom{d(k-3)}{i-j-\ell} + \binom{d}{j}\binom{d(k-2)}{i-j} \\ &=(k-2)\binom{d}{j}\binom{d(k-2)}{i-j} + \binom{d}{j}\binom{d(k-2)}{i-j} = (k-1)\binom{d}{j}\binom{d(k-2)}{i-j},\end{aligned}$$

where $\Gamma \in \mathcal{ST}^{\times}((k-1)d, d)$ and the fourth equality follows from Vandermonde's identity after observing that shifting the upper index of the sum to $i-j$ does not change the sum.

The statement in the last line ($j = d-1$) follows from graded Poincaré duality (see (5.18)). $\square$

Example 5.46. For stacked cross-polytopal 3-spheres on 12 vertices Theorem 5.45 yields the following Betti numbers for the linear strand: If we compare them with the

$j \setminus i$	0	1	2	3	4	5	6	7	8
1	0	24	80	116	88	36	8	1	0

TABLE 5.7: The graded Betti numbers $\beta_{i,i+1}$ of stacked cross-polytopal 3-spheres on 12 vertices.

bounds for the Betti numbers of a 3-dimensional balanced normal pseudomanifold on 12 vertices from Theorem 5.37, displayed in Example 5.38, we see that they are smaller in almost all places.

5.7 Concluding remarks

In light of Lemma 5.35 and the analogy between stacked and cross-polytopal stacked spheres, the following conjecture appears to be natural:

Conjecture 5.47. Let Δ be a $(d-1)$-dimensional balanced normal pseudomanifold, with $d \geq 4$ and let $f_0(\Delta) = kd$, for some integer $k \geq 2$. Then

$$\beta_{i,i+1}(\mathbb{F}[\Delta]) \leq \beta_{i,i+1}(\mathbb{F}[\Gamma]),$$

for $\Gamma \in \mathcal{ST}^{\times}(kd, d)$, and for every $i \geq 0$. Moreover if equality is attained for any $i \geq 0$, then $\Delta \in \mathcal{ST}^{\times}(kd, d)$.

Conjecture 5.47 mimics once more the behaviour of graded Betti numbers for arbitrary normal pseudomanifolds, which is the main ingredient in the proof of Theorem 1.49 in [Mur15]. Indeed, from a sharp upper bound for the numbers $\beta_{i,i+1}$, it follows directly a sharp upper bound for the *sigma number* σ_0 (see e.g., [Mur15, Corollary 5.8]), which yields the inequality in Theorem 1.49. Therefore we ask if the bound in Conjecture 5.47 implies, possibly via a modification of the sigma numbers, the balanced analog of Theorem 1.49.

Question 5.48. Can a positive answer to Conjecture 5.47 be used to extend the validity of Theorem 2.23 to all normal d-pseudomanifolds, for $d \geq 3$?

Finally, we mention another way to obtain interesting bounds for the graded Betti numbers of a balanced complex. It is well known that the graded Betti numbers of an homogeneous ideal are bounded from above by those of any of its initial ideals ([HH11, Theorem 3.3.1]), and therefore also by those of its *generic initial ideal* (see [HH11, Chapter 4]). One can consider a multigraded version of the generic initial ideal, on which the action considered on the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ is not due to $GL_n(\mathbb{F})$, but to products of smaller groups $GL_{n_i}(\mathbb{F})$. In [Mur08] a certain squarefree operator is described, preserving the multigrading (i.e., the coloring) and the graded Betti numbers. Applying this operator on the multigraded generic initial ideal of the Stanley-Reisner ideal of a balanced complex Δ, we can associate to Δ a *color shifted* balanced simplicial complex with larger Betti numbers and several interesting properties, for which we refer to [BN04; Mur08]. For example, if Δ is pure then the simplicial complex obtained from this process is shellable [Mur08, Proposition 4.2].

Chapter 6

Balanced manifolds on few vertices and an implementation of cross-flips

6.1 Introduction

The study of the number of faces in each dimension that a triangulation of a manifold M can have is a very classical and hard problem in combinatorial topology. On an apparently simpler level one can ask the following question: what is the minimum number of vertices needed to triangulate a manifold M? Again the picture is far from being complete. An interesting tool to approach these and other kind of problems are bistellar flips as defined in Chapter 4, a finite set of local moves which preserves the PL-homeomorphism type, and suffices to connect any two combinatorial triangulations of a given manifold (equivalently, triangulations of PL-manifolds). Björner and Lutz [BL00] designed a computer program called BISTELLAR, employing bistellar flips in order to obtain triangulations on few vertices and heuristically recognize the homeomorphism type. This tool led to a significant number of small or even vertex-minimal triangulations which are listed in The Manifold Page [Lut], along with many other interesting examples (see also [BL14] and [BL00]).
In this chapter we ask ourselves the following question: what is the minimum number of vertices that a balanced triangulation of a manifold M can have? As we saw in Chapter 4, Izmestiev, Klee and Novik introduced a finite set of local moves called *cross-flips*, which preserves balancedness, the PL-homeomorphism type, and suffices to connect any two balanced combinatorial triangulations of a manifold. We provide a primitive computer program implemented in Sage [The17] to search through the set of balanced triangulations of a manifold and we obtain the following results:

- We find balanced triangulations of surfaces on few vertices. In particular we describe the unique vertex-minimal balanced triangulation of $\mathbb{R}\mathbf{P}^2$ on 9 vertices.
- We find a balanced triangulation of the dunce hat on 11 vertices. Section 6.3.2 is devoted to the proof of its vertex-minimality.
- In Section 6.4 we discuss balanced triangulations of 3-manifolds on few vertices. In particular we exhibit a vertex-minimal balanced triangulation of $\mathbb{R}\mathbf{P}^3$ on 16 vertices with interesting symmetries, and triangulations of the connected sums $(S^2 \times S^1)^{\#2}$ and $(S^2 \underline{\times} S^1)^{\#2}$ that belong to the balanced Walkup class.
- Finally in Section 6.4.4 we construct balanced 3-spheres on few vertices that are non-shellable or shellable but not vertex decomposable, using results in knot theory.

The source code and the list of facets of all simplicial complexes appearing in this paper are made available at [Ven].

Remember that balancedness is not a topological property: in general it is possible to turn any triangulation Δ of a topological space into a balanced one by considering its barycentric subdivision $\mathrm{Bd}(\Delta) = \{\{v_{F_1}, \dots, v_{F_m}\} : \emptyset \neq F_1 \subsetneq \cdots \subsetneq F_m, F_i \in \Delta\}$. Indeed in Chapter 2 we remarked that for the order complex of a ranked poset, the rank function gives a coloring which yields balancedness. Moreover in Definition 4.9 we defined a cross-flips on Δ to be a transformations of the form

$$\Delta \longmapsto \chi_\Phi(\Delta) := (\Delta \setminus \Phi) \cup (\partial \mathcal{C}_{d+1} \setminus \Phi)\,,$$

for an induced subcomplex $\Phi \subseteq \Delta$ that is isomorphic to a shellable and coshellable subcomplex of $\partial\mathcal{C}_{d+1}$. The work of Izmestiev, Klee and Novik [IKN17] specializes the theory of flips to the balanced setting using these operations.

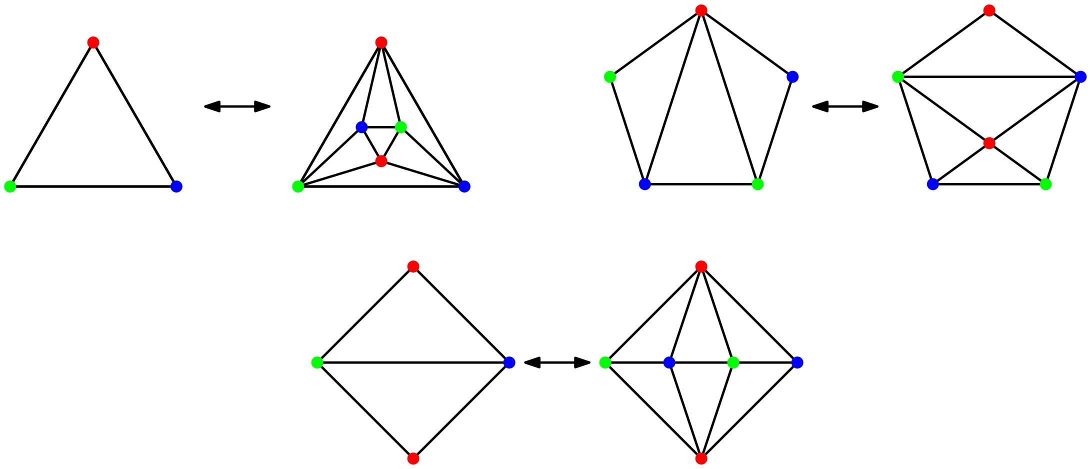

FIGURE 6.1: All non-trivial basic cross-flips for $d = 2$.

We now recall the family of subcomplexes $\lozenge^d(\Gamma_I^d)$ defined in Chapter 4, which describe all basic cross-flips (see Theorem 4.42). In order to shorten the notation we set $\Phi_I := \lozenge^d(\Gamma_I^d)$. For $0 \leq i \leq d+1$ it follows from Lemma 4.35 that

$$\Phi_{\{i\}} = \begin{cases} \langle\{v_0\}\rangle * \langle\{i+1\}, \{v_{i+1}\}\rangle * \cdots * \langle\{d\}, \{v_d\}\rangle & \text{for } i = 0 \\ \langle\{0, \dots, i-1, v_i\}\rangle * \langle\{i+1\}, \{v_{i+1}\}\rangle * \cdots * \langle\{d\}, \{v_d\}\rangle & \text{for } 1 \leq i \leq d\,, \\ \langle\{0, \dots, d\}\rangle & \text{for } i = d+1 \end{cases}$$

and let $\Phi_I := \bigcup_{i \in I} \Phi_{\{i\}}$, for every $I \subseteq [d+1]$. As discussed in Chapter 4 those complexes are indeed shellable subcomplexes of the boundary of the $(d+1)$-dimensional cross-polytope, and a cross-flip replacing a subcomplex Φ_I with its complement Φ_J (note that this family is closed under taking complement w.r.t. $\partial\mathcal{C}_{d+1}$) is called a basic cross-flip. The basic cross-flip replacing $\Phi_{\{0\}}$ with $\partial\mathcal{C}_{d+1} \setminus \Phi_{\{0\}} \cong \Phi_{\{0\}}$ is referred to as trivial flip, because it clearly does not affect the combinatorics, and every non-trivial basic cross-flips either increases or decreases the number of vertices. We refer to the former as *up-flips* and to the latter as *down-flips*. Moreover two distinct sets $I \neq J \subseteq [d+1]$ might lead to isomorphic subcomplexes $\Phi_I \cong \Phi_J$, and certain basic cross-flips can be generated (i.e. written as combination) by some others. As depicted in Figure 4.14, the flips in the second line of Figure 6.1 (we count the arrows separately) can be obtained via a combination of the four moves in the first row. These issues, as well as a description of the possible f-vectors of the complexes Φ_I, have been studied in Chapter 4. In particular in Theorem 4.42 we established that there are precisely $2^{d+1} - 2$ non isomorphic non-trivial basic cross-flips in dimension d. Moreover 2^d of them suffice to generate them all.

The case of surfaces has been also studied in [MS18; KMS19]. The interest in cross-flips, and in particular in basic cross-flips, is mainly due to Theorem 4.10. This result by Izmestiev, Klee and Novik [IKN17] states that any two balanced PL-homeomorphic combinatorial manifolds can be transformed one into the other by a finite sequence of cross-flips, or even basic cross-flips. This serves as a motivation to develop an implementation of these moves, as it was done in the setting of bistellar flips by Björner and Lutz in [BL00] with BISTELLAR. In particular our goal is to find balanced triangulations of a given manifold on few vertices, since taking barycentric subdivision typically leads to large complexes.

Remark 6.1. We conclude this section by offering a way to visualize the results above. Consider a graph whose vertices are all the balanced combinatorial triangulation of a certain manifold M, and whose edges are basic cross-flips. We call this graph the *cross-flip graph* of M, and observe that **??** states that this graph is connected. Furthermore we can associate to M another connected graph on the same vertex set, but with edges the sufficient flips guaranteed in Theorem 4.42. We call this graph the *reduced cross-flip graph*. Figure 6.2 shows a plot of a subgraph of the reduced cross-flip graph, displayed by ranking the (all non-isomorphic) complexes according to the number of vertices (see the numbers on the left). Here we stop performing up-flips on a sphere Δ if $f_0(\Delta) \geq 14$. Note that there is no guarantee of enumerating *all* the balanced spheres in this way: as an example all the spheres in Figure 6.2 with $f_0(\Delta) = 16$ satisfy $f_1(\Delta) \leq 72$, whereas in [Zhe16] a balanced 3-sphere on 16 vertices with 96 edges is constructed.

J	$f(\Phi_J)$	K	$f(\partial\mathcal{C}_{d+1} \setminus \Phi_J) = f(\Phi_K)$
$[3]$	$(1,4,6,4,1)$	$[0,1,2,3]$	$(1,8,24,32,15)$
$[2]$	$(1,5,9,7,2)$	$[0,1,2]$	$(1,8,24,31,14)$
$[2,3]$	$(1,6,12,10,3)$	$[0,1,3]$	$(1,8,24,30,13)$
$[1]$	$(1,6,13,12,4)$	$[0,1]$	$(1,8,23,28,12)$
$[1,3]$	$(1,7,16,15,5)$	$[0,2,3]$	$(1,8,23,27,11)$
$[1,2]$	$(1,7,17,17,6)$	$[0,2]$	$(1,8,22,25,10)$
$[1,2,3]$	$(1,7,18,19,7)$	$[0,3]$	$(1,8,21,23,9)$
$[0]$	$(1,7,18,20,8)$	$[0]$	$(1,7,18,20,8)$

Table 6.1: The f-vectors of 3-dimensional basic cross-flips.

6.2 The implementation

The main purpose of our implementation is to obtain small, possibly vertex-minimal, balanced triangulations of surfaces and 3-manifolds. To achieve this we start from the barycentric subdivision of a non-balanced triangulation, many of which can be found in [Lut], and reduce them using cross-flips. We first establish some notations: a vertex $v \in \Delta$ is called *removable* if there exists a down flip χ_Φ such that $v \notin \chi_\Phi(\Delta)$. A balanced simplicial complex without removable vertices is called *irreducible*. In Figure 6.2 irreducible triangulations of S^3 can be visualized as vertices not connected with any lower vertex.

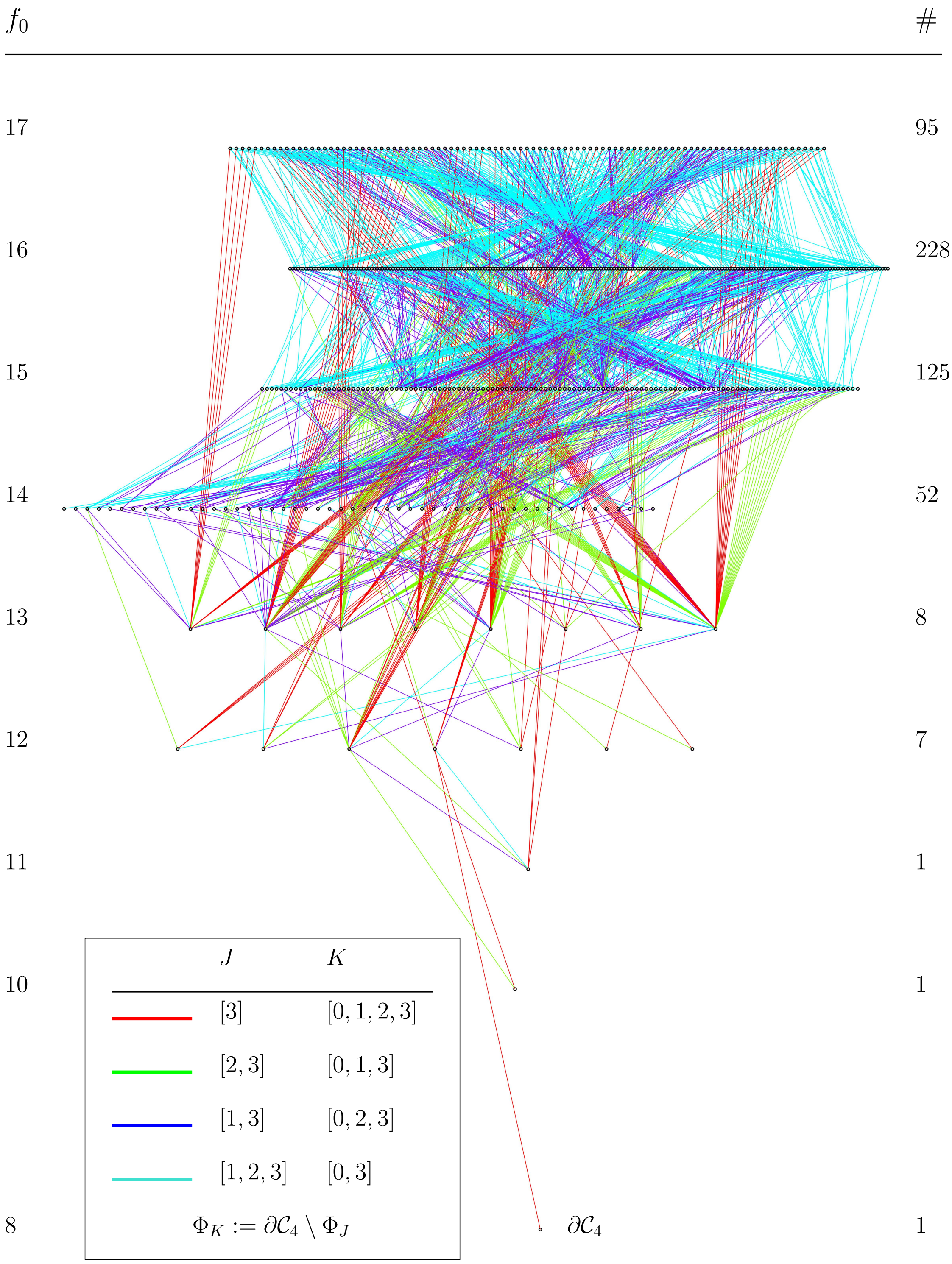

FIGURE 6.2: The subgraph of the reduced cross-flip graph of S^3 obtained applying the 4 sufficient cross-flips $\Delta \longmapsto \chi_{\Phi_J}(\Delta)$, starting from $\partial\mathcal{C}_4$. Edges represent flips, and different colors correspond to different flips. Up-flips are performed for $f_0 < 14$.

Remark 6.2. While a vertex-minimal balanced triangulation is clearly irreducible, the converse is not true. Indeed there are quite many irreducible triangulations, and they can have a large vertex set, as shown in Corollary 6.4.

Lemma 6.3. *Let Δ be a pure d-dimensional balanced simplicial complex. If a vertex $v \in \Delta$ is removable, then $f_0(\mathrm{lk}_\Delta(v)) = 2d$.*

Proof. If the vertex v is removable, then there exists an induced subcomplex $\Gamma \subseteq \Delta$ that is isomorphic to a subcomplex of $\partial\mathcal{C}_{d+1}$, such that Γ is a d-ball and v is in the interior of Γ, because vertices on the boundary are preserved. Since the link of a vertex in the interior of a balanced d-ball is a balanced $(d-1)$-sphere, and the only such subcomplex of $\partial\mathcal{C}_{d+1}$ is isomorphic to $\partial\mathcal{C}_d$ it follows that $\mathrm{lk}_\Delta(v) \cong \partial\mathcal{C}_d$, hence $f_0(\mathrm{lk}_\Delta(v)) = 2d$. □

Corollary 6.4. *Let Δ be a combinatorial d-manifold, with $d \geq 3$. Then the barycentric subdivision $Bd(\Delta)$ is irreducible.*

Proof. For every vertex $v_F \in \mathrm{Bd}(\Delta)$, corresponding to a k-face $F \in \Delta$ we have $\mathrm{lk}_{\mathrm{Bd}(\Delta)}(v_F) \cong \mathrm{Bd}(\partial\Delta_k) * \mathrm{Bd}(\mathrm{lk}_\Delta(F))$. Moreover, since $\mathrm{lk}_{\mathrm{Bd}(\Delta)}(v_F)$ is a combinatorial $(d-k-1)$-sphere, it has at least $f_i(\partial\Delta_{d-k})$ i-faces. Hence

$$\begin{aligned} f_0(\mathrm{lk}_{\mathrm{Bd}(\Delta)}(v_F)) &= 2^{k+1} - 2 + \sum_{i=0}^{d-k-1} f_i(\mathrm{lk}_\Delta(F)) \\ &\geq 2^{k+1} - 2 + \sum_{i=0}^{d-k-1} f_i(\partial\Delta_{d-k}) \\ &= 2^{d-k+1} + 2^{k+1} - 4. \end{aligned}$$

For a fixed d the last expression is minimized when $k = \frac{d}{2}$, and in that case we have $f_0(\mathrm{lk}_{\mathrm{Bd}(\Delta)}(v_F)) \geq 4\left(2^{\frac{d}{2}} - 1\right)$, which is strictly larger than $2d$, for $d \geq 3$. □

Note that Corollary 6.4 holds for more general classes, e.g., homology manifolds, for which the inequality $f_i(\mathrm{lk}_\Delta(F)) \geq f_i(\partial\Delta_{d-k})$ still holds for every k-face F and for every i. The computation above also shows that for $d = 2$ since every vertex v_F arising from the subdivision of an edge F has degree 4, it is potentially removable. Since the barycentric subdivision is the standard way of turning any triangulation into a balanced triangulation of the same space, for $d \geq 3$ the result above represents a bad news. Indeed Corollary 6.4 states that to reduce such a subdivision we are forced to start with some up-flips and to increase the number of vertices, which for the case of barycentric subdivisions is typically quite large. Our code meets two main challenges:

- List all the flippable subcomplexes of any type;
- Decide which type of move to apply and which subcomplex to flip.

Already in dimension 1 the problem of deciding if a fixed complex has a subcomplex isomorphic to a given one, known as the *subgraph isomorphism problem*, is NP-complete. However since graphs are computationally well studied it is convenient to reduce the problem to the one dimensional case, to employ structures and algorithms designed for graphs. Recall that a pure strongly connected d-dimensional simplicial complex is a pseudomanifold if every $(d-1)$-face is contained in exactly two facets.

Definition 6.5. For a pure d-dimensional pseudomanifold Δ the *dual graph* $G(\Delta)$ is the graph on the vertex set $\{F \in \Delta : \dim(F) = d\}$ and with edge set $\{\{F_i, F_j\} : \dim(F_i \cap F_j) = d-1\}$.

Given a d-dimensional pseudomanifold Δ and a pure subcomplex $\Phi \subseteq \partial\mathcal{C}_{d+1}$ that is a ball, we list all subgraphs of $G(\Delta)$ that are isomorphic to $G(\Phi)$ using an algorithm such as the VF2 algorithm [Cor+04], from which we only keep those that correspond to induced subcomplexes. Moreover once a flip $\Delta \longmapsto \chi_\Phi(\Delta) =: \Delta'$ is performed we do not need to rerun the check on the entire complex to list all the flippable subcomplexes of Δ', but it suffices to update the list locally, by considering only the induced subcomplexes of Δ' that are not induced subcomplexes of Δ. Even though this idea allows to deal with relatively large 3-dimensional complexes, higher dimensions appear to be still out of reach.
For the second problem, namely to decide which subcomplex to flip, we propose and combine two very naive strategies: given a balanced pseudomanifold Δ we choose a subcomplex Φ among those which

- maximize $\left|\left\{v \in \chi_\Phi(\Delta) : f_0(\mathrm{lk}_{\chi_\Phi(\Delta)}(v)) = 2d\right\}\right|$,
- maximize $\displaystyle\sum_{v\in\chi_\Phi(\Delta),\dim(v)=0} (f_0(\mathrm{lk}_{\chi_\Phi(\Delta)}(v)))^2$.

The two optimization steps are applied following the order above. With the first condition we simply maximize the number of potentially removable vertices, while maximizing the sum of squares of the vertex degrees forces the new triangulation to have an inhomogeneous degree distribution, and hence some very poorly connected vertices.

Remark 6.6. Typically, starting from a large triangulation, we cannot hope to reduce drastically the number of vertices through a sequence consisting only of down-flips, because irreducible triangulations are quite frequent. Even a restricted example like Figure 6.2 reveals several irreducible triangulations of S^3 on few vertices. We can overcome this inconvenience by interposing a certain number of random up-flips to avoid local minima.

Remark 6.7. We make no claim of efficiency, and we do not take into account the time needed to reduce a triangulation. Undoubtedly many details in the implementation can be improved, and the strategies refined. So far we were able to obtain small balanced f-vectors of all the 3-dimensional examples considered.

6.3 Surfaces and the Dunce Hat

The first complexes we consider are triangulations of compact 2-manifolds. In this case the number of vertices uniquely determines the remaining entries of the f-vector. In Table 6.2 we display a list of minimal known f-vectors of several surfaces, as well as the f-vectors of the corresponding barycentric subdivisions, which is always the starting input for our procedure. Finally in the fourth column we report the smallest known f-vectors of balanced triangulations found via the program.

6.3.1 Real projective plane

We found a unique vertex-minimal balanced triangulation $\Delta_9^{\mathbb{R}\mathbf{P}^2}$ of the real projective plane, which is depicted in two ways in Figure 6.3. The f-vector is $f(\Delta_9^{\mathbb{R}\mathbf{P}^2}) = (1, 9, 24, 16)$. The non-balanced minimal triangulation has 6 vertices.

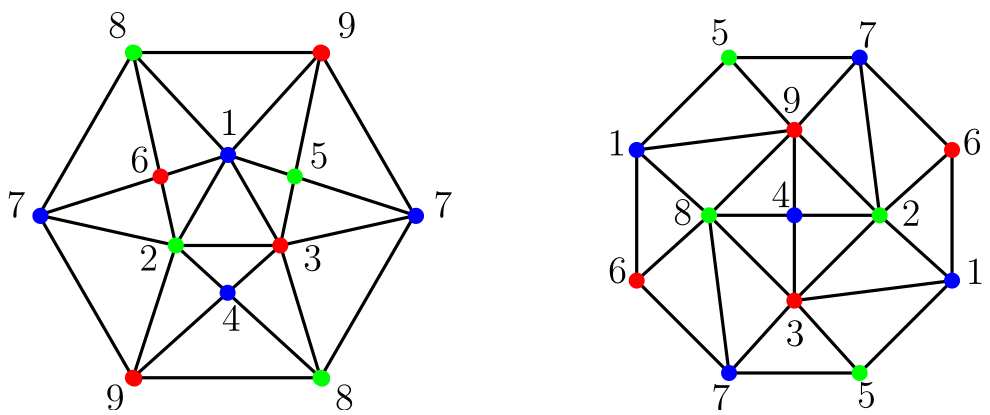

FIGURE 6.3: The simplicial complex $\Delta_9^{\mathbb{R}\mathbf{P}^2}$ represented as the quotient of a disk in two different ways.

Lemma 6.8. *The simplicial complex $\Delta_9^{\mathbb{R}\mathbf{P}^2}$ is a vertex (hence every f_i) minimal balanced triangulation of the projective plane.*

Proof. The claim follows from Corollary 2.14. □

Remark 6.9. Since triangulated surfaces on 9 vertices are listed in [Lut], we can check that $\Delta_9^{\mathbb{R}\mathbf{P}^2}$ is indeed the unique balanced triangulation of $\mathbb{R}\mathbf{P}^2$ on 9 vertices.

$\lvert\Delta\rvert$	Min $f(\Delta)$	$f(\mathrm{Bd}(\Delta))$	Min. Bal. f known	Notes
S^2	$(1,4,6,4)$	$(1,14,36,24)$	$(1,6,12,8)^*$	$\partial\mathcal{C}_3$
$\mathbb{T}$	$(1,7,21,14)$	$(1,42,126,84)$	$(1,9,24,16)^*$	see [KN16b]
$\mathbb{T}^{\#2}$	$(1,10,36,24)$	$(1,70,216,144)$	$(1,12,42,28)$	
$\mathbb{T}^{\#3}$	$(1,10,42,28)$	$(1,80,252,168)$	$(1,14,54,36)$	
$\mathbb{T}^{\#4}$	$(1,11,51,34)$	$(1,96,306,204)$	$(1,14,60,36)$	
$\mathbb{T}^{\#5}$	$(1,12,60,40)$	$(1,112,360,240)$	$(1,16,72,48)$	
$\mathbb{R}\mathbf{P}^2$	$(1,6,15,10)$	$(1,31,90,60)$	$(1,9,24,16)^*$	$\Delta_9^{\mathbb{R}\mathbf{P}^2}$
$(\mathbb{R}\mathbf{P}^2)^{\#2}$	$(1,8,24,16)$	$(1,48,144,96)$	$(1,11,33,22)$	
$(\mathbb{R}\mathbf{P}^2)^{\#3}$	$(1,9,30,20)$	$(1,59,180,120)$	$(1,12,39,26)$	
$(\mathbb{R}\mathbf{P}^2)^{\#4}$	$(1,9,33,22)$	$(1,64,198,132)$	$(1,12,42,28)$	
$(\mathbb{R}\mathbf{P}^2)^{\#5}$	$(1,9,36,24)$	$(1,69,216,144)$	$(1,13,48,32)$	

TABLE 6.2: Some small f-vectors of balanced surfaces.

6.3.2 Dunce hat

The dunce hat is a topological space which exhibits interesting properties: it is contractible but non-collapsible, and its triangulations are Cohen-Macaulay over any field but not shellable. It can be visualized as a triangular disk whose edges are identified via a non-coherent orientation. Surprisingly, even if it is possible to construct a balanced triangulation by allowing only 3 vertices to be on the boundary of such disk, the vertex-minimal one, which is depicted in Figure 6.4, is achieved when the singularity contains 4 vertices. Its f-vector is $f(\Delta^{\mathrm{DH}}) = (1,11,34,24)$. In the rest of this section we prove that this is indeed the least number of vertices that a balanced triangulation of the dunce hat can have. We call the *singularity* of a triangulation of the dunce hat the 1-dimensional subcomplex of faces whose link is not a sphere. Note that the link of any edge in the singularity consists of three isolated vertices. The set of faces whose link is a sphere is called the *interior* of the dunce hat, and it coincides with the interior of the triangular disk.

Since the dunce hat is not a manifold, the number of vertices of a triangulation does not uniquely determine the other face numbers, but the number of vertices involved in the singularity also plays a role. If we let f_0^{sing} be this number, then the f-vector $(1, f_0, f_1, f_2)$ of any triangulation of the dunce hat satisfies the following equations:

$$\begin{cases} f_0 - f_1 + f_2 = 1 \\ f_0^{\text{sing}} + 2f_1 - 3f_2 = 0 \end{cases} . \tag{6.1}$$

In particular it holds that $f_1 = f_0^{\text{sing}} + 3f_0 - 3$. We proceed now with a sequence of lemmas leading to Proposition 6.16, proving that the triangulation in Figure 6.4 is indeed a balanced vertex-minimal triangulation of the dunce hat.

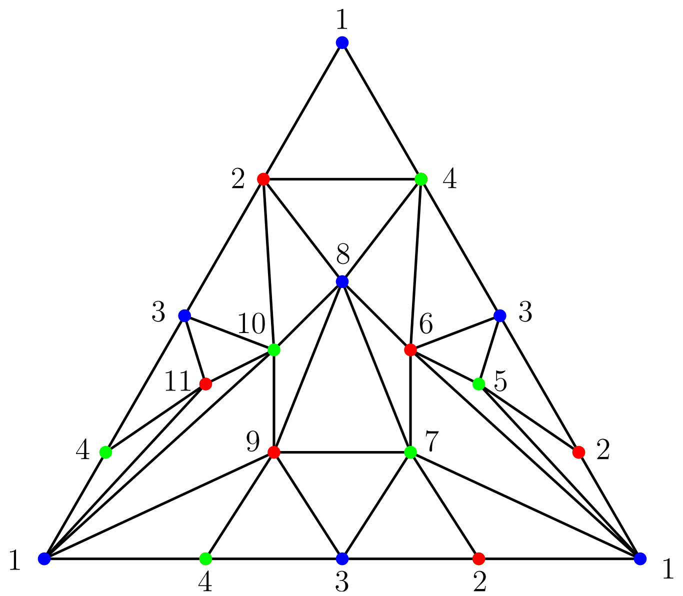

FIGURE 6.4: Minimal balanced triangulation Δ^{DH} of the dunce hat.

Lemma 6.10. *Let Δ be a balanced 2-dimensional Cohen-Macaulay complex that is not shellable. Then each color class contains at least two vertices.*

Proof. Assume there exists a color class containing only one vertex v. As discussed in Lemma 2.13 it follows that Δ is a cone over the 1-dimensional subcomplex $\text{lk}_\Delta(v)$. It is well known that the links of a Cohen-Macaulay simplicial complex are Cohen-Macaulay (see e.g., [Sta96]). But since every 1-dimensional Cohen-Macaulay complex is shellable, and coning preserves shellability this implies that Δ is shellable. □

Lemma 6.11. *Let Δ be a balanced 2-dimensional Cohen-Macaulay complex that is not shellable. Assume moreover that every edge of Δ is contained in at least two triangles. Then each color class contains at least three vertices.*

Proof. By Lemma 6.10 we can assume that there are only two vertices of color 1, say v_1 and v_2. Let $\Delta_{[23]}$ be the subcomplex generated by all faces of Δ not containing color 1. Since we assumed that every edge of Δ is contained in at least two triangles it follows that every edge e of $\Delta_{[23]}$ is contained in the two triangles $e \cup \{v_1\}$ and

$e \cup \{v_2\}$. This implies that Δ is obtained taking the suspension of $\Delta_{[23]}$, and hence $\Delta_{[23]} = \mathrm{lk}_\Delta(v_1) = \mathrm{lk}_\Delta(v_2)$. In particular $\Delta_{[23]}$ is Cohen-Macaulay and 1-dimensional, hence shellable. We deduce that Δ is the suspension over the shellable complex $\Delta_{[23]}$ and hence shellable, since suspension preserves shellability. $\square$

Lemma 6.12. *If $f_0^{sing}(\Delta) = 3$, then $f_0(\Delta) \geq 10$.*

Proof. Observe that by Lemma 6.11 we need at least 9 vertices to triangulate the dunce hat in a balanced way, and the only possible configuration is $(n_1, n_2, n_3) = (3, 3, 3)$, where n_i is the number of vertices of color i. Moreover note that in this case the singularity consists of one vertex per color class. Let us consider the edge e in the singularity containing the colors 1 and 2. Since there are three copies of e in the boundary of the disk, each of which needs to be completed to a triangle using an interior vertex of color 3, we infer that $n_3 \geq 4$, because another vertex of color 3 already lies in the singularity. $\square$

Remark 6.13. The bound in Lemma 6.12 is far from being tight. Using our computer program the vertex-minimal balanced triangulation obtained for the case $f_0^{\mathrm{sing}}(\Delta) = 3$ has 14 vertices. However Lemma 6.12 combined with Lemma 6.15 suffices for our purpose.

We call a pair of vertices i, j, such that $\{i, j\} \notin \Delta$ and $\kappa(i) \neq \kappa(j)$ a *bichromatic missing edge.*

Lemma 6.14. *Let Δ be a balanced triangulation of the dunce hat. Then $f_0(\Delta) \geq 10$. Let m be the number of bichromatic missing edges. If $f_0^{sing}(\Delta) + m \geq 7$, then $f_0(\Delta) \geq 11$.*

Proof. For any balanced 2-dimensional simplicial complex with n_i vertices of color i, for $i = 1, 2, 3$, the number of edges is clearly bounded from above by the number of edges of the complete 3-partite graph K_{n_1,n_2,n_3}, which equals $|E(K_{n_1,n_2,n_3})| = n_1n_2 + n_1n_3 + n_2n_3$. Combined with Lemma 6.12, which allows us to assume $f_0^{\mathrm{sing}}(\Delta) \geq 4$, this yields

$$f_1(\Delta) = f_0^{\mathrm{sing}}(\Delta) + 3f_0(\Delta) - 3 \leq n_1n_2 + n_1n_3 + n_2n_3 \leq \frac{f_0(\Delta)^2}{3}, \tag{6.2}$$

where the last inequality follows by maximizing the function $n_1n_2 + n_1n_3 + n_2n_3$, under the constraint $\sum_{i=1}^{3} n_i = f_0(\Delta)$. Solving the inequality $f_0^{\mathrm{sing}}(\Delta) + 3f_0(\Delta) - 3 \leq \frac{f_0(\Delta)^2}{3}$ for $f_0(\Delta)$ we obtain

$$f_0(\Delta) \geq \frac{9 + \sqrt{81 + 12f_0^{\mathrm{sing}}(\Delta) - 36}}{2}.$$

By Lemma 6.12 we can assume $f_0^{\mathrm{sing}}(\Delta) \geq 4$, from which it follows that $f_0(\Delta) \geq 9,32$. The second statement follows in the same way by imposing $f_1(\Delta) \leq n_1n_2 + n_1n_3 + n_2n_3 - m$ in the first inequality, which yields

$$f_0(\Delta) \geq \frac{9 + \sqrt{81 + 12(f_0^{\mathrm{sing}}(\Delta) + m) - 36}}{2}.$$

Under the assumption $f_0^{\mathrm{sing}}(\Delta) + m \geq 7$ we obtain $f_0(\Delta) \geq 10,18$. $\square$

In order to show that the minimum number of vertices for a balanced triangulation of the dunce hat is 11 it remains to show that no such simplicial complex exists with $f_0(\Delta) = 10$ and $f_0^{\text{sing}}(\Delta) \in \{3, 4, 5, 6\}$.

Lemma 6.15. *No balanced triangulation of the dunce hat on 10 vertices exists.*

Proof. Since any triangulation Δ of the dunce hat is Cohen-Macaulay, non-shellable, and has the property that every edge is contained in two or three triangles, Lemma 6.10 and Lemma 6.11 imply that every color class of Δ contains at least three vertices. Assume a balanced triangulation Δ on 10 vertices exists. There is a unique way to partition 10 vertices in three classes, such that every class contains at least three, namely $(n_1, n_2, n_3) = (3, 3, 4)$, where n_i is the number of vertices of color i. Moreover, due to Lemma 6.14, we can assume that $f_0^{\text{sing}}(\Delta) \in \{3, 4, 5, 6\}$. In what follows we denote by n_i^{sing} the number of vertices in the singularity of color i. Note the following facts:

Claim 1: If $f_0^{\text{sing}}(\Delta) = 3$ then there are at least four missing bichromatic edges.

- Since the singularity is a triangle it must be colored using all three colors. This implies that $n_1^{\text{sing}} + n_2^{\text{sing}} \leq 2$, and hence there are at least four interior vertices of color 1 or 2. Denote with v one of these vertices and assume $\kappa(v) = 1$, where κ is the coloring map of Δ. Since the link of v is a polygon with an even number of vertices, but there are 7 remaining vertices of color 2 and 3 (3 and 4 respectively), then there exists a vertex a of color 3 such that $\{v, a\} \notin \Delta$. We obtain in this way a bichromatic missing edge for each of the four interior vertices of color 1 and 2.

Claim 2: If $f_0^{\text{sing}}(\Delta) = 4$ then there are at least three missing bichromatic edges.

- If $(n_1^{\text{sing}}, n_2^{\text{sing}}) \neq (2, 2)$, then there are at least three vertices of color 1 and 2 in the interior of the disk and the link of these vertices is a polygon with an even number of vertices. Let v be one of these vertices, and assume w.l.o.g. that $\kappa(v) = 1$. Since $n_2 + n_3 (= n_1 + n_3) = 7$ only three of the vertices colored with 3 can appear in the link of v. Hence there is at least one missing bichromatic edge for each of the three vertices.
- If $(n_1^{\text{sing}}, n_2^{\text{sing}}) = (2, 2)$, then there are exactly two vertices of color 1 and 2 (say v and w) in the interior of Δ and their link is an even polygon. Again since $n_2 + n_3 = n_1 + n_3 = 7$ each of these two vertices avoids at least one vertex of color 3 and there is at least one missing bichromatic edge for each of the two vertices. Let us denote by $\{v, a\}$ and $\{w, b\}$ these missing edges. If $a \neq b$, then since a is in the interior and since the link of a contains at most two vertices of color 1, it can only contain two vertices of color 2. Hence there is at least a third bichromatic missing edge $\{z, a\}$. If $a = b$, then the link of a is the whole singularity (a square) and, in particular, there exists an edge $\{x, y\}$ in the interior of the disk whose endpoints are in the singularity. This yields a contradiction, because in the case considered, two vertices in the singularity are either connected by an edge in the boundary of the disk, or they have the same color.

Claim 3: If $f_0^{\text{sing}}(\Delta) = 5$, then there are at least two missing bichromatic edges.

- Since the singularity is a 5-gon it must be colored using all the three color classes. This implies that $n_1^{\text{sing}} + n_2^{\text{sing}} \leq 4$, and hence there are at least two interior vertices of color 1 or 2. As in the previous paragraph each of these vertices must avoid at least one vertex of color 3, giving rise to two bichromatic missing edges.

Claim 4: If $f_0^{\text{sing}}(\Delta) = 6$, then there is at least one missing bichromatic edge.

- If $(n_1^{\text{sing}}, n_2^{\text{sing}}) = (3,3)$, then the link of every interior vertex is the whole singularity, hence Δ is the join of a triangulation of S^1 with 4 isolated vertices. This is clearly a contradiction.
- If $(n_1^{\text{sing}}, n_2^{\text{sing}}) \neq (3,3)$, then there is at least one interior vertex of color 1 or 2. Once more its link cannot contain all the 7 remaining vertices of a different color, so it must miss at least one vertex from the color class 3. This produces a bichromatic missing edge.

If we let m be the number of bichromatic missing edges, then the four claims above imply $f_0^{\text{sing}}(\Delta) + m \geq 7$ for any $f_0^{\text{sing}}(\Delta)$. We conclude using Lemma 6.14. □

Proposition 6.16. *The simplicial complex in Figure 6.4 is a vertex-minimal balanced triangulation of the dunce hat.*

Proof. The claim follows combining Lemma 6.12, Lemma 6.14 and Lemma 6.15, which show that no such triangulation exists on less than 11 vertices, for any value of $f_0^{\text{sing}}(\Delta)$. □

6.4 3-manifolds

In this section we report some interesting and small balanced triangulations of 3-manifolds found using our computer program.

6.4.1 Real projective space

We present a peculiar balanced triangulation $\Delta_{16}^{\mathbb{R}\mathbf{P}^3}$ of the real projective space on 16 vertices. An interesting feature of this complex is its strong symmetry: it is *centrally symmetric* (i.e., there is a free involution acting) and all the vertex links are isomorphic to the 2-sphere in Figure 6.5. The list of facets of $\Delta_{16}^{\mathbb{R}\mathbf{P}^3}$ can be found in Appendix A. Since the projective space is homeomorphic to the *lens space* $L(2,1)$, a particular case of the following result of Zheng shows that our triangulation is vertex-minimal.

Proposition 6.17. [Zhe16, Proposition 4.3] *Any balanced triangulation of the lens space $L(p,q)$, with $p > 1$, has at least 16 vertices.*

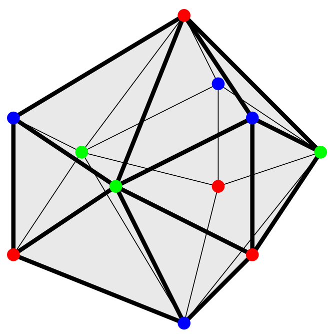

FIGURE 6.5: The (all isomorphic) vertex links of the triangulation $\Delta_{16}^{\mathbb{R}\mathbf{P}^3}$.

6.4.2 Small balanced triangulations of 3-manifolds

In Table 6.3 we report the smallest known f-vectors of balanced triangulations of several 3-manifolds. We point out that some of these triangulations were previously known, and they are referenced through this section. For instance Klee and Novik [KN16b] proved the existence of a d-dimensional simplicial complex on $3d$ vertices which provides a vertex-minimal balanced triangulation of $S^{d-1} \times S^1$ when d is even, and of the twisted bundle $S^{d-1} \underline{\times} S^1$ when d is odd. Moreover they construct balanced triangulations of both $S^{d-1} \times S^1$ and $S^{d-1} \underline{\times} S^1$ on $3d+2$ vertices. This, combined with a result of Zheng ([Zhe16]) proving that any balanced triangulation Δ of a d-manifold with $\beta_1(\Delta;\mathbb{Q}) \neq 0$ and $\beta_2(\Delta;\mathbb{Q}) = 0$ has at least $3d+2$ vertices, shows that the second and third line in Table 6.3 correspond to vertex-minimal balanced triangulations. As previously discussed, via Proposition 6.17 we can conclude that also $\Delta_{16}^{\mathbb{R}\mathbf{P}^3}$ and a triangulation of the lens space $L(3,1)$ constructed in [Zhe16] are balanced vertex-minimal. Most notably the mentioned triangulations of $S^{d-1} \times S^1$ (d even), $S^{d-1} \underline{\times} S^1$ (d odd) and of $L(3,1)$ are *balanced neighborly*, that is they do not have bichromatic missing edges.
In the rest of the table we report the minimal balanced f-vectors achieved for different 3-manifolds, such as several lens spaces $L(p,q)$, connected sums, two additional spherical 3-manifolds called the octahedral space and the cube space, and the Poincaré homology 3-sphere. For a more extensive treatment of this topic we refer to [Lut99].
A classical theorem in topology by Edwards and Cannon states that the k-fold suspension of any homology d-sphere is homeomorphic to S^{d+k}, even though it is not a combinatorial sphere. Since balancedness is preserved by taking suspension we obtain a family of non-combinatorial balanced triangulations of S^d, for $d \geq 5$. For $d = 5$ we report the smallest f-vector known for a non-combinatorial balanced sphere.

Corollary 6.18. *There exists a balanced non-combinatorial 5-sphere with f-vector $(1, 30, 288, 1132, 2106, 1848, 616)$. Moreover by taking further suspensions we obtain balanced non-combinatorial d-spheres on $2d + 20$ vertices, for every $d \geq 5$.*

Remark 6.19. There exists a procedure introduced by Datta to construct the topological suspension by increasing the number of vertices only by one. Unfortunately this one point suspension does not preserve balancedness.

6.4.3 The connected sum of S^2 bundles over S^1 and the balanced Walkup class

The lower bound theorem for manifolds (Theorem 1.49) gives a bound for the number of edges of a triangulation Δ of an $\mathbb{F}$-homology manifold with a certain number of vertices, depending of $\widetilde{\beta}_1(\Delta;\mathbb{F})$. As we discussed it is an interesting refinement of the lower bound theorem for homology spheres, obtained from the study of algebraic invariants of Buchsbaum graded rings. In [JK+18] Juhnke-Kubitzke, Murai, Novik and Sawaske proved Theorem 2.23, a balanced analog of this bound and established a conjecture of Klee and Novik [KN16b, Conjecture 4.14] for the characterization of the case of equality, when the dimension is greater than or equal to 4. In terms of the f-numbers Theorem 2.23 reads as follows.

Theorem 6.20. [JK+18] *Let Δ be a connected d-dimensional balanced $\mathbb{F}$-homology manifold, with $d \geq 3$. Then*

$$2f_1(\Delta) - 3df_0(\Delta) \geq 4\binom{d+1}{2}(\widetilde{\beta}_1(\Delta;\mathbb{F}) - 1). \tag{6.3}$$

Moreover if $d \geq 4$ equality holds if and only if Δ is in the balanced Walkup class $\mathcal{BH}_{d+1}$.

The balanced Walkup class is described in Definition 2.25. Theorem 6.20 leaves unsolved the case of equality when $d = 3$, which is still part of Conjecture 4.14 in [KN16b].

Conjecture 6.21. [KN16b] Let Δ be a connected 3-dimensional balanced $\mathbb{F}$-homology manifold. Then $2f_1(\Delta) - 9f_0(\Delta) = 24(\widetilde{\beta}_1(\Delta;\mathbb{F}) - 1)$ if and only if Δ is in the balanced Walkup class.

Using our computer program we found two balanced triangulation of $(S^2 \times S^1)^{\#2}$ and $(S^2 \underline{\times} S^1)^{\#2}$ respectively with f-vector $(1, 16, 84, 136, 68)$. Since $\widetilde{\beta}_1((S^2 \times S^1)^{\#2};\mathbb{F}) = \widetilde{\beta}_1((S^2 \underline{\times} S^1)^{\#2};\mathbb{F}) = 2$, it is easy to see that both triangulations attain equality in (6.3). In light of Conjecture 6.21 it is natural to ask if these two simplicial complexes belong to the balanced Walkup class. We answer positively by giving an explicit decomposition. In what follows we denote by $\partial\mathcal{C}_4(v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4)$ the boundary of the cross-polytope on the vertex set $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$, such that $\{v_i, w_i\}$ is not an edge for any $i = 1, \dots, 4$.

- $(S^2 \underline{\times} S^1)^{\#2}$:
  - As it was pointed out in the first paragraph of Section 6.4.3 there is a balanced simplicial complex $\Delta_{12}^{S^2 \underline{\times} S^1}$ on 12 vertices which triangulates $S^2 \underline{\times} S^1$. It can be obtained from three copies of $\partial\mathcal{C}_4$, namely $\partial\mathcal{C}_4(x_1, \dots, x_4, y_1', \dots, y_4')$, $\partial\mathcal{C}_4(y_1, \dots, y_4, z_1', \dots, z_4')$ and $\partial\mathcal{C}_4(z_1, \dots, z_4, x_1', \dots, x_4')$, via two connected sums and one handle addition, identifying the vertices v_i and v_i' (see [KN16b]). Clearly it belongs to $\mathcal{BH}_4$.
  - For any facet $F = \{r_1, r_2, r_3, r_4\}$ of $\Delta_{12}^{S^2 \underline{\times} S^1}$ we can take the connected sum with $\partial\mathcal{C}_4(s_1, \dots, s_4, r_1', \dots, r_4')$ and $\partial\mathcal{C}_4(t_1, \dots, t_4, s_1', \dots, s_4')$, again identifying r_i, with r_i' and s_i with s_i'.
  - Finally we can choose any other facet $G = \{u_1, u_2, u_3, u_4\}$ of $\Delta_{12}^{S^2 \underline{\times} S^1}$ such that $F \cap G = \emptyset$ and such that the distance from F and G (measured on the dual graph) is even, and perform balanced handle addition identifying the vertices t_i with u_i. Since the vertices in the link of t_i are a subset of $\{s_1, s_2, s_3, s_4\}$ and $F \cap G = \emptyset$, we conclude that the links of t_i and u_i do not intersect, and hence the handle addition is well defined. Note that the last choice can produce non-isomorphic triangulations of $(S^2 \underline{\times} S^1)^{\#2}$, all of which sit inside $\mathcal{BH}_4$.
- $(S^2 \times S^1)^{\#2}$:
  - With a similar construction as in the previous case we can triangulate the orientable bundle $S^2 \times S^1$ with four copies of $\partial\mathcal{C}_4$, namely $\partial\mathcal{C}_4(x_1, \dots, x_4, y_1', \dots, y_4')$, $\partial\mathcal{C}_4(y_1, \dots, y_4, z_1, \dots, z_4)$, $\partial\mathcal{C}_4(w_1, \dots, w_4, x_1'', y_2'', y_3'', y_4'')$ and $\partial\mathcal{C}_4(w_1', \dots, w_4', y_1'', z_2, z_3, z_4)$. Here we perform three connected sums and a balanced handle addition identifying vertices v_i, v_i' and v_i''. We denote this simplicial complex on 16 vertices by $\Delta_{16}^{S^2 \times S^1}$ (note that it is not the vertex-minimal balanced triangulation of $S^2 \times S^1$).
  - We now pick the facets $F = \{x_1, x_2, x_3, x_4\}$ and $G = \{w_1, z_2, z_3, z_4\}$ of $\Delta_{16}^{S^2 \times S^1}$, and we observe that the link of any vertex in F (respectively G) does not contain any vertex in G (respectively F). Moreover F and G have an even distance (with respect to the dual graph of $\Delta_{16}^{S^2 \times S^1}$).

– Finally we perform a connected sum and subsequent handle addition using $\partial\mathcal{C}_4(x_1''', \dots, x_4''', w_1''', z_2''', z_3''', z_4''')$, where the identifications are between v_i and v_i'''.

Remark 6.22. The description of the construction for $(S^2 \underline{\times} S^1)^{\#2}$ mostly relies on the simplicial complex $\Delta_{12}^{S^2 \underline{\times} S^1}$. Since in [KN16b] the authors showed that the analog construction in dimension d provides a triangulation on $3d$ vertices of $S^{d-1} \underline{\times} S^1$ if d is even and of $S^{d-1} \times S^1$ if d is odd, the extra handle that we add provides triangulations on $4d$ vertices of $(S^{d-1} \underline{\times} S^1)^{\#2}$ if d is odd and of $(S^{d-1} \times S^1)^{\#2}$ if d is even.

$\lvert\Delta\rvert$	Min $f(\Delta)$	$f(\mathrm{Bd}(\Delta))$	Min. Bal. f known	Notes
S^3	$(1, 5, 10, 10, 5)$	$(1, 30, 150, 240, 120)$	$(1, 8, 24, 32, 16)^*$	$\partial\mathcal{C}_4$
$S^2 \times S^1$	$(1, 10, 42, 64, 32)$	$(1, 148, 916, 1536, 768)$	$(1, 14, 64, 100, 50)^*$	see [KN16b]
$S^2 \underline{\times} S^1$	$(1, 9, 36, 54, 27)$	$(1, 126, 774, 1296, 648)$	$(1, 12, 54, 84, 42)^*$	see [KN16b]
$\mathbb{R}\mathbf{P}^3$	$(1, 11, 51, 80, 40)$	$(1, 182, 1142, 1920, 960)$	$(1, 16, 88, 144, 72)^*$	$\Delta_{16}^{\mathbb{R}\mathbf{P}^3}$
$L(3, 1)$	$(1, 12, 66, 108, 54)$	$(1, 240, 1536, 2592, 1296)$	$(1, 16, 96, 160, 80)^*$	see [Zhe16]
$L(4, 1)$	$(1, 14, 84, 140, 70)$	$(1, 308, 1988, 3360, 1680)$	$(1, 20, 132, 224, 112)$	
$L(5, 1)$	$(1, 15, 97, 164, 82)$	$(1, 358, 2326, 3936, 1968)$	$(1, 22, 152, 260, 130)$	
$L(5, 2)$	$(1, 14, 86, 144, 72)$	$(1, 316, 2044, 3456, 1728)$	$(1, 20, 132, 224, 112)$	
$L(6, 1)$	$(1, 16, 110, 188, 94)$	$(1, 408, 2664, 4512, 2256)$	$(1, 24, 176, 304, 152)$	
$(S^2 \times S^1)^{\#2}$	$(1, 12, 58, 92, 46)$	$(1, 208, 1312, 2208, 1104)$	$(1, 16, 84, 136, 68)$	see 6.4.3
$(S^2 \underline{\times} S^1)^{\#2}$	$(1, 12, 58, 92, 46)$	$(1, 208, 1312, 2208, 1104)$	$(1, 16, 84, 136, 68)$	see 6.4.3
$(S^2 \times S^1)\#\mathbb{R}\mathbf{P}^3$	$(1, 14, 73, 118, 59)$	$(1, 264, 1680, 2832, 1416)$	$(1, 20, 118, 196, 98)$	
$(\mathbb{R}\mathbf{P}^3)^{\#2}$	$(1, 15, 86, 142, 71)$	$(1, 314, 2018, 3408, 1704)$	$(1, 21, 137, 232, 116)$	
$(S^2 \times S^1)^{\#3}$	$(1, 13, 72, 118, 59)$	$(1, 262, 1678, 2832, 1416)$	$(1, 20, 118, 196, 98)$	
$(S^2 \underline{\times} S^1)^{\#3}$	$(1, 13, 72, 118, 59)$	$(1, 262, 1678, 2832, 1416)$	$(1, 19, 111, 184, 92)$	
$S^1 \times S^1 \times S^1$	$(1, 15, 105, 180, 90)$	$(1, 390, 2550, 4320, 2160)$	$(1, 24, 168, 288, 144)$	
Oct. space	$(1, 15, 102, 174, 87)$	$(1, 378, 2466, 4176, 2088)$	$(1, 24, 168, 288, 144)$	
Cube space	$(1, 15, 90, 150, 75)$	$(1, 330, 2130, 3600, 1800)$	$(1, 23, 157, 268, 134)$	
Poincaré	$(1, 16, 106, 180, 90)$	$(1, 392, 2552, 4320, 2160)$	$(1, 26, 180, 308, 154)$	
$\mathbb{R}\mathbf{P}^2 \times S^1$	$(1, 14, 84, 140, 70)$	$(1, 308, 1988, 3360, 1680)$	$(1, 24, 156, 264, 132)$	
Triple-trefoil	$(1, 18, 143, 250, 125)$	$(1, 536, 3536, 6000, 3000)$	$(1, 28, 204, 352, 176)$	Δ_{28}^{3T}
Double-trefoil	$(1, 16, 108, 184, 92)$	$(1, 400, 2608, 4416, 2208)$	$(1, 22, 136, 228, 114)$	Δ_{22}^{2T}

TABLE 6.3: Some small f-vectors of balanced 3-manifolds.

6.4.4 Non-vertex decomposable and non-shellable balanced 3-spheres.

In this paragraph we exhibit two interesting balanced triangulations of the 3-sphere, namely one that is shellable but not vertex decomposable and a second one which is not constructible, and hence not shellable. In Section 1.3 we discussed the hierarchy which relates these three properties: every vertex decomposable simplicial complex is indeed shellable, and every shellable complex is constructible. In particular while there exist shellable 3-spheres which are not vertex decomposable, the existence of constructible, but not shellable 3-spheres is still open. In order to obtain interesting, possibly small balanced triangulations we again start from the barycentric subdivision of two distinct triangulations of the 3-sphere with a sufficiently complicated knot embedded in their skeleton. For instance we turn our attention to the connected sum of 2 or 3 *trefoil knots*, called a *double-trefoil* and a *triple-trefoil*. The reason for this choice is that in general the barycentric subdivision might turn non-shellable simplicial complexes into shellable ones, while complicated knots are obstructions to shellability even after the subdivision. We employ the following rephrasing of results by Ehrenborg and Hachimori [EH01], and Hachimori and Ziegler [HZ00].

Theorem 6.23. *Let Δ be a triangulation of a 3-sphere.*

- [HZ00] *If the skeleton of Δ contains a double-trefoil knot on 6 edges, then Δ is not vertex decomposable.*
- [EH01] *If the skeleton of Δ contains a triple-trefoil knot on 6 edges, then Δ is not constructible (hence not shellable).*

For an introduction to knot theory and a rigorous definition of complicatedness of knots we defer to a work of Benedetti and Lutz ([BL13]), where triangulations of the 3-sphere containing the double- and triple-trefoil knot on 3 edges were constructed: the first one has 16 vertices (see $S_{16,92}$ in [BL13]), while the second one has 18 vertices ($S_{18,125}$). Using our computer program we take the barycentric subdivision of these two complexes and we reduce them only applying cross-flips preserving the subdivision of the knots, which consist of 6 vertices. More precisely we only allow flips of the form $\Delta \longmapsto \chi_\Phi(\Delta)$, where the interior of Φ does not contain any of the 6 edges of the knot. Theorem 6.23 guarantees that in this way the obstructions for vertex decomposability and shellability are preserved, which yields the following result.

Proposition 6.24. *There exist balanced triangulations Δ_{22}^{2T} and Δ_{28}^{3T} of the 3-sphere that are:*

- *Shellable but not vertex decomposable (Δ_{22}^{2T}), and $f(\Delta_{22}^{2T}) = (1, 22, 136, 228, 114)$.*
- *Non constructible, hence not shellable (Δ_{28}^{3T}),and $f(\Delta_{28}^{3T}) = (1, 28, 204, 352, 176)$.*

In Appendix A.5 we report the list of facets of these two simplicial complexes.

6.5 Normal 3-pseudomanifolds

We conclude discussing the case of balanced pseudomanifolds. As defined in Definition 1.6 a pure d-dimensional pseudomanifold Δ is normal if the link of each face of dimension at most $(d-2)$ is connected. For $d = 2$ this class of simplicial complexes coincides with that of triangulated surfaces. Moreover, since the vertex links of a normal d-pseudomanifold are normal $(d-1)$-pseudomanifolds, it follows that for a balanced normal 3-pseudomanifold the vertex links are balanced triangulated surfaces. In this section we report small balanced normal 3-pseudomanifolds which are not combinatorial 3-manifolds, obtained applying cross-flips to the barycentric subdivision of the complexes on 9 vertices enumerated by Akhmejanov [Akh] modifying a computer program by Sulanke. It is very important to observe that since **??** holds only for combinatorial manifolds there is no connectivity result for the set of balanced normal 3-pseudomanifolds under corss-flips and hence it might be the case that the barycentric subdivision we start from and the balanced vertex-minimal triangulation lie in a different connected component (of the cross-flip graph). In Table 6.4 we exhibit the f-vector of the complex with minimum number of vertices among those that our program returned after a fixed number of iterations. Since cross-flips clearly preserve the PL-homeomorphism type, the complexes whose f-vectors appear in Table 6.4 are indeed balanced triangulations of the spaces in [Akh]. The first column of Table 6.4 reports the number of vertices whose link is homeomorphic to S^2, $\mathbb{R}\mathbf{P}^2$, $S^1 \times S^1$ and $\mathbb{R}\mathbf{P}^2 \# \mathbb{R}\mathbf{P}^2$ respectively, since those are the only homeomorphism types that can appear as vertex links of (non-balanced) normal 3-pseudomanifolds with up to 9 vertices. Note that, except for the case $(3, 4, 0, 2)$, this determines the homeomorphism type. From this numbers one can easily infer the singularity type of the corresponding balanced triangulation, since the number of vertex links not homeomorphic to S^2 does not change in the reduction process.

Hom. type	Min $f(\Delta)$	Hom. type	Min $f(\Delta)$
$8,0,1,0$	$(1,12,51,80,40)$	$2,4,1,2$	$(1,24,155,272,136)$
$8,0,0,1$	$(1,14,61,96,48)$	$2,2,2,3$	$(1,24,155,272,136)$
$7,2,0,0$	$(1,11,42,64,32)$	$2,2,1,4$	$(1,28,194,344,172)$
$7,0,2,0$	$(1,11,45,72,36)$	$2,2,0,5$	$(1,27,181,320,160)$
$7,0,0,2$	$(1,13,55,88,44)$	$1,8,0,0$	$(1,25,155,268,134)$
$6,2,0,1$	$(1,15,69,112,56)$	$1,6,1,1$	$(1,28,193,340,170)$
$6,0,3,0$	$(1,15,72,120,60)$	$1,4,4,0$	$(1,28,190,336,168)$
$6,0,1,2$	$(1,15,76,128,64)$	$1,4,2,2$	$(1,28,188,332,166)$
$6,0,0,3$	$(1,16,83,140,70)$	$1,4,1,3$	$(1,29,193,340,170)$
$5,4,0,0$	$(1,16,78,128,64)$	$1,4,0,4$	$(1,28,194,344,172)$
$5,2,1,1$	$(1,17,94,160,80)$	$1,2,4,2$	$(1,31,214,380,190)$
$5,2,0,2$	$(1,18,91,152,76)$	$1,2,3,3$	$(1,30,205,364,182)$
$5,0,2,2$	$(1,19,109,188,94)$	$1,2,2,4$	$(1,30,207,368,184)$
$5,0,0,4$	$(1,18,110,192,96)$	$1,2,0,6$	$(1,31,218,388,194)$
$4,4,1,0$	$(1,19,104,176,88)$	$1,0,8,0$	$(1,30,214,384,192)$
$4,4,0,1$	$(1,19,104,176,88)$	$1,0,4,4$	$(1,33,235,420,210)$
$4,2,2,1$	$(1,20,120,208,104)$	$0,8,1,0$	$(1,31,196,340,170)$
$4,2,1,2$	$(1,20,120,208,104)$	$0,6,0,3$	$(1,32,214,376,188)$
$4,2,0,3$	$(1,20,122,212,106)$	$0,4,4,1$	$(1,32,217,384,192)$
$4,0,5,0$	$(1,22,139,244,122)$	$0,4,3,2$	$(1,31,210,372,186)$
$4,0,1,4$	$(1,23,144,252,126)$	$0,4,0,5$	$(1,33,240,428,214)$
$3,6,0,0$	$(1,21,114,192,96)$	$0,2,4,3$	$(1,38,290,520,260)$
$3,4,2,0$	$(1,22,128,220,110)$	$0,2,3,4$	$(1,32,244,440,220)$
$3,4,0,2_a$	$(1,22,134,232,116)$	$0,2,2,5$	$(1,37,263,468,234)$
$3,4,0,2_b$	$(1,22,138,240,120)$	$0,0,9,0$	$(1,41,296,528,264)$
$3,2,2,2$	$(1,24,159,280,140)$	$0,0,5,4$	$(1,36,281,508,254)$
$3,2,1,3$	$(1,23,144,252,126)$	$0,0,3,6$	$(1,33,260,472,236)$
$2,6,0,1$	$(1,23,139,240,120)$	$0,0,1,8$	$(1,36,265,476,238)$
$2,4,2,1$	$(1,26,179,316,158)$	$0,0,0,9$	$(1,36,265,476,238)$

TABLE 6.4: The minimal f-vector found of a balanced triangulation of some normal 3-pseudomanifolds with singularities.

Remark 6.25. Observe that reducing the barycentric subdivisions of the normal pseudomanifolds of type $(7, 2, 0, 0)$, $(7, 0, 2, 0)$ and $(7, 0, 0, 2)$ the program returned the suspension of $\Delta_9^{\mathbb{R}\mathbf{P}^2}$, of the balanced vertex-minimal triangulation of $S^1 \times S^1$ and of a balanced vertex-minimal triangulation of $\mathbb{R}\mathbf{P}^2 \# \mathbb{R}\mathbf{P}^2$ respectively.

6.6 Concluding remarks

We conclude the chapter with some questions and problems which could be approached using our program, possibly after increasing the effectiveness of the search strategy. It is known (see e.g., [Lut08]) that all the simplicial 3-spheres with less than 11 vertices are shellable, and the smallest non-shellable 3-sphere known has 13 vertices. On the other hand the picture for the case of 3-balls is more complete: every non-shellable 3-ball has at least 9 vertices. In Proposition 6.24 we find a balanced non-shellable 3-sphere on 28 vertices, and the removal of the star of any vertex yields a balanced non-shellable 3-ball on 27 vertices. There is no reason to believe this is the minimum number of vertices for such a simplicial sphere/ball.

Question 6.26. What is the minimum number of vertices that a non-shellable balanced 3-sphere/3-ball can have?

Even in the general setting very little is known on the minimum number of vertices needed to triangulate the real projective space $\mathbb{R}\mathbf{P}^n$, for $n \geq 5$. For $\mathbb{R}\mathbf{P}^4$ the minimum number of vertices is 16 [Lut99].

Question 6.27. Find a small balanced triangulations of $\mathbb{R}\mathbf{P}^4$.

What about the opposite direction, namely the search for balanced manifolds with a high number of faces on a fixed number of vertices? We discussed in Chapter 2 a result of Zheng [Zhe16] which poses restrictions on the number of vertices that a balanced 2-neighborly 3-sphere can have (see Proposition 2.29). Therefore we ask the following questions, the first of which can be hopefully investigated via our program.

Question 6.28. Is there a balanced 2-neighborly 3-sphere on $4m$ vertices, for all $m \geq 5$? Is there a balanced 2-neighborly 4-sphere, which is not the suspension of a balanced 2-neighborly 3-sphere?

The *Pachner graph* $\mathcal{F}_n$ of flag 2-spheres on n-vertices is the graph whose vertices are flag 2-spheres on n-vertices and two spheres are connected if they can be transformed one into the other via an edge flip. Recently, Burton, Datta and Spreer [BDS17] proved that for $n \geq 8$ $\mathcal{F}_n$ consists of an isolated vertex and a second connected component. Motivated by this result, we consider the subgraph $\mathcal{B}_n$ of the cross-flip graph of balanced 2-spheres on n vertices, i.e., the induced subgraph corresponding to 2-spheres on n vertices. From Figure 4.2 we can observe that edges of this graph correspond to the two flips (four if we count the inverses) in the last row.

Question 6.29. How many connected components does $\mathcal{B}_n$ have for every $n \geq 6$?

Appendix A

Lists of facets

We report the list of facets of a balanced triangulation of some 2- and 3-dimensional manifolds with the minimum number of vertices known, as presented in Chapter 6. When the triangulation is vertex-minimal we add a reference. For the sake of completeness we also include the list of facets of the minimal balanced triangulation of the dunce hat in Proposition 6.16, the minimal balanced triangulation of the dunce hat found when the singularity is prescribed to contain three vertices, as well as the balanced non-partitionable Cohen-Macaulay simplicial complex C_3 from Theorem 3.14.

A.1 Balanced 2-manifolds

[0, 1, 5]	[0, 2, 7]	[1, 2, 3]	[1, 6, 8]	[3, 4, 8]	[4, 5, 6]
[0, 1, 8]	[0, 4, 5]	[1, 2, 6]	[2, 3, 4]	[3, 5, 7]	[4, 6, 8]
[0, 2, 4]	[0, 7, 8]	[1, 3, 5]	[2, 6, 7]	[3, 7, 8]	[5, 6, 7]

$\mathbb{T}$

[0, 1, 3]	[0, 7, 8]	[1, 4, 11]	[2, 6, 8]	[3, 5, 10]	[5, 6, 10]
[0, 1, 6]	[0, 7, 9]	[1, 10, 11]	[2, 6, 9]	[4, 5, 7]	[6, 8, 10]
[0, 3, 9]	[0, 8, 11]	[2, 3, 5]	[2, 7, 9]	[4, 5, 11]	[8, 10, 11]
[0, 5, 6]	[1, 3, 10]	[2, 3, 8]	[3, 4, 8]	[4, 6, 9]	
[0, 5, 11]	[1, 4, 6]	[2, 5, 7]	[3, 4, 9]	[4, 7, 8]	

$\mathbb{T}^{\#2} = \mathbb{T}\#\mathbb{T}$

[0, 4, 6]	[1, 4, 5]	[2, 3, 8]	[3, 4, 11]	[4, 8, 13]	[6, 9, 11]
[0, 4, 10]	[1, 4, 10]	[2, 3, 12]	[3, 4, 12]	[4, 12, 13]	[7, 8, 10]
[0, 6, 7]	[1, 5, 7]	[2, 6, 12]	[3, 7, 8]	[5, 7, 12]	[7, 11, 13]
[0, 7, 10]	[1, 6, 9]	[2, 8, 13]	[3, 7, 11]	[5, 8, 9]	[8, 9, 10]
[1, 2, 6]	[1, 7, 13]	[2, 10, 11]	[4, 5, 8]	[5, 9, 12]	[9, 10, 11]
[1, 2, 10]	[1, 9, 13]	[2, 11, 13]	[4, 6, 11]	[6, 7, 12]	[9, 12, 13]

$\mathbb{T}^{\#3}$

[0, 3, 6]	[0, 12, 13]	[1, 7, 12]	[2, 7, 8]	[4, 6, 10]	[6, 9, 13]
[0, 3, 12]	[1, 3, 8]	[1, 11, 13]	[2, 7, 11]	[4, 9, 11]	[7, 9, 11]
[0, 4, 5]	[1, 3, 11]	[2, 3, 5]	[2, 11, 13]	[4, 9, 12]	[7, 10, 12]
[0, 4, 6]	[1, 4, 8]	[2, 3, 6]	[3, 5, 10]	[4, 10, 11]	[8, 9, 13]
[0, 5, 7]	[1, 4, 12]	[2, 4, 5]	[3, 8, 9]	[5, 10, 13]	[10, 12, 13]
[0, 7, 8]	[1, 5, 7]	[2, 4, 8]	[3, 9, 12]	[6, 7, 9]	
[0, 8, 13]	[1, 5, 13]	[2, 6, 13]	[3, 10, 11]	[6, 7, 10]	

$\mathbb{T}^{\#4}$

[0, 2, 11]	[0, 10, 14]	[1, 8, 15]	[3, 4, 5]	[4, 5, 12]	[7, 9, 14]
[0, 2, 13]	[0, 11, 12]	[1, 9, 12]	[3, 4, 15]	[4, 8, 9]	[7, 12, 15]
[0, 3, 10]	[1, 2, 5]	[2, 4, 9]	[3, 5, 11]	[5, 10, 12]	[8, 13, 15]
[0, 3, 13]	[1, 2, 15]	[2, 4, 15]	[3, 6, 11]	[5, 11, 14]	[9, 10, 12]
[0, 4, 8]	[1, 5, 14]	[2, 5, 10]	[3, 6, 13]	[6, 7, 8]	[9, 13, 14]
[0, 4, 12]	[1, 6, 12]	[2, 6, 10]	[3, 7, 9]	[6, 7, 12]	[11, 12, 15]
[0, 7, 8]	[1, 6, 14]	[2, 6, 11]	[3, 7, 15]	[6, 8, 13]	[11, 14, 15]
[0, 7, 14]	[1, 8, 9]	[2, 9, 13]	[3, 9, 10]	[6, 10, 14]	[13, 14, 15]

$\mathbb{T}^{\#5}$

[1, 2, 4]	[1, 3, 8]	[2, 3, 6]	[2, 7, 9]	[4, 5, 7]	[7, 8, 9]
[1, 2, 9]	[1, 4, 5]	[2, 3, 7]	[3, 5, 7]	[4, 6, 8]	
[1, 3, 5]	[1, 8, 9]	[2, 4, 6]	[3, 6, 8]	[4, 7, 8]	

$\mathbb{R}\mathbf{P}^2$

[0, 2, 4]	[0, 5, 10]	[1, 2, 8]	[1, 5, 6]	[2, 3, 4]	[3, 5, 10]
[0, 2, 8]	[0, 6, 9]	[1, 2, 9]	[1, 5, 7]	[2, 3, 9]	[3, 6, 9]
[0, 4, 6]	[0, 7, 9]	[1, 4, 6]	[1, 7, 9]	[3, 4, 10]	
[0, 5, 7]	[0, 8, 10]	[1, 4, 10]	[1, 8, 10]	[3, 5, 6]	

$(\mathbb{R}\mathbf{P}^2)^{\#2}$

[0, 1, 3]	[0, 4, 11]	[1, 8, 9]	[2, 6, 8]	[3, 9, 11]	[8, 10, 11]
[0, 1, 7]	[0, 5, 8]	[1, 8, 10]	[2, 7, 10]	[4, 6, 11]	
[0, 2, 4]	[0, 8, 11]	[2, 3, 9]	[2, 8, 9]	[5, 6, 8]	
[0, 2, 7]	[1, 3, 10]	[2, 3, 10]	[3, 5, 6]	[7, 9, 11]	
[0, 3, 5]	[1, 7, 9]	[2, 4, 6]	[3, 6, 11]	[7, 10, 11]	

$(\mathbb{R}\mathbf{P}^2)^{\#3}$

[0, 1, 4]	[0, 4, 8]	[2, 3, 9]	[3, 4, 9]	[4, 5, 10]	[6, 8, 10]
[0, 1, 6]	[1, 4, 10]	[2, 5, 9]	[3, 6, 10]	[4, 8, 9]	[7, 8, 11]
[0, 2, 3]	[1, 6, 9]	[2, 5, 10]	[3, 7, 11]	[5, 6, 7]	[8, 9, 11]
[0, 2, 8]	[1, 9, 11]	[2, 8, 10]	[3, 10, 11]	[5, 6, 9]	
[0, 3, 6]	[1, 10, 11]	[3, 4, 7]	[4, 5, 7]	[6, 7, 8]	

$(\mathbb{R}\mathbf{P}^2)^{\#4}$

$[0,1,2]$	$[0,6,11]$	$[1,6,10]$	$[2,10,11]$	$[4,8,10]$	$[7,9,11]$
$[0,1,12]$	$[0,11,12]$	$[1,7,12]$	$[3,4,5]$	$[5,6,10]$	$[10,11,12]$
$[0,2,5]$	$[1,2,3]$	$[2,3,11]$	$[3,5,6]$	$[5,7,12]$	
$[0,4,5]$	$[1,3,4]$	$[2,5,7]$	$[3,6,8]$	$[5,10,12]$	
$[0,4,8]$	$[1,4,10]$	$[2,7,8]$	$[3,8,9]$	$[6,7,11]$	
$[0,6,8]$	$[1,6,7]$	$[2,8,10]$	$[3,9,11]$	$[7,8,9]$	

$(\mathbb{R}\mathbf{P}^2)^{\#5}$

A.2 Dunce hat

$[1,2,4]$	$[1,4,11]$	$[1,10,11]$	$[2,4,8]$	$[3,4,11]$	$[4,6,8]$
$[1,2,5]$	$[1,5,6]$	$[2,3,5]$	$[2,8,10]$	$[3,5,6]$	$[6,7,8]$
$[1,2,7]$	$[1,6,7]$	$[2,3,7]$	$[3,4,6]$	$[3,7,9]$	$[7,8,9]$
$[1,4,9]$	$[1,9,10]$	$[2,3,10]$	$[3,4,9]$	$[3,10,11]$	$[8,9,10]$

The vertex-minimal balanced triangulation of the dunce hat as in Proposition 6.16. The vertices in the singularity are $\{1,2,3,4\}$.

$[1,2,4]$	$[1,3,12]$	$[1,12,13]$	$[2,8,10]$	$[4,5,6]$	$[5,6,13]$
$[1,2,10]$	$[1,4,7]$	$[2,3,5]$	$[2,9,14]$	$[4,6,8]$	$[5,12,13]$
$[1,2,14]$	$[1,6,13]$	$[2,3,8]$	$[3,5,12]$	$[4,7,9]$	$[8,11,10]$
$[1,3,6]$	$[1,11,10]$	$[2,3,9]$	$[3,6,8]$	$[4,8,11]$	$[11,9,14]$
$[1,3,7]$	$[1,11,14]$	$[2,4,5]$	$[3,7,9]$	$[4,11,9]$	

The vertex-minimal balanced triangulation of the dunce hat with three vertices prescribed in the singularity. The vertices in the singularity are $\{1,2,3\}$.

A.3 A balanced non-partitionable Cohen-Macaulay complex

[0, 1, 2, 3]	[1, 2, 5, 6]	[1, 5, 8, 11]	[2, 6, 18, 19]	[4, 7, 13, 15]	[4, 10, 14, 16]
[0, 1, 2, 5]	[1, 2, 6, 9]	[1, 6, 9, 11]	[2, 6, 18, 20]	[4, 7, 13, 19]	[4, 10, 18, 20]
[0, 2, 3, 14]	[1, 2, 9, 10]	[1, 8, 9, 11]	[2, 10, 14, 16]	[4, 7, 14, 15]	[6, 14, 15, 17]
[0, 2, 3, 18]	[1, 3, 4, 7]	[2, 3, 10, 14]	[2, 10, 18, 20]	[4, 7, 18, 19]	[6, 14, 16, 17]
[0, 2, 5, 12]	[1, 3, 4, 10]	[2, 3, 10, 18]	[3, 4, 7, 14]	[4, 8, 13, 15]	[6, 18, 19, 21]
[0, 2, 12, 15]	[1, 4, 5, 7]	[2, 5, 6, 12]	[3, 4, 7, 18]	[4, 8, 13, 19]	[6, 18, 20, 21]
[0, 2, 12, 19]	[1, 4, 5, 8]	[2, 6, 12, 15]	[3, 4, 10, 14]	[4, 8, 14, 15]	[8, 14, 15, 17]
[0, 2, 14, 15]	[1, 4, 8, 9]	[2, 6, 12, 19]	[3, 4, 10, 18]	[4, 8, 14, 16]	[8, 14, 16, 17]
[0, 2, 18, 19]	[1, 4, 9, 10]	[2, 6, 14, 15]	[4, 5, 7, 13]	[4, 8, 18, 19]	[8, 18, 19, 21]
[1, 2, 3, 10]	[1, 5, 6, 11]	[2, 6, 14, 16]	[4, 5, 8, 13]	[4, 8, 18, 20]	[8, 18, 20, 21]

The balanced non-partitionable Cohen-Macaulay simplicial complex C_3 from Theorem 3.14.

A.4 Balanced 3-manifolds

[0, 1, 5, 6]	[0, 4, 5, 13]	[1, 2, 5, 6]	[2, 4, 5, 13]	[3, 6, 8, 9]	[5, 9, 11, 12]
[0, 1, 5, 10]	[0, 4, 7, 12]	[1, 2, 5, 10]	[2, 4, 7, 12]	[3, 6, 9, 13]	[6, 7, 8, 9]
[0, 1, 6, 7]	[0, 4, 7, 13]	[1, 2, 6, 7]	[2, 4, 7, 13]	[3, 8, 9, 10]	[6, 7, 9, 13]
[0, 1, 7, 10]	[0, 5, 6, 13]	[1, 2, 7, 10]	[2, 5, 6, 13]	[3, 9, 10, 12]	[7, 8, 9, 11]
[0, 3, 8, 10]	[0, 5, 8, 10]	[2, 3, 6, 8]	[2, 5, 10, 12]	[3, 9, 11, 12]	[7, 9, 11, 13]
[0, 3, 8, 11]	[0, 5, 8, 11]	[2, 3, 6, 13]	[2, 6, 7, 8]	[3, 9, 11, 13]	
[0, 3, 10, 12]	[0, 5, 11, 12]	[2, 3, 8, 11]	[2, 7, 8, 11]	[5, 8, 9, 10]	
[0, 3, 11, 12]	[0, 6, 7, 13]	[2, 3, 11, 13]	[2, 7, 10, 12]	[5, 8, 9, 11]	
[0, 4, 5, 12]	[0, 7, 10, 12]	[2, 4, 5, 12]	[2, 7, 11, 13]	[5, 9, 10, 12]	

$S^2 \times S^1$

[0, 1, 2, 7]	[0, 2, 3, 9]	[1, 2, 3, 4]	[1, 8, 10, 11]	[4, 5, 6, 11]	[5, 6, 7, 8]
[0, 1, 2, 11]	[0, 2, 5, 7]	[1, 2, 3, 8]	[2, 3, 4, 5]	[4, 5, 7, 10]	[5, 6, 8, 11]
[0, 1, 3, 6]	[0, 2, 9, 11]	[1, 2, 4, 7]	[2, 3, 8, 9]	[4, 5, 10, 11]	[5, 7, 8, 10]
[0, 1, 3, 10]	[0, 3, 5, 6]	[1, 2, 8, 11]	[2, 4, 5, 7]	[4, 6, 7, 9]	[5, 8, 10, 11]
[0, 1, 6, 7]	[0, 3, 9, 10]	[1, 3, 4, 6]	[2, 8, 9, 11]	[4, 6, 9, 11]	[6, 7, 8, 9]
[0, 1, 10, 11]	[0, 5, 6, 7]	[1, 3, 8, 10]	[3, 4, 5, 6]	[4, 7, 9, 10]	[6, 8, 9, 11]
[0, 2, 3, 5]	[0, 9, 10, 11]	[1, 4, 6, 7]	[3, 8, 9, 10]	[4, 9, 10, 11]	[7, 8, 9, 10]

$S^2 \underline{\times} S^1$

[0, 2, 3, 7]	[0, 5, 10, 14]	[1, 4, 7, 13]	[2, 3, 7, 11]	[3, 5, 8, 10]	[4, 6, 11, 14]
[0, 2, 3, 12]	[0, 6, 9, 14]	[1, 4, 7, 15]	[2, 3, 11, 12]	[3, 6, 8, 9]	[4, 7, 8, 13]
[0, 2, 7, 15]	[0, 6, 9, 15]	[1, 5, 10, 14]	[2, 5, 8, 13]	[3, 6, 8, 10]	[5, 8, 10, 15]
[0, 2, 12, 15]	[0, 9, 12, 14]	[1, 5, 10, 15]	[2, 5, 8, 15]	[3, 6, 9, 11]	[6, 8, 9, 15]
[0, 3, 4, 5]	[0, 9, 12, 15]	[1, 6, 10, 13]	[2, 5, 11, 13]	[3, 6, 10, 11]	[6, 8, 10, 15]
[0, 3, 4, 7]	[0, 10, 12, 14]	[1, 6, 10, 15]	[2, 5, 11, 14]	[3, 7, 8, 9]	[6, 9, 11, 14]
[0, 3, 5, 10]	[1, 2, 5, 14]	[1, 7, 9, 13]	[2, 7, 11, 14]	[3, 7, 9, 11]	[6, 10, 11, 13]
[0, 3, 10, 12]	[1, 2, 5, 15]	[1, 7, 9, 14]	[2, 8, 12, 13]	[3, 10, 11, 12]	[7, 8, 9, 13]
[0, 4, 5, 14]	[1, 2, 7, 14]	[1, 9, 12, 13]	[2, 8, 12, 15]	[4, 5, 8, 13]	[7, 9, 11, 14]
[0, 4, 6, 14]	[1, 2, 7, 15]	[1, 9, 12, 14]	[2, 11, 12, 13]	[4, 5, 11, 13]	[8, 9, 12, 13]
[0, 4, 6, 15]	[1, 4, 6, 13]	[1, 10, 12, 13]	[3, 4, 5, 8]	[4, 5, 11, 14]	[8, 9, 12, 15]
[0, 4, 7, 15]	[1, 4, 6, 15]	[1, 10, 12, 14]	[3, 4, 7, 8]	[4, 6, 11, 13]	[10, 11, 12, 13]

$\mathbb{R}\mathbf{P}^3$ ($\Delta_{16}^{\mathbb{R}\mathbf{P}^3}$)

[1, 5, 9, 15]	[1, 8, 9, 15]	[2, 7, 9, 14]	[3, 5, 11, 13]	[3, 8, 9, 14]	[4, 6, 12, 13]
[1, 5, 9, 16]	[1, 8, 9, 16]	[2, 7, 9, 15]	[3, 5, 11, 14]	[3, 8, 9, 16]	[4, 6, 12, 14]
[1, 5, 12, 15]	[1, 8, 10, 13]	[2, 7, 10, 15]	[3, 5, 12, 13]	[3, 8, 10, 14]	[4, 7, 9, 13]
[1, 5, 12, 16]	[1, 8, 10, 16]	[2, 7, 10, 16]	[3, 5, 12, 15]	[3, 8, 10, 16]	[4, 7, 9, 15]
[1, 6, 11, 14]	[1, 8, 11, 13]	[2, 7, 12, 14]	[3, 6, 9, 13]	[4, 5, 9, 15]	[4, 7, 10, 13]
[1, 6, 11, 15]	[1, 8, 11, 15]	[2, 7, 12, 16]	[3, 6, 9, 16]	[4, 5, 9, 16]	[4, 7, 10, 15]
[1, 6, 12, 14]	[2, 5, 11, 13]	[2, 8, 9, 14]	[3, 6, 10, 15]	[4, 5, 10, 14]	[4, 8, 10, 13]
[1, 6, 12, 15]	[2, 5, 11, 16]	[2, 8, 9, 15]	[3, 6, 10, 16]	[4, 5, 10, 15]	[4, 8, 10, 14]
[1, 7, 10, 13]	[2, 5, 12, 13]	[2, 8, 11, 13]	[3, 6, 12, 13]	[4, 5, 11, 14]	[4, 8, 12, 13]
[1, 7, 10, 16]	[2, 5, 12, 16]	[2, 8, 11, 15]	[3, 6, 12, 15]	[4, 5, 11, 16]	[4, 8, 12, 14]
[1, 7, 11, 13]	[2, 6, 10, 15]	[2, 8, 12, 13]	[3, 7, 9, 13]	[4, 6, 9, 13]	
[1, 7, 11, 14]	[2, 6, 10, 16]	[2, 8, 12, 14]	[3, 7, 9, 14]	[4, 6, 9, 16]	
[1, 7, 12, 14]	[2, 6, 11, 15]	[3, 5, 10, 14]	[3, 7, 11, 13]	[4, 6, 11, 14]	
[1, 7, 12, 16]	[2, 6, 11, 16]	[3, 5, 10, 15]	[3, 7, 11, 14]	[4, 6, 11, 16]	

$L(3, 1)$

[0, 1, 4, 6]	[0, 11, 12, 15]	[1, 7, 12, 18]	[2, 5, 16, 19]	[3, 8, 10, 14]	[7, 9, 15, 18]
[0, 1, 4, 16]	[0, 11, 12, 17]	[1, 7, 14, 16]	[2, 6, 9, 17]	[3, 8, 12, 13]	[7, 11, 12, 15]
[0, 1, 6, 14]	[0, 14, 16, 19]	[1, 8, 9, 13]	[2, 6, 12, 19]	[3, 10, 11, 14]	[7, 11, 12, 19]
[0, 1, 14, 16]	[0, 14, 17, 18]	[1, 9, 13, 18]	[2, 6, 14, 17]	[3, 10, 12, 16]	[7, 11, 14, 19]
[0, 4, 6, 15]	[0, 14, 18, 19]	[1, 10, 12, 16]	[2, 8, 14, 15]	[4, 6, 10, 19]	[7, 12, 15, 18]
[0, 4, 15, 16]	[1, 2, 5, 6]	[1, 10, 12, 18]	[2, 9, 11, 17]	[4, 6, 13, 15]	[7, 12, 16, 19]
[0, 5, 8, 15]	[1, 2, 5, 8]	[2, 3, 4, 11]	[2, 12, 16, 19]	[4, 6, 13, 19]	[7, 14, 16, 19]
[0, 5, 8, 17]	[1, 2, 6, 14]	[2, 3, 4, 16]	[2, 14, 15, 18]	[4, 10, 11, 19]	[8, 10, 12, 15]
[0, 5, 15, 16]	[1, 2, 8, 14]	[2, 3, 6, 9]	[2, 14, 17, 18]	[4, 11, 13, 17]	[8, 10, 14, 15]
[0, 5, 16, 19]	[1, 4, 6, 10]	[2, 3, 6, 12]	[3, 4, 10, 11]	[4, 11, 13, 19]	[8, 12, 13, 17]
[0, 5, 17, 18]	[1, 4, 10, 16]	[2, 3, 9, 11]	[3, 4, 10, 16]	[4, 13, 15, 18]	[9, 13, 15, 18]
[0, 5, 18, 19]	[1, 5, 6, 10]	[2, 3, 12, 16]	[3, 6, 9, 13]	[4, 13, 17, 18]	[10, 11, 14, 19]
[0, 6, 9, 15]	[1, 5, 8, 13]	[2, 4, 11, 17]	[3, 6, 12, 13]	[5, 6, 10, 19]	[10, 12, 15, 18]
[0, 6, 9, 17]	[1, 5, 10, 18]	[2, 4, 15, 16]	[3, 7, 8, 9]	[5, 8, 13, 17]	[10, 14, 15, 18]
[0, 6, 14, 17]	[1, 5, 13, 18]	[2, 4, 15, 18]	[3, 7, 8, 14]	[5, 10, 18, 19]	[10, 14, 18, 19]
[0, 8, 12, 15]	[1, 7, 8, 9]	[2, 4, 17, 18]	[3, 7, 9, 11]	[5, 13, 17, 18]	[11, 12, 13, 17]
[0, 8, 12, 17]	[1, 7, 8, 14]	[2, 5, 6, 19]	[3, 7, 11, 14]	[6, 9, 13, 15]	[11, 12, 13, 19]
[0, 9, 11, 15]	[1, 7, 9, 18]	[2, 5, 8, 15]	[3, 8, 9, 13]	[6, 12, 13, 19]	
[0, 9, 11, 17]	[1, 7, 12, 16]	[2, 5, 15, 16]	[3, 8, 10, 12]	[7, 9, 11, 15]	

$L(4, 1)$

[0, 1, 2, 3]	[0, 9, 12, 21]	[1, 14, 15, 16]	[3, 11, 13, 20]	[4, 10, 11, 20]	[8, 11, 14, 21]
[0, 1, 2, 21]	[0, 12, 13, 21]	[1, 14, 15, 19]	[3, 11, 14, 15]	[5, 7, 16, 20]	[8, 12, 14, 16]
[0, 1, 3, 7]	[0, 13, 17, 19]	[2, 3, 8, 12]	[3, 11, 14, 18]	[5, 7, 19, 20]	[8, 14, 16, 17]
[0, 1, 4, 9]	[0, 13, 17, 21]	[2, 3, 11, 20]	[3, 12, 13, 20]	[5, 8, 9, 19]	[8, 14, 17, 21]
[0, 1, 4, 10]	[1, 2, 3, 8]	[2, 3, 12, 20]	[3, 13, 17, 18]	[5, 8, 9, 21]	[9, 12, 15, 21]
[0, 1, 7, 16]	[1, 2, 8, 21]	[2, 4, 6, 11]	[3, 14, 15, 17]	[5, 8, 10, 21]	[9, 15, 16, 17]
[0, 1, 9, 21]	[1, 3, 6, 7]	[2, 4, 6, 17]	[3, 14, 17, 18]	[5, 8, 13, 19]	[9, 16, 17, 20]
[0, 1, 10, 16]	[1, 3, 6, 13]	[2, 4, 11, 20]	[4, 5, 8, 10]	[5, 9, 15, 16]	[10, 11, 20, 21]
[0, 2, 3, 11]	[1, 3, 8, 13]	[2, 4, 17, 20]	[4, 5, 8, 13]	[5, 9, 15, 21]	[10, 12, 15, 19]
[0, 2, 11, 19]	[1, 4, 9, 20]	[2, 6, 11, 19]	[4, 6, 7, 12]	[5, 9, 16, 20]	[10, 12, 15, 21]
[0, 2, 17, 19]	[1, 4, 10, 20]	[2, 6, 17, 19]	[4, 6, 7, 17]	[5, 9, 19, 20]	[10, 12, 19, 20]
[0, 2, 17, 21]	[1, 6, 7, 16]	[2, 8, 12, 16]	[4, 6, 11, 14]	[5, 10, 15, 16]	[10, 12, 20, 21]
[0, 3, 7, 11]	[1, 6, 13, 19]	[2, 8, 16, 17]	[4, 6, 12, 14]	[5, 10, 15, 21]	[11, 13, 18, 21]
[0, 4, 5, 10]	[1, 6, 14, 16]	[2, 8, 17, 21]	[4, 7, 12, 15]	[6, 7, 12, 16]	[11, 13, 20, 21]
[0, 4, 5, 13]	[1, 6, 14, 19]	[2, 12, 16, 20]	[4, 7, 15, 17]	[6, 11, 14, 19]	[11, 14, 15, 19]
[0, 4, 9, 12]	[1, 8, 9, 19]	[2, 16, 17, 20]	[4, 8, 10, 11]	[6, 12, 14, 16]	[11, 14, 18, 21]
[0, 4, 12, 13]	[1, 8, 9, 21]	[3, 6, 7, 17]	[4, 8, 11, 14]	[6, 13, 17, 19]	[12, 13, 20, 21]
[0, 5, 7, 16]	[1, 8, 13, 19]	[3, 6, 13, 17]	[4, 8, 12, 13]	[7, 11, 15, 19]	[13, 17, 18, 21]
[0, 5, 7, 19]	[1, 9, 19, 20]	[3, 7, 11, 15]	[4, 8, 12, 14]	[7, 12, 15, 19]	[14, 15, 16, 17]
[0, 5, 10, 16]	[1, 10, 15, 16]	[3, 7, 15, 17]	[4, 9, 12, 15]	[7, 12, 16, 20]	[14, 17, 18, 21]
[0, 5, 13, 19]	[1, 10, 15, 19]	[3, 8, 12, 13]	[4, 9, 15, 17]	[7, 12, 19, 20]	
[0, 7, 11, 19]	[1, 10, 19, 20]	[3, 11, 13, 18]	[4, 9, 17, 20]	[8, 10, 11, 21]	

$L(5, 1)$

[0, 1, 7, 14]	[0, 6, 15, 19]	[1, 11, 12, 15]	[3, 5, 11, 12]	[5, 6, 9, 11]	[8, 9, 17, 19]
[0, 1, 7, 18]	[0, 7, 8, 18]	[1, 11, 12, 18]	[3, 7, 12, 15]	[5, 6, 11, 12]	[8, 10, 14, 16]
[0, 1, 14, 19]	[0, 7, 13, 14]	[1, 11, 15, 16]	[3, 7, 12, 17]	[5, 6, 12, 19]	[8, 10, 16, 18]
[0, 1, 18, 19]	[0, 8, 10, 15]	[1, 11, 16, 18]	[3, 7, 14, 16]	[5, 6, 16, 19]	[8, 11, 12, 14]
[0, 3, 10, 15]	[0, 8, 10, 18]	[2, 3, 7, 14]	[3, 7, 16, 17]	[5, 7, 8, 9]	[8, 11, 12, 18]
[0, 3, 10, 18]	[0, 11, 13, 14]	[2, 3, 7, 15]	[3, 9, 11, 15]	[5, 7, 13, 16]	[8, 11, 14, 16]
[0, 3, 15, 19]	[0, 11, 13, 17]	[2, 3, 10, 14]	[3, 9, 15, 19]	[5, 12, 13, 19]	[8, 11, 16, 18]
[0, 3, 18, 19]	[1, 2, 4, 14]	[2, 3, 10, 15]	[3, 9, 18, 19]	[5, 13, 16, 19]	[8, 12, 14, 19]
[0, 4, 5, 8]	[1, 2, 4, 15]	[2, 4, 8, 14]	[3, 10, 14, 16]	[6, 7, 9, 17]	[8, 12, 17, 19]
[0, 4, 5, 13]	[1, 2, 7, 14]	[2, 4, 8, 15]	[3, 10, 16, 18]	[6, 7, 16, 17]	[9, 11, 13, 15]
[0, 4, 6, 15]	[1, 2, 7, 15]	[2, 8, 10, 14]	[3, 11, 12, 15]	[6, 9, 11, 17]	[9, 11, 13, 17]
[0, 4, 6, 17]	[1, 4, 9, 14]	[2, 8, 10, 15]	[4, 5, 8, 9]	[6, 11, 12, 14]	[9, 13, 15, 19]
[0, 4, 8, 15]	[1, 4, 9, 18]	[3, 4, 5, 9]	[4, 5, 12, 13]	[6, 12, 14, 19]	[9, 13, 17, 19]
[0, 4, 13, 17]	[1, 4, 15, 16]	[3, 4, 5, 12]	[4, 6, 15, 16]	[6, 15, 16, 19]	[11, 13, 14, 16]
[0, 5, 7, 8]	[1, 4, 16, 18]	[3, 4, 9, 18]	[4, 6, 16, 17]	[7, 8, 9, 17]	[11, 13, 15, 16]
[0, 5, 7, 13]	[1, 7, 12, 15]	[3, 4, 12, 17]	[4, 8, 9, 14]	[7, 8, 12, 17]	[12, 13, 17, 19]
[0, 6, 11, 14]	[1, 7, 12, 18]	[3, 4, 16, 17]	[4, 12, 13, 17]	[7, 8, 12, 18]	[13, 15, 16, 19]
[0, 6, 11, 17]	[1, 9, 14, 19]	[3, 4, 16, 18]	[5, 6, 7, 9]	[7, 13, 14, 16]	
[0, 6, 14, 19]	[1, 9, 18, 19]	[3, 5, 9, 11]	[5, 6, 7, 16]	[8, 9, 14, 19]	

$L(5, 2)$

[0, 2, 3, 9]	[1, 3, 12, 15]	[2, 6, 17, 20]	[3, 6, 12, 17]	[5, 9, 15, 20]	[8, 11, 12, 18]
[0, 2, 3, 16]	[1, 3, 12, 17]	[2, 6, 17, 21]	[3, 6, 15, 19]	[5, 9, 18, 23]	[8, 11, 12, 22]
[0, 2, 4, 10]	[1, 3, 13, 17]	[2, 6, 20, 23]	[3, 8, 19, 23]	[5, 9, 21, 23]	[8, 19, 20, 23]
[0, 2, 4, 22]	[1, 3, 13, 23]	[2, 6, 21, 23]	[3, 9, 11, 19]	[5, 11, 16, 20]	[9, 10, 14, 17]
[0, 2, 9, 10]	[1, 3, 15, 19]	[2, 7, 8, 20]	[4, 5, 7, 20]	[5, 15, 16, 18]	[9, 11, 18, 19]
[0, 2, 16, 22]	[1, 3, 19, 23]	[2, 7, 8, 22]	[4, 5, 7, 21]	[5, 15, 16, 20]	[9, 14, 17, 21]
[0, 3, 5, 8]	[1, 7, 10, 12]	[2, 8, 11, 18]	[4, 5, 17, 20]	[6, 11, 14, 18]	[9, 15, 18, 19]
[0, 3, 5, 16]	[1, 7, 10, 14]	[2, 8, 11, 22]	[4, 5, 21, 23]	[6, 11, 14, 20]	[9, 15, 19, 20]
[0, 3, 8, 19]	[1, 7, 12, 22]	[2, 8, 18, 23]	[4, 7, 14, 21]	[6, 11, 18, 19]	[10, 13, 15, 16]
[0, 3, 9, 19]	[1, 7, 14, 22]	[2, 8, 20, 23]	[4, 7, 14, 22]	[6, 12, 15, 21]	[10, 13, 16, 17]
[0, 4, 10, 12]	[1, 10, 12, 15]	[2, 9, 10, 17]	[4, 10, 12, 15]	[6, 12, 17, 21]	[10, 14, 15, 16]
[0, 4, 12, 18]	[1, 10, 14, 15]	[2, 9, 11, 18]	[4, 10, 13, 15]	[6, 13, 15, 18]	[10, 14, 16, 17]
[0, 4, 14, 18]	[1, 12, 17, 22]	[2, 9, 17, 21]	[4, 10, 13, 17]	[6, 13, 15, 21]	[11, 12, 16, 21]
[0, 4, 14, 22]	[1, 13, 17, 22]	[2, 9, 18, 23]	[4, 11, 12, 18]	[6, 13, 21, 23]	[11, 12, 16, 22]
[0, 5, 8, 18]	[1, 13, 22, 23]	[2, 9, 21, 23]	[4, 11, 12, 21]	[6, 13, 22, 23]	[11, 14, 16, 20]
[0, 5, 16, 18]	[1, 14, 15, 20]	[2, 11, 16, 22]	[4, 11, 14, 18]	[6, 14, 20, 23]	[11, 14, 16, 21]
[0, 6, 13, 18]	[1, 14, 20, 23]	[3, 4, 5, 17]	[4, 11, 14, 21]	[6, 14, 22, 23]	[12, 16, 17, 21]
[0, 6, 13, 22]	[1, 14, 22, 23]	[3, 4, 5, 23]	[4, 12, 15, 21]	[6, 15, 18, 19]	[12, 16, 17, 22]
[0, 6, 14, 18]	[1, 15, 19, 20]	[3, 4, 13, 17]	[4, 13, 15, 21]	[7, 8, 10, 12]	[13, 15, 16, 18]
[0, 6, 14, 22]	[1, 19, 20, 23]	[3, 4, 13, 23]	[4, 13, 21, 23]	[7, 8, 10, 19]	[13, 16, 17, 22]
[0, 8, 10, 12]	[2, 3, 9, 11]	[3, 5, 6, 11]	[5, 6, 11, 20]	[7, 8, 12, 22]	[14, 15, 16, 20]
[0, 8, 10, 19]	[2, 3, 11, 16]	[3, 5, 6, 17]	[5, 6, 17, 20]	[7, 8, 19, 20]	[14, 16, 17, 21]
[0, 8, 12, 18]	[2, 4, 7, 20]	[3, 5, 8, 23]	[5, 7, 9, 20]	[7, 9, 10, 14]	
[0, 9, 10, 19]	[2, 4, 7, 22]	[3, 5, 11, 16]	[5, 7, 9, 21]	[7, 9, 10, 19]	
[0, 13, 16, 18]	[2, 4, 10, 17]	[3, 6, 11, 19]	[5, 8, 18, 23]	[7, 9, 14, 21]	
[0, 13, 16, 22]	[2, 4, 17, 20]	[3, 6, 12, 15]	[5, 9, 15, 18]	[7, 9, 19, 20]	

$L(6, 1)$

[0, 3, 6, 11]	[1, 3, 7, 12]	[2, 7, 12, 16]	[3, 7, 8, 12]	[4, 12, 20, 23]	[8, 9, 11, 22]
[0, 3, 6, 15]	[1, 3, 7, 17]	[2, 7, 12, 23]	[3, 7, 8, 17]	[4, 12, 21, 23]	[8, 9, 21, 22]
[0, 3, 9, 11]	[1, 3, 11, 12]	[2, 9, 15, 16]	[3, 8, 9, 11]	[4, 17, 20, 23]	[8, 10, 16, 19]
[0, 3, 9, 15]	[1, 3, 17, 20]	[2, 9, 15, 18]	[3, 8, 9, 21]	[4, 17, 21, 23]	[8, 10, 18, 19]
[0, 5, 6, 11]	[1, 5, 6, 11]	[2, 9, 18, 19]	[3, 8, 11, 12]	[5, 6, 14, 15]	[8, 11, 12, 22]
[0, 5, 6, 15]	[1, 5, 6, 21]	[2, 9, 19, 23]	[3, 8, 17, 21]	[5, 6, 14, 21]	[8, 12, 16, 20]
[0, 5, 10, 11]	[1, 5, 11, 12]	[2, 12, 16, 20]	[4, 5, 9, 19]	[5, 9, 14, 19]	[8, 12, 20, 22]
[0, 5, 10, 19]	[1, 5, 12, 21]	[2, 12, 20, 23]	[4, 5, 9, 21]	[5, 9, 14, 21]	[8, 13, 16, 19]
[0, 5, 13, 15]	[1, 6, 20, 22]	[2, 13, 15, 16]	[4, 5, 10, 11]	[5, 13, 14, 15]	[8, 13, 16, 20]
[0, 5, 13, 19]	[1, 6, 21, 22]	[2, 13, 15, 18]	[4, 5, 10, 19]	[5, 13, 14, 19]	[8, 13, 19, 23]
[0, 7, 9, 16]	[1, 7, 12, 23]	[2, 13, 16, 20]	[4, 5, 11, 12]	[6, 14, 15, 22]	[8, 13, 20, 22]
[0, 7, 9, 22]	[1, 7, 13, 18]	[2, 13, 18, 20]	[4, 5, 12, 21]	[6, 14, 21, 22]	[8, 13, 21, 22]
[0, 7, 10, 16]	[1, 7, 13, 23]	[2, 17, 18, 19]	[4, 6, 15, 22]	[7, 8, 10, 16]	[8, 13, 21, 23]
[0, 7, 10, 22]	[1, 7, 17, 18]	[2, 17, 18, 20]	[4, 6, 20, 22]	[7, 8, 10, 18]	[8, 17, 18, 19]
[0, 9, 11, 22]	[1, 12, 21, 23]	[2, 17, 19, 23]	[4, 9, 15, 18]	[7, 8, 12, 16]	[8, 17, 19, 23]
[0, 9, 15, 16]	[1, 13, 18, 20]	[2, 17, 20, 23]	[4, 9, 18, 19]	[7, 8, 17, 18]	[8, 17, 21, 23]
[0, 10, 11, 22]	[1, 13, 20, 22]	[3, 4, 6, 15]	[4, 10, 11, 22]	[7, 9, 14, 22]	[9, 14, 19, 23]
[0, 10, 16, 19]	[1, 13, 21, 22]	[3, 4, 6, 20]	[4, 10, 15, 18]	[7, 9, 14, 23]	[9, 14, 21, 22]
[0, 13, 15, 16]	[1, 13, 21, 23]	[3, 4, 9, 15]	[4, 10, 15, 22]	[7, 10, 14, 18]	[10, 14, 15, 18]
[0, 13, 16, 19]	[1, 17, 18, 20]	[3, 4, 9, 21]	[4, 10, 18, 19]	[7, 10, 14, 22]	[10, 14, 15, 22]
[1, 3, 6, 11]	[2, 7, 9, 16]	[3, 4, 17, 20]	[4, 11, 12, 22]	[7, 13, 14, 18]	[13, 14, 15, 18]
[1, 3, 6, 20]	[2, 7, 9, 23]	[3, 4, 17, 21]	[4, 12, 20, 22]	[7, 13, 14, 23]	[13, 14, 19, 23]

$\mathbb{R}\mathbf{P}^2 \times S^1$

[0, 2, 10, 16]	[1, 2, 3, 4]	[2, 3, 14, 16]	[3, 14, 16, 23]	[6, 7, 14, 22]	[9, 13, 16, 20]
[0, 2, 10, 25]	[1, 2, 3, 18]	[2, 3, 14, 18]	[3, 15, 19, 23]	[6, 10, 12, 24]	[9, 13, 20, 25]
[0, 2, 16, 20]	[1, 2, 4, 12]	[2, 4, 10, 12]	[3, 15, 19, 24]	[6, 10, 22, 24]	[9, 13, 21, 25]
[0, 2, 20, 25]	[1, 2, 12, 18]	[2, 9, 14, 15]	[3, 19, 24, 25]	[6, 13, 14, 17]	[9, 15, 19, 23]
[0, 5, 6, 13]	[1, 3, 4, 24]	[2, 9, 14, 16]	[4, 10, 12, 24]	[6, 14, 22, 23]	[9, 18, 19, 23]
[0, 5, 6, 24]	[1, 3, 13, 18]	[2, 9, 15, 21]	[5, 6, 7, 12]	[6, 21, 22, 23]	[10, 11, 13, 17]
[0, 5, 13, 16]	[1, 3, 13, 25]	[2, 9, 16, 20]	[5, 6, 7, 17]	[6, 21, 22, 24]	[10, 11, 17, 23]
[0, 5, 16, 24]	[1, 3, 24, 25]	[2, 9, 20, 25]	[5, 6, 12, 24]	[7, 9, 14, 15]	[10, 13, 15, 22]
[0, 6, 13, 19]	[1, 4, 12, 24]	[2, 9, 21, 25]	[5, 6, 13, 17]	[7, 9, 14, 16]	[10, 13, 17, 18]
[0, 6, 19, 23]	[1, 7, 9, 16]	[2, 10, 12, 25]	[5, 7, 12, 25]	[7, 9, 15, 19]	[10, 13, 18, 22]
[0, 6, 21, 23]	[1, 7, 9, 18]	[2, 12, 18, 21]	[5, 7, 17, 25]	[7, 9, 18, 19]	[10, 15, 22, 24]
[0, 6, 21, 24]	[1, 7, 16, 22]	[2, 12, 21, 25]	[5, 9, 11, 13]	[7, 10, 12, 25]	[10, 16, 17, 23]
[0, 7, 10, 18]	[1, 7, 18, 22]	[2, 14, 15, 17]	[5, 9, 11, 23]	[7, 10, 18, 22]	[12, 18, 21, 23]
[0, 7, 10, 25]	[1, 9, 16, 24]	[2, 14, 17, 18]	[5, 9, 13, 16]	[7, 14, 15, 17]	[12, 21, 23, 25]
[0, 7, 18, 19]	[1, 9, 18, 24]	[2, 15, 17, 21]	[5, 9, 16, 24]	[7, 14, 16, 22]	[13, 14, 17, 18]
[0, 7, 19, 25]	[1, 12, 18, 24]	[2, 17, 18, 21]	[5, 9, 18, 23]	[7, 15, 17, 19]	[13, 15, 21, 22]
[0, 8, 10, 16]	[1, 13, 18, 22]	[3, 4, 10, 24]	[5, 9, 18, 24]	[7, 17, 19, 25]	[13, 21, 22, 25]
[0, 8, 10, 18]	[1, 13, 22, 25]	[3, 6, 13, 14]	[5, 11, 13, 17]	[8, 10, 16, 17]	[14, 16, 22, 23]
[0, 8, 16, 21]	[1, 16, 17, 23]	[3, 6, 13, 19]	[5, 11, 17, 23]	[8, 10, 17, 18]	[15, 17, 19, 24]
[0, 8, 18, 21]	[1, 16, 17, 24]	[3, 6, 14, 23]	[5, 12, 18, 23]	[8, 16, 17, 21]	[15, 17, 21, 24]
[0, 13, 16, 20]	[1, 16, 22, 23]	[3, 6, 19, 23]	[5, 12, 18, 24]	[8, 17, 18, 21]	[15, 21, 22, 24]
[0, 13, 19, 25]	[1, 17, 23, 25]	[3, 10, 15, 23]	[5, 12, 23, 25]	[9, 10, 11, 13]	[16, 17, 21, 24]
[0, 13, 20, 25]	[1, 17, 24, 25]	[3, 10, 15, 24]	[5, 17, 23, 25]	[9, 10, 11, 23]	[17, 19, 24, 25]
[0, 16, 21, 24]	[1, 22, 23, 25]	[3, 10, 16, 23]	[6, 7, 10, 12]	[9, 10, 13, 15]	[21, 22, 23, 25]
[0, 18, 19, 23]	[2, 3, 4, 10]	[3, 13, 14, 18]	[6, 7, 10, 22]	[9, 10, 15, 23]	
[0, 18, 21, 23]	[2, 3, 10, 16]	[3, 13, 19, 25]	[6, 7, 14, 17]	[9, 13, 15, 21]	

Poincaré homology 3-sphere

[0, 1, 5, 9]	[0, 7, 10, 13]	[1, 6, 9, 11]	[2, 8, 9, 10]	[4, 5, 7, 14]	[5, 8, 9, 12]
[0, 1, 5, 13]	[0, 7, 10, 15]	[1, 6, 11, 15]	[2, 8, 10, 13]	[4, 5, 7, 15]	[5, 8, 12, 15]
[0, 1, 9, 10]	[1, 3, 4, 14]	[1, 8, 9, 10]	[3, 4, 7, 9]	[4, 7, 9, 11]	[6, 7, 9, 11]
[0, 1, 10, 13]	[1, 3, 4, 15]	[1, 8, 9, 11]	[3, 4, 7, 14]	[4, 7, 10, 13]	[6, 7, 11, 13]
[0, 2, 3, 13]	[1, 3, 8, 14]	[1, 8, 10, 13]	[3, 4, 9, 12]	[4, 7, 10, 15]	[6, 11, 12, 13]
[0, 2, 3, 15]	[1, 3, 8, 15]	[1, 8, 11, 15]	[3, 4, 12, 13]	[4, 7, 11, 13]	[6, 11, 12, 15]
[0, 2, 5, 9]	[1, 4, 5, 14]	[2, 3, 4, 13]	[3, 6, 7, 9]	[4, 9, 11, 12]	[8, 9, 11, 12]
[0, 2, 5, 13]	[1, 4, 5, 15]	[2, 3, 4, 15]	[3, 6, 7, 13]	[4, 11, 12, 13]	[8, 11, 12, 15]
[0, 2, 9, 10]	[1, 5, 6, 9]	[2, 4, 10, 13]	[3, 6, 9, 12]	[5, 6, 9, 12]	
[0, 2, 10, 15]	[1, 5, 6, 15]	[2, 4, 10, 15]	[3, 6, 12, 13]	[5, 6, 12, 15]	
[0, 3, 7, 13]	[1, 5, 8, 13]	[2, 5, 8, 9]	[3, 7, 8, 14]	[5, 7, 8, 14]	
[0, 3, 7, 15]	[1, 5, 8, 14]	[2, 5, 8, 13]	[3, 7, 8, 15]	[5, 7, 8, 15]	

$(S^2 \times S^1)^{\#2}$

[0, 1, 5, 9]	[0, 8, 10, 13]	[1, 4, 8, 9]	[2, 5, 7, 12]	[3, 12, 13, 14]	[6, 7, 12, 14]
[0, 1, 5, 13]	[0, 9, 10, 11]	[1, 4, 11, 13]	[2, 5, 10, 12]	[3, 13, 14, 15]	[6, 7, 14, 15]
[0, 1, 8, 9]	[1, 2, 4, 11]	[1, 6, 8, 15]	[2, 7, 8, 12]	[4, 5, 9, 10]	[7, 8, 12, 13]
[0, 1, 8, 13]	[1, 2, 4, 14]	[1, 6, 14, 15]	[2, 10, 11, 12]	[4, 5, 10, 13]	[7, 9, 11, 12]
[0, 2, 5, 7]	[1, 2, 8, 12]	[1, 8, 12, 13]	[3, 4, 6, 8]	[4, 7, 8, 13]	[7, 11, 13, 15]
[0, 2, 5, 10]	[1, 2, 8, 15]	[1, 11, 12, 13]	[3, 4, 6, 14]	[4, 7, 11, 13]	[7, 12, 13, 14]
[0, 2, 7, 11]	[1, 2, 11, 12]	[2, 3, 4, 8]	[3, 6, 8, 15]	[4, 8, 9, 10]	[7, 13, 14, 15]
[0, 2, 10, 11]	[1, 2, 14, 15]	[2, 3, 4, 14]	[3, 6, 11, 12]	[4, 8, 10, 13]	[9, 10, 11, 12]
[0, 5, 7, 9]	[1, 4, 5, 9]	[2, 3, 8, 15]	[3, 6, 11, 15]	[5, 7, 9, 12]	
[0, 5, 10, 13]	[1, 4, 5, 13]	[2, 3, 14, 15]	[3, 6, 12, 14]	[5, 9, 10, 12]	
[0, 7, 9, 11]	[1, 4, 6, 8]	[2, 4, 7, 8]	[3, 11, 12, 13]	[6, 7, 11, 12]	
[0, 8, 9, 10]	[1, 4, 6, 14]	[2, 4, 7, 11]	[3, 11, 13, 15]	[6, 7, 11, 15]	

$(S^2 \underline{\times} S^1)^{\#2}$

[0, 3, 6, 11]	[0, 8, 14, 19]	[2, 7, 10, 12]	[3, 9, 17, 18]	[4, 13, 14, 15]	[7, 14, 17, 18]
[0, 3, 6, 19]	[1, 2, 5, 10]	[2, 7, 12, 16]	[3, 11, 13, 14]	[4, 14, 15, 16]	[8, 9, 10, 19]
[0, 3, 11, 14]	[1, 2, 5, 16]	[2, 12, 15, 16]	[3, 13, 14, 19]	[5, 7, 10, 11]	[8, 9, 13, 17]
[0, 3, 14, 19]	[1, 2, 10, 12]	[2, 12, 15, 18]	[4, 5, 8, 13]	[5, 7, 16, 17]	[8, 9, 17, 18]
[0, 4, 5, 8]	[1, 2, 12, 18]	[2, 14, 15, 16]	[4, 5, 13, 15]	[5, 8, 13, 17]	[8, 9, 18, 19]
[0, 4, 5, 15]	[1, 2, 14, 16]	[2, 14, 15, 18]	[4, 6, 7, 10]	[5, 10, 11, 15]	[8, 14, 17, 18]
[0, 4, 6, 8]	[1, 2, 14, 18]	[3, 5, 10, 17]	[4, 6, 7, 18]	[5, 10, 15, 19]	[8, 14, 18, 19]
[0, 4, 6, 15]	[1, 4, 12, 16]	[3, 5, 10, 19]	[4, 6, 8, 10]	[5, 13, 15, 19]	[9, 10, 11, 15]
[0, 5, 7, 11]	[1, 4, 12, 18]	[3, 5, 13, 17]	[4, 6, 15, 18]	[6, 7, 10, 17]	[9, 10, 15, 19]
[0, 5, 7, 17]	[1, 4, 14, 16]	[3, 5, 13, 19]	[4, 7, 9, 10]	[6, 7, 17, 18]	[9, 11, 15, 18]
[0, 5, 8, 17]	[1, 4, 14, 18]	[3, 6, 10, 17]	[4, 7, 9, 13]	[6, 8, 10, 19]	[9, 15, 18, 19]
[0, 5, 11, 15]	[1, 5, 10, 17]	[3, 6, 10, 19]	[4, 7, 13, 14]	[6, 11, 15, 18]	[13, 14, 15, 19]
[0, 6, 8, 19]	[1, 5, 16, 17]	[3, 6, 11, 18]	[4, 7, 14, 18]	[7, 9, 10, 11]	[14, 15, 18, 19]
[0, 6, 11, 15]	[1, 10, 12, 17]	[3, 6, 17, 18]	[4, 8, 9, 10]	[7, 9, 11, 13]	
[0, 7, 11, 14]	[1, 12, 16, 17]	[3, 9, 11, 13]	[4, 8, 9, 13]	[7, 10, 12, 17]	
[0, 7, 14, 17]	[2, 5, 7, 10]	[3, 9, 11, 18]	[4, 12, 15, 16]	[7, 11, 13, 14]	
[0, 8, 14, 17]	[2, 5, 7, 16]	[3, 9, 13, 17]	[4, 12, 15, 18]	[7, 12, 16, 17]	

$(S^2 \times S^1)\#\mathbb{R}\mathbf{P}^3$

[0, 3, 9, 15]	[1, 2, 4, 10]	[2, 4, 9, 16]	[3, 14, 15, 20]	[5, 8, 15, 17]	[8, 9, 16, 20]
[0, 3, 9, 18]	[1, 2, 4, 16]	[2, 4, 9, 18]	[3, 14, 17, 18]	[5, 8, 16, 17]	[8, 9, 18, 20]
[0, 3, 14, 15]	[1, 3, 6, 16]	[2, 5, 8, 16]	[3, 14, 17, 19]	[5, 10, 12, 20]	[9, 11, 15, 17]
[0, 3, 14, 18]	[1, 3, 6, 18]	[2, 5, 8, 18]	[3, 14, 19, 20]	[5, 12, 15, 17]	[9, 11, 15, 20]
[0, 4, 9, 18]	[1, 3, 10, 20]	[2, 5, 10, 12]	[4, 6, 9, 16]	[5, 12, 17, 19]	[9, 11, 16, 20]
[0, 4, 9, 19]	[1, 3, 18, 20]	[2, 5, 12, 18]	[4, 6, 9, 19]	[5, 12, 19, 20]	[9, 11, 17, 19]
[0, 4, 13, 18]	[1, 4, 6, 15]	[2, 7, 10, 12]	[4, 6, 14, 15]	[6, 7, 12, 16]	[9, 12, 15, 17]
[0, 4, 13, 19]	[1, 4, 6, 16]	[2, 7, 12, 18]	[4, 6, 14, 19]	[6, 7, 12, 18]	[9, 12, 17, 19]
[0, 9, 12, 15]	[1, 4, 10, 17]	[2, 8, 9, 16]	[4, 7, 10, 17]	[6, 9, 11, 16]	[11, 13, 15, 17]
[0, 9, 12, 19]	[1, 4, 15, 17]	[2, 8, 9, 18]	[4, 7, 17, 18]	[6, 9, 11, 19]	[11, 13, 15, 20]
[0, 11, 13, 16]	[1, 6, 8, 15]	[3, 5, 10, 20]	[4, 13, 15, 17]	[6, 11, 14, 16]	[11, 13, 16, 20]
[0, 11, 13, 18]	[1, 6, 8, 18]	[3, 5, 16, 17]	[4, 13, 15, 20]	[6, 11, 14, 19]	[11, 13, 17, 18]
[0, 11, 14, 16]	[1, 8, 10, 17]	[3, 5, 17, 19]	[4, 13, 17, 18]	[6, 12, 14, 15]	[11, 14, 17, 18]
[0, 11, 14, 18]	[1, 8, 10, 20]	[3, 5, 19, 20]	[4, 13, 19, 20]	[6, 12, 14, 16]	[11, 14, 17, 19]
[0, 12, 13, 16]	[1, 8, 15, 17]	[3, 6, 7, 16]	[4, 14, 15, 20]	[7, 8, 10, 17]	[12, 13, 16, 20]
[0, 12, 13, 19]	[1, 8, 18, 20]	[3, 6, 7, 18]	[4, 14, 19, 20]	[7, 8, 10, 20]	[12, 13, 19, 20]
[0, 12, 14, 15]	[2, 3, 5, 10]	[3, 7, 16, 17]	[5, 6, 8, 15]	[7, 8, 16, 17]	
[0, 12, 14, 16]	[2, 3, 5, 16]	[3, 7, 17, 18]	[5, 6, 8, 18]	[7, 8, 16, 20]	
[1, 2, 3, 10]	[2, 4, 7, 10]	[3, 9, 15, 20]	[5, 6, 12, 15]	[7, 10, 12, 20]	
[1, 2, 3, 16]	[2, 4, 7, 18]	[3, 9, 18, 20]	[5, 6, 12, 18]	[7, 12, 16, 20]	

$\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$

[0, 5, 6, 7]	[1, 3, 6, 18]	[2, 3, 9, 10]	[3, 6, 18, 21]	[5, 6, 7, 8]	[8, 9, 16, 19]
[0, 5, 6, 16]	[1, 3, 6, 23]	[2, 3, 9, 11]	[3, 6, 21, 23]	[5, 6, 8, 21]	[8, 10, 12, 16]
[0, 5, 7, 20]	[1, 3, 10, 17]	[2, 3, 10, 20]	[3, 9, 10, 22]	[5, 6, 15, 16]	[8, 12, 16, 18]
[0, 5, 9, 16]	[1, 3, 10, 20]	[2, 3, 11, 12]	[3, 9, 11, 22]	[5, 6, 15, 21]	[8, 14, 16, 18]
[0, 5, 9, 22]	[1, 3, 17, 18]	[2, 3, 12, 23]	[3, 10, 17, 22]	[5, 7, 8, 20]	[8, 14, 16, 19]
[0, 5, 20, 22]	[1, 3, 20, 23]	[2, 3, 20, 23]	[3, 11, 12, 21]	[5, 8, 17, 21]	[8, 14, 18, 21]
[0, 6, 7, 11]	[1, 4, 5, 9]	[2, 4, 5, 17]	[3, 11, 14, 21]	[5, 9, 15, 16]	[8, 14, 19, 21]
[0, 6, 11, 16]	[1, 4, 5, 17]	[2, 4, 5, 20]	[3, 11, 14, 22]	[5, 15, 17, 21]	[8, 17, 19, 21]
[0, 7, 11, 14]	[1, 4, 6, 11]	[2, 4, 12, 18]	[3, 12, 21, 23]	[6, 7, 8, 18]	[9, 15, 16, 19]
[0, 7, 14, 19]	[1, 4, 6, 18]	[2, 4, 12, 23]	[3, 14, 18, 21]	[6, 8, 18, 21]	[10, 12, 15, 21]
[0, 7, 19, 20]	[1, 4, 9, 11]	[2, 4, 17, 18]	[3, 14, 18, 22]	[6, 11, 13, 16]	[10, 15, 17, 21]
[0, 9, 10, 16]	[1, 4, 17, 18]	[2, 4, 20, 23]	[3, 17, 18, 22]	[6, 13, 16, 23]	[11, 12, 13, 16]
[0, 9, 10, 22]	[1, 5, 9, 15]	[2, 5, 8, 17]	[4, 5, 9, 22]	[6, 15, 16, 23]	[12, 13, 16, 18]
[0, 10, 12, 16]	[1, 5, 15, 17]	[2, 5, 8, 20]	[4, 5, 20, 22]	[6, 15, 21, 23]	[12, 15, 21, 23]
[0, 10, 12, 21]	[1, 6, 11, 13]	[2, 8, 9, 10]	[4, 6, 7, 11]	[7, 8, 10, 12]	[13, 14, 16, 18]
[0, 10, 17, 21]	[1, 6, 13, 23]	[2, 8, 9, 19]	[4, 6, 7, 18]	[7, 8, 10, 20]	[13, 14, 16, 23]
[0, 10, 17, 22]	[1, 9, 11, 13]	[2, 8, 10, 20]	[4, 7, 11, 14]	[7, 8, 12, 18]	[13, 14, 18, 22]
[0, 11, 12, 16]	[1, 9, 13, 19]	[2, 8, 17, 19]	[4, 7, 12, 18]	[7, 10, 12, 15]	[13, 14, 22, 23]
[0, 11, 12, 21]	[1, 9, 15, 19]	[2, 9, 11, 13]	[4, 7, 12, 23]	[7, 10, 15, 20]	[13, 17, 18, 22]
[0, 11, 14, 21]	[1, 10, 15, 17]	[2, 9, 13, 19]	[4, 7, 14, 23]	[7, 12, 15, 23]	[13, 17, 19, 22]
[0, 14, 19, 21]	[1, 10, 15, 20]	[2, 11, 12, 13]	[4, 9, 11, 22]	[7, 14, 15, 19]	[13, 19, 20, 22]
[0, 17, 19, 21]	[1, 13, 19, 20]	[2, 12, 13, 18]	[4, 11, 14, 22]	[7, 14, 15, 23]	[13, 20, 22, 23]
[0, 17, 19, 22]	[1, 13, 20, 23]	[2, 13, 17, 18]	[4, 14, 22, 23]	[7, 15, 19, 20]	[14, 15, 16, 19]
[0, 19, 20, 22]	[1, 15, 19, 20]	[2, 13, 17, 19]	[4, 20, 22, 23]	[8, 9, 10, 16]	[14, 15, 16, 23]

$S^1 \times S^1 \times S^1$

[0, 2, 3, 4]	[1, 2, 5, 12]	[1, 16, 17, 21]	[3, 5, 21, 22]	[5, 8, 21, 23]	[7, 9, 16, 23]
[0, 2, 3, 10]	[1, 2, 5, 23]	[1, 16, 21, 23]	[3, 10, 21, 22]	[5, 9, 12, 13]	[7, 14, 17, 20]
[0, 2, 4, 9]	[1, 2, 6, 10]	[2, 3, 4, 17]	[3, 12, 13, 14]	[5, 9, 13, 22]	[7, 15, 19, 22]
[0, 2, 9, 10]	[1, 2, 6, 15]	[2, 3, 10, 22]	[3, 12, 14, 21]	[5, 11, 17, 19]	[7, 15, 20, 22]
[0, 3, 4, 13]	[1, 2, 10, 12]	[2, 3, 16, 17]	[3, 13, 16, 17]	[5, 11, 17, 20]	[7, 16, 19, 22]
[0, 3, 10, 21]	[1, 2, 15, 23]	[2, 3, 16, 22]	[3, 13, 16, 22]	[5, 11, 19, 23]	[8, 10, 21, 22]
[0, 3, 13, 14]	[1, 4, 6, 11]	[2, 4, 9, 17]	[4, 6, 11, 19]	[5, 11, 20, 22]	[8, 11, 12, 15]
[0, 3, 14, 21]	[1, 4, 6, 13]	[2, 5, 9, 12]	[4, 6, 13, 19]	[6, 7, 8, 14]	[8, 11, 15, 22]
[0, 4, 9, 18]	[1, 4, 7, 12]	[2, 5, 9, 17]	[4, 7, 8, 12]	[6, 7, 8, 16]	[8, 12, 13, 14]
[0, 4, 13, 19]	[1, 4, 7, 17]	[2, 5, 17, 19]	[4, 7, 8, 23]	[6, 7, 14, 19]	[8, 12, 13, 15]
[0, 4, 18, 19]	[1, 4, 11, 12]	[2, 5, 19, 23]	[4, 7, 9, 17]	[6, 7, 16, 19]	[8, 16, 21, 23]
[0, 7, 14, 19]	[1, 4, 13, 17]	[2, 6, 8, 10]	[4, 7, 9, 23]	[6, 8, 10, 21]	[9, 10, 11, 12]
[0, 7, 14, 20]	[1, 5, 12, 21]	[2, 6, 8, 15]	[4, 8, 11, 12]	[6, 8, 13, 14]	[9, 11, 12, 15]
[0, 7, 15, 19]	[1, 5, 21, 23]	[2, 8, 10, 22]	[4, 8, 11, 23]	[6, 8, 13, 15]	[9, 12, 13, 15]
[0, 7, 15, 20]	[1, 6, 10, 11]	[2, 8, 15, 22]	[4, 9, 18, 23]	[6, 8, 16, 21]	[9, 13, 15, 23]
[0, 9, 10, 11]	[1, 6, 13, 15]	[2, 9, 10, 12]	[4, 11, 19, 23]	[6, 10, 11, 20]	[9, 13, 16, 22]
[0, 9, 11, 15]	[1, 7, 12, 14]	[2, 15, 19, 22]	[4, 18, 19, 23]	[6, 10, 20, 21]	[9, 13, 16, 23]
[0, 9, 15, 18]	[1, 7, 14, 17]	[2, 15, 19, 23]	[5, 7, 9, 17]	[6, 11, 16, 19]	[9, 15, 18, 23]
[0, 10, 11, 20]	[1, 10, 11, 12]	[2, 16, 17, 19]	[5, 7, 9, 22]	[6, 11, 16, 20]	[11, 15, 20, 22]
[0, 10, 20, 21]	[1, 12, 14, 21]	[2, 16, 19, 22]	[5, 7, 17, 20]	[6, 13, 14, 19]	[11, 16, 17, 19]
[0, 11, 15, 20]	[1, 13, 15, 23]	[3, 4, 13, 17]	[5, 7, 20, 22]	[6, 16, 20, 21]	[11, 16, 17, 20]
[0, 13, 14, 19]	[1, 13, 16, 17]	[3, 5, 12, 13]	[5, 8, 11, 22]	[7, 8, 12, 14]	[14, 17, 20, 21]
[0, 14, 20, 21]	[1, 13, 16, 23]	[3, 5, 12, 21]	[5, 8, 11, 23]	[7, 8, 16, 23]	[15, 18, 19, 23]
[0, 15, 18, 19]	[1, 14, 17, 21]	[3, 5, 13, 22]	[5, 8, 21, 22]	[7, 9, 16, 22]	[16, 17, 20, 21]

Octahedral space

[0, 1, 6, 11]	[0, 6, 20, 21]	[2, 5, 9, 18]	[3, 4, 14, 15]	[4, 9, 13, 15]	[7, 8, 14, 16]
[0, 1, 6, 21]	[0, 7, 9, 16]	[2, 5, 9, 22]	[3, 9, 15, 16]	[4, 9, 17, 18]	[7, 9, 15, 16]
[0, 1, 11, 17]	[0, 7, 14, 16]	[2, 5, 10, 11]	[3, 10, 11, 19]	[4, 10, 11, 17]	[7, 10, 14, 20]
[0, 1, 17, 21]	[0, 7, 14, 20]	[2, 5, 10, 12]	[3, 10, 14, 20]	[4, 10, 12, 17]	[7, 10, 20, 21]
[0, 2, 7, 9]	[0, 7, 20, 21]	[2, 5, 12, 22]	[3, 10, 19, 21]	[4, 12, 17, 18]	[7, 11, 15, 16]
[0, 2, 7, 21]	[1, 6, 8, 11]	[2, 5, 14, 18]	[3, 10, 20, 21]	[4, 13, 14, 15]	[8, 9, 13, 19]
[0, 2, 9, 17]	[1, 6, 8, 12]	[2, 7, 9, 22]	[3, 12, 15, 16]	[5, 8, 11, 16]	[8, 11, 13, 19]
[0, 2, 17, 21]	[1, 6, 12, 22]	[2, 7, 10, 12]	[3, 12, 15, 20]	[5, 8, 14, 16]	[9, 13, 19, 22]
[0, 3, 4, 9]	[1, 6, 21, 22]	[2, 7, 10, 21]	[3, 12, 19, 22]	[5, 9, 18, 19]	[10, 11, 13, 19]
[0, 3, 4, 11]	[1, 7, 8, 11]	[2, 7, 12, 22]	[3, 12, 20, 22]	[5, 9, 19, 22]	[10, 11, 16, 17]
[0, 3, 9, 16]	[1, 7, 8, 12]	[2, 8, 11, 13]	[3, 14, 15, 20]	[5, 10, 11, 16]	[10, 12, 16, 17]
[0, 3, 11, 19]	[1, 7, 9, 15]	[2, 8, 13, 14]	[3, 19, 21, 22]	[5, 10, 12, 16]	[10, 13, 19, 21]
[0, 3, 12, 16]	[1, 7, 9, 22]	[2, 9, 17, 18]	[3, 20, 21, 22]	[5, 12, 19, 22]	[11, 15, 16, 17]
[0, 3, 12, 19]	[1, 7, 11, 15]	[2, 10, 11, 13]	[4, 6, 8, 9]	[5, 14, 18, 19]	[12, 15, 16, 17]
[0, 4, 9, 17]	[1, 7, 12, 22]	[2, 10, 13, 21]	[4, 6, 8, 12]	[6, 8, 9, 19]	[12, 15, 17, 20]
[0, 4, 11, 17]	[1, 9, 13, 15]	[2, 13, 14, 15]	[4, 6, 9, 18]	[6, 8, 11, 19]	[12, 17, 18, 20]
[0, 5, 12, 16]	[1, 9, 13, 22]	[2, 13, 15, 21]	[4, 6, 12, 18]	[6, 9, 18, 19]	[13, 19, 21, 22]
[0, 5, 12, 19]	[1, 11, 15, 17]	[2, 14, 15, 17]	[4, 7, 8, 12]	[6, 12, 18, 20]	[14, 15, 17, 20]
[0, 5, 14, 16]	[1, 13, 15, 21]	[2, 14, 17, 18]	[4, 7, 8, 14]	[6, 12, 20, 22]	[14, 17, 18, 20]
[0, 5, 14, 19]	[1, 13, 21, 22]	[2, 15, 17, 21]	[4, 7, 10, 12]	[6, 14, 18, 19]	
[0, 6, 11, 19]	[1, 15, 17, 21]	[3, 4, 9, 15]	[4, 7, 10, 14]	[6, 14, 18, 20]	
[0, 6, 14, 19]	[2, 5, 8, 11]	[3, 4, 10, 11]	[4, 8, 9, 13]	[6, 20, 21, 22]	
[0, 6, 14, 20]	[2, 5, 8, 14]	[3, 4, 10, 14]	[4, 8, 13, 14]	[7, 8, 11, 16]	

Cube space

A.5 Balanced non-vertex decomposable and non-shellable 3-spheres

$[6,19,v_3,v_{12}]$	$[14,19,21,v_{23}]$	$[6,16,v_3,v_{12}]$	$[14,19,v_3,v_{12}]$	$[10,11,19,v_2]$	$[14,19,v_1,v_{23}]$	$[7,9,14,v_1]$	$[7,8,13,v_{23}]$
$[6,19,v_3,v_{13}]$	$[15,19,v_2,v_{13}]$	$[6,16,v_3,v_{13}]$	$[6,9,13,17]$	$[10,15,19,v_2]$	$[14,20,v_1,v_{23}]$	$[7,9,14,v_3]$	$[7,8,13,v_{13}]$
$[6,19,21,v_{12}]$	$[17,18,21,v_{12}]$	$[11,16,v_3,v_{13}]$	$[10,16,18,v_2]$	$[9,11,13,17]$	$[12,15,v_1,v_{23}]$	$[7,14,v_3,v_{12}]$	$[12,14,v_2,v_{12}]$
$[14,19,21,v_{12}]$	$[16,18,21,v_{12}]$	$[11,16,v_3,v_{12}]$	$[10,17,18,v_2]$	$[9,11,17,v_1]$	$[8,12,v_1,v_{23}]$	$[7,15,v_3,v_{12}]$	$[12,15,v_2,v_{12}]$
$[6,17,21,v_{12}]$	$[6,16,v_1,v_{12}]$	$[11,16,v_2,v_{13}]$	$[17,18,v_2,v_{23}]$	$[9,11,12,13]$	$[8,20,v_1,v_{23}]$	$[7,15,v_3,v_{23}]$	$[7,15,v_2,v_{12}]$
$[6,17,v_1,v_{12}]$	$[6,10,16,v_1]$	$[6,16,v_2,v_{13}]$	$[16,18,v_2,v_{23}]$	$[11,12,13,v_{12}]$	$[8,13,20,v_{23}]$	$[7,13,15,v_{23}]$	$[7,14,v_2,v_{12}]$
$[17,18,v_1,v_{12}]$	$[14,16,21,v_{12}]$	$[6,10,16,v_2]$	$[11,16,v_2,v_{23}]$	$[12,13,14,v_{12}]$	$[8,9,12,v_3]$	$[7,13,15,v_{13}]$	$[7,14,v_2,v_{13}]$
$[16,18,v_1,v_{12}]$	$[6,19,21,v_{23}]$	$[6,10,12,v_2]$	$[11,17,v_2,v_{23}]$	$[11,12,v_3,v_{12}]$	$[7,8,9,v_3]$	$[7,15,v_2,v_{13}]$	$[7,14,v_1,v_{13}]$
$[6,9,17,v_1]$	$[7,8,9,v_1]$	$[6,12,v_2,v_{13}]$	$[11,13,17,v_{23}]$	$[9,11,12,v_3]$	$[8,12,v_3,v_{23}]$	$[12,13,14,v_{13}]$	$[7,8,v_1,v_{13}]$
$[6,9,12,v_1]$	$[10,16,18,v_1]$	$[6,12,13,v_{13}]$	$[11,13,16,v_{23}]$	$[9,11,19,v_3]$	$[12,15,v_3,v_{23}]$	$[13,14,20,v_{13}]$	
$[6,10,12,v_1]$	$[10,17,18,v_1]$	$[6,13,19,v_{13}]$	$[13,14,16,v_{23}]$	$[9,14,19,v_3]$	$[12,15,v_3,v_{12}]$	$[7,8,v_3,v_{23}]$	
$[8,9,12,v_1]$	$[14,16,21,v_{23}]$	$[13,15,19,v_{13}]$	$[11,13,16,v_{12}]$	$[10,12,15,v_2]$	$[10,11,17,v_1]$	$[8,13,20,v_{13}]$	
$[6,9,12,13]$	$[16,18,21,v_{23}]$	$[6,13,19,v_{23}]$	$[10,11,17,v_2]$	$[10,12,15,v_1]$	$[10,11,19,v_1]$	$[14,20,v_1,v_{13}]$	
$[11,19,v_3,v_{13}]$	$[17,18,21,v_{23}]$	$[13,15,19,v_{23}]$	$[13,14,20,v_{23}]$	$[10,15,19,v_1]$	$[9,11,19,v_1]$	$[8,20,v_1,v_{13}]$	
$[11,19,v_2,v_{13}]$	$[6,17,21,v_{23}]$	$[6,13,17,v_{23}]$	$[13,14,16,v_{12}]$	$[15,19,v_1,v_{23}]$	$[9,14,19,v_1]$	$[12,14,v_2,v_{13}]$	

The list of facets of Δ^{2T}_{22} (see Proposition 6.24). The order given by the columns (top to bottom and left to right) is a shelling order. The 6 vertices of the double-trefoil knot are labeled v_F, according to the face F they correspond to in the barycentric subdivision.

$[0,2,6,v_{13}]$	$[0,10,21,v_3]$	$[5,11,12,v_{23}]$	$[8,17,20,v_{12}]$	$[0,8,17,v_{12}]$	$[3,7,11,v_1]$	$[0,10,21,v_1]$	$[14,18,19,v_{13}]$
$[1,11,19,v_1]$	$[10,20,v_3,v_{13}]$	$[1,14,18,19]$	$[5,16,v_2,v_{13}]$	$[11,19,21,v_1]$	$[2,4,v_2,v_{13}]$	$[5,12,16,v_{13}]$	$[5,6,14,v_{12}]$
$[8,16,19,21]$	$[2,4,v_1,v_{13}]$	$[5,17,18,v_{23}]$	$[0,17,v_2,v_{13}]$	$[3,7,16,v_1]$	$[11,13,v_1,v_{23}]$	$[0,1,8,17]$	$[0,11,21,v_3]$
$[6,14,19,v_{13}]$	$[1,8,17,20]$	$[1,10,18,20]$	$[4,16,v_1,v_{23}]$	$[2,5,6,v_{23}]$	$[5,11,18,v_{23}]$	$[5,14,v_1,v_{12}]$	$[11,12,13,v_{23}]$
$[1,13,17,v_2]$	$[6,17,19,v_{13}]$	$[5,6,10,v_{23}]$	$[13,17,21,v_2]$	$[5,17,v_3,v_{23}]$	$[1,14,19,v_1]$	$[0,16,v_1,v_{23}]$	$[2,6,20,v_{13}]$
$[14,19,21,v_1]$	$[10,18,20,v_{13}]$	$[0,15,17,v_3]$	$[7,15,16,v_3]$	$[6,14,19,21]$	$[0,6,10,15]$	$[2,18,19,v_{12}]$	$[1,10,18,19]$
$[3,7,11,v_3]$	$[17,19,v_3,v_{23}]$	$[5,14,21,v_1]$	$[6,14,20,v_{13}]$	$[5,6,16,21]$	$[9,15,16,v_1]$	$[10,20,21,v_2]$	$[5,16,21,v_2]$
$[0,1,8,16]$	$[0,2,v_2,v_{13}]$	$[9,12,15,16]$	$[7,11,18,v_{13}]$	$[1,14,20,v_1]$	$[10,12,13,v_{12}]$	$[4,16,v_1,v_{13}]$	$[5,10,12,v_{23}]$
$[8,11,20,21]$	$[17,18,19,v_{13}]$	$[9,16,v_1,v_{13}]$	$[2,12,20,v_{12}]$	$[10,18,19,v_{12}]$	$[0,6,10,v_{23}]$	$[2,4,v_2,v_{23}]$	$[6,14,20,v_{12}]$
$[11,20,v_3,v_{13}]$	$[5,6,10,15]$	$[3,7,16,v_3]$	$[11,12,13,v_{12}]$	$[0,10,15,v_3]$	$[0,2,v_2,v_{23}]$	$[11,19,v_2,v_{12}]$	$[13,17,21,v_1]$
$[5,12,15,16]$	$[10,13,21,v_1]$	$[0,6,15,17]$	$[8,16,20,21]$	$[0,1,17,v_2]$	$[1,8,16,20]$	$[5,11,12,v_{13}]$	$[2,6,20,v_{12}]$
$[0,2,6,v_{23}]$	$[14,20,v_1,v_{12}]$	$[2,19,v_3,v_{12}]$	$[0,3,11,v_1]$	$[1,10,20,v_2]$	$[5,10,12,15]$	$[5,17,v_2,v_{13}]$	$[2,7,12,15]$
$[1,11,13,v_1]$	$[5,11,18,v_{13}]$	$[2,7,18,v_{23}]$	$[0,8,16,v_{12}]$	$[1,11,19,v_2]$	$[8,11,19,v_{12}]$	$[4,16,v_2,v_{13}]$	$[5,17,v_1,v_{12}]$
$[5,17,21,v_2]$	$[7,10,18,v_{13}]$	$[1,10,19,v_2]$	$[2,5,v_3,v_{12}]$	$[0,3,16,v_3]$	$[7,10,12,v_{12}]$	$[0,17,v_3,v_{12}]$	$[0,16,v_3,v_{12}]$
$[7,11,v_3,v_{13}]$	$[17,18,19,v_{23}]$	$[11,13,v_2,v_{12}]$	$[2,7,15,v_1]$	$[2,5,6,v_{12}]$	$[2,7,v_1,v_{23}]$	$[8,16,19,v_{12}]$	$[4,16,v_2,v_{23}]$
$[8,11,19,21]$	$[10,20,21,v_3]$	$[8,11,20,v_{12}]$	$[5,6,15,16]$	$[2,7,12,v_{12}]$	$[6,16,19,21]$	$[15,16,19,v_3]$	$[1,14,18,20]$
$[1,17,20,v_1]$	$[2,9,15,v_1]$	$[2,12,20,v_{13}]$	$[10,19,v_2,v_{12}]$	$[16,20,21,v_2]$	$[0,1,16,v_2]$	$[1,16,20,v_2]$	$[0,11,21,v_1]$
$[5,17,21,v_1]$	$[2,19,v_3,v_{23}]$	$[7,11,v_1,v_{23}]$	$[10,12,13,v_{23}]$	$[10,13,21,v_2]$	$[15,17,19,v_3]$	$[0,10,v_1,v_{23}]$	$[1,13,17,v_1]$
$[0,16,v_2,v_{23}]$	$[2,4,v_1,v_{23}]$	$[14,18,20,v_{13}]$	$[7,15,16,v_1]$	$[17,20,v_1,v_{12}]$	$[9,12,16,v_{13}]$	$[7,11,18,v_{23}]$	$[11,12,20,v_{12}]$
$[5,17,v_3,v_{12}]$	$[11,20,21,v_3]$	$[2,18,19,v_{23}]$	$[7,10,15,v_3]$	$[7,10,12,15]$	$[5,6,14,21]$	$[11,12,20,v_{13}]$	$[0,3,16,v_1]$
$[2,9,12,15]$	$[1,11,13,v_2]$	$[2,9,12,v_{13}]$	$[7,10,18,v_{12}]$	$[2,9,v_1,v_{13}]$	$[6,15,16,19]$	$[2,5,v_3,v_{23}]$	$[5,17,18,v_{13}]$
$[0,3,11,v_3]$	$[10,13,v_2,v_{12}]$	$[7,10,v_3,v_{13}]$	$[0,6,17,v_{13}]$	$[2,7,18,v_{12}]$	$[10,13,v_1,v_{23}]$	$[16,19,v_3,v_{12}]$	$[6,15,17,19]$

The list of facets of Δ^{3T}_{28} (see Proposition 6.24). The 6 vertices of the double-trefoil knot are labeled v_F, according to the face F they correspond to in the barycentric subdivision.

Bibliography

[Adi18] K. Adiprasito. "Combinatorial Lefschetz theorems beyond positivity". In: *ArXiv e-prints* (Dec. 2018). arXiv: 1812.10454 [math.CO].

[AI15] K. A. Adiprasito and I. Izmestiev. "Derived subdivisions make every PL sphere polytopal". In: *Israel J. Math.* 208.1 (2015), pp. 443–450. ISSN: 0021-2172.

[AS16] K. A. Adiprasito and R. Sanyal. "Relative Stanley-Reisner theory and upper bound theorems for Minkowski sums". In: *Publ. Math. Inst. Hautes Études Sci.* 124 (2016), pp. 99–163. ISSN: 0073-8301.

[Akh] T. Akhmejanov. Personal page. https://http://pi.math.cornell.edu/~takhmejanov/pseudoManifolds.html.

[Ale30] J. W. Alexander. "The combinatorial theory of complexes". In: *Ann. of Math. (2)* 31.2 (1930), pp. 292–320. ISSN: 0003-486X.

[BN04] E. Babson and I. Novik. "Face numbers and nongeneric initial ideals". In: *Electron. J. Combin.* 11.2 (2004/06), Research Paper 25, 23. ISSN: 1077-8926.

[Bar73] D. Barnette. "A proof of the lower bound conjecture for convex polytopes". In: *Pacific J. Math.* 46 (1973), pp. 349–354. ISSN: 0030-8730.

[BL13] B. Benedetti and F. H. Lutz. "Knots in collapsible and non-collapsible balls". In: *Electron. J. Combin.* 20.3 (2013), Paper 31, 29. ISSN: 1077-8926.

[BL14] B. Benedetti and F. H. Lutz. "Random discrete Morse theory and a new library of triangulations". In: *Exp. Math.* 23.1 (2014), pp. 66–94. ISSN: 1058-6458.

[Big93] A. Bigatti. "Upper bounds for the Betti numbers of a given Hilbert function". In: *Comm. Algebra* 21.7 (1993), pp. 2317–2334. ISSN: 0092-7872.

[BL81] L. J. Billera and C. W. Lee. "A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes". In: *J. Combin. Theory Ser. A* 31.3 (1981), pp. 237–255. ISSN: 0097-3165.

[Bin83] R. H. Bing. *The geometric topology of 3-manifolds.* Vol. 40. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1983, pp. x+238. ISBN: 0-8218-1040-5.

[Bjö95] A. Björner. "Topological methods". In: *Handbook of combinatorics, Vol. 1, 2.* Elsevier Sci. B. V., Amsterdam, 1995, pp. 1819–1872.

[BFS87] A. Björner, P. Frankl, and R. Stanley. "The number of faces of balanced Cohen-Macaulay complexes and a generalized Macaulay theorem". In: *Combinatorica* 7.1 (1987), pp. 23–34. ISSN: 0209-9683.

[BL00] A. Björner and F. H. Lutz. "Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere". In: *Experiment. Math.* 9.2 (2000), pp. 275–289. ISSN: 1058-6458.

[BW96] A. Björner and M. L. Wachs. “Shellable nonpure complexes and posets. I”. In: *Trans. Amer. Math. Soc.* 348.4 (1996), pp. 1299–1327. ISSN: 0002-9947.

[BW97] A. Björner and M. L. Wachs. “Shellable nonpure complexes and posets. II”. In: *Trans. Amer. Math. Soc.* 349.10 (1997), pp. 3945–3975. ISSN: 0002-9947.

[BM07] G. Brinkmann and B. D. McKay. “Fast generation of planar graphs”. In: *MATCH Commun. Math. Comput. Chem.* 58.2 (2007), pp. 323–357. ISSN: 0340-6253.

[BK11] J. Browder and S. Klee. “Lower bounds for Buchsbaum complexes”. In: *European J. Combin.* 32.1 (2011), pp. 146–153. ISSN: 0195-6698.

[BM71] H. Bruggesser and P. Mani. “Shellable decompositions of cells and spheres”. In: *Math. Scand.* 29 (1971), 197–205 (1972). ISSN: 0025-5521.

[BH93] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993, pp. xii+403. ISBN: 0-521-41068-1.

[BDS17] B. A. Burton, B. Datta, and J. Spreer. “The Pachner graph of 2-spheres”. In: *ArXiv e-prints* (2017). arXiv: `1701.05144 [math.CO]`.

[Can79] J. W. Cannon. “Shrinking Cell-Like Decompositions of Manifolds. Codimension Three”. In: *Annals of Mathematics* 110.1 (1979), pp. 83–112.

[Cas95] M. R. Casali. “A note about bistellar operations on PL-manifolds with boundary”. In: *Geom. Dedicata* 56.3 (1995), pp. 257–262. ISSN: 0046-5755.

[CK10] S. Choi and J. S. Kim. “A combinatorial proof of a formula for Betti numbers of a stacked polytope”. In: *Electron. J. Combin.* 17.1 (2010), Research Paper 9, 8. ISSN: 1077-8926.

[CL69] G. F. Clements and B. Lindström. “A generalization of a combinatorial theorem of Macaulay”. In: *J. Combinatorial Theory* 7 (1969), pp. 230–238.

[CKS17] G. Codenotti, L. Katthän, and R. Sanyal. “On f- and h- vectors of relative simplicial complexes”. In: *ArXiv e-prints* (Nov. 2017). arXiv: `1711.02729 [math.CO]`.

[CV19] G. Codenotti and L. Venturello. “Octahedralizing 3-colorable 3-polytopes”. In: *ArXiv e-prints* (Mar. 2019). arXiv: `1903.10178 [math.CO]`.

[Cor+04] L. P. Cordella, P. Foggia, C. Sansone, and M. Vento. “A (sub)graph isomorphism algorithm for matching large graphs”. In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 26.10 (2004), pp. 1367–1372. ISSN: 0162-8828.

[DK78] G. Danaraj and V. Klee. “A representation of 2-dimensional pseudomanifolds and its use in the design of a linear-time shelling algorithm”. In: *Ann. Discrete Math.* 2 (1978). Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976), pp. 53–63.

[DM17] B. Datta and S. Murai. “On stacked triangulated manifolds”. In: *Electron. J. Combin.* 24.4 (2017), Paper 4.12, 14. ISSN: 1077-8926.

[Duv+16] A. M. Duval, B. Goeckner, C. J. Klivans, and J. L. Martin. “A non-partitionable Cohen-Macaulay simplicial complex”. In: *Adv. Math.* 299 (2016), pp. 381–395. ISSN: 0001-8708.

[Edw75] R. D. Edwards. "The double suspension of a certain homology 3-sphere is S^5". In: *Notices AMS* 22.A-334 (1975).

[EH01] R. Ehrenborg and M. Hachimori. "Non-constructible complexes and the bridge index". In: *European J. Combin.* 22.4 (2001), pp. 475–489. ISSN: 0195-6698.

[EK90] S. Eliahou and M. Kervaire. "Minimal resolutions of some monomial ideals". In: *J. Algebra* 129.1 (1990), pp. 1–25. ISSN: 0021-8693.

[ER02] E. G. Evans Jr. and B. P. Richert. "Possible resolutions for a given Hilbert function". In: *Comm. Algebra* 30.2 (2002), pp. 897–906. ISSN: 0092-7872.

[Fis77a] S. Fisk. "Geometric coloring theory". In: *Advances in Math.* 24.3 (1977), pp. 298–340. ISSN: 0001-8708.

[Fis77b] S. Fisk. "Variations on coloring, surfaces and higher-dimensional manifolds". In: *Advances in Math.* 25.3 (1977), pp. 226–266.

[Fog88] A. Fogelsanger. *The generic rigidity of minimal cycles*. Thesis (Ph.D.)–Cornell University. ProQuest LLC, Ann Arbor, MI, 1988, p. 68.

[Fra04] C. A. Francisco. "Almost complete intersections and the lex-plus-powers conjecture". In: *J. Algebra* 276.2 (2004), pp. 737–760. ISSN: 0021-8693.

[FR07] C. A. Francisco and B. P. Richert. "Lex-plus-powers ideals". In: *Syzygies and Hilbert functions*. Vol. 254. Lect. Notes Pure Appl. Math. Chapman & Hall/CRC, Boca Raton, FL, 2007, pp. 113–144.

[Gar80] A. M. Garsia. "Combinatorial methods in the theory of Cohen-Macaulay rings". In: *Adv. in Math.* 38.3 (1980), pp. 229–266. ISSN: 0001-8708.

[GKN11] M. Goff, S. Klee, and I. Novik. "Balanced complexes and complexes without large missing faces". In: *Ark. Mat.* 49.2 (2011), pp. 335–350. ISSN: 0004-2080.

[Got78] G. Gotzmann. "Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes". In: *Math. Z.* 158.1 (1978), pp. 61–70. ISSN: 0025-5874.

[GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Grü03] B. Grünbaum. *Convex polytopes*. Vol. 221. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003, pp. xvi+468. ISBN: 0-387-40409-0.

[Hac00a] M. Hachimori. "Combinatorics of constructible complexes". Tokio University. PhD thesis. 2000.

[Hac00b] M. Hachimori. "Constructible complexes and recursive division of posets". In: *Theoret. Comput. Sci.* 235.2 (2000). Combinatorics and optimization (Okinawa, 1996), pp. 225–237. ISSN: 0304-3975.

[HZ00] M. Hachimori and G. M. Ziegler. "Decompositons of simplicial balls and spheres with knots consisting of few edges". In: *Math. Z.* 235.1 (2000), pp. 159–171. ISSN: 0025-5874.

[Hal04] H. T. Hall. *Counterexamples in discrete geometry*. Thesis (Ph.D.)–University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 2004, p. 55. ISBN: 978-0542-00821-4.

[HH99] J. Herzog and T. Hibi. "Componentwise linear ideals". In: *Nagoya Math. J.* 153 (1999), pp. 141–153. ISSN: 0027-7630.

[HH11] J. Herzog and T. Hibi. *Monomial ideals*. Vol. 260. Graduate Texts in Mathematics. Springer-Verlag London, Ltd., London, 2011, pp. xvi+305. ISBN: 978-0-85729-105-9.

[Hul93] H. A. Hulett. "Maximum Betti numbers of homogeneous ideals with a given Hilbert function". In: *Comm. Algebra* 21.7 (1993), pp. 2335–2350. ISSN: 0092-7872.

[IJ03] I. Izemestiev and M. Joswig. "Branched coverings, triangulations, and 3-manifolds". In: *Adv. Geom.* 3.2 (2003), pp. 191–225.

[IKN17] I. Izmestiev, S. Klee, and I. Novik. "Simplicial moves on balanced complexes". In: *Adv. Math.* 320 (2017), pp. 82–114. ISSN: 0001-8708.

[Jon08] J. Jonsson. *Simplicial complexes of graphs*. Vol. 1928. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2008, pp. xiv+378. ISBN: 978-3-540-75858-7.

[JKM17] M. Juhnke-Kubitzke and S. Murai. "Balanced generalized lower bound inequality for simplicial polytopes". In: *Sel. Math. New Ser.* (2017), pp. 1–13.

[JV17] M. Juhnke-Kubitzke and L. Venturello. "A balanced non-partitionable Cohen-Macaulay complex". In: *ArXiv e-prints* (Nov. 2017). arXiv: 1711.05529 [math.CO].

[JV18a] M. Juhnke-Kubitzke and L. Venturello. "Balanced shellings and moves on balanced manifolds". In: *ArXiv e-prints* (Apr. 2018). arXiv: 1804.06270 [math.CO].

[JV18b] M. Juhnke-Kubitzke and L. Venturello. "Graded Betti numbers of balanced simplicial complexes". In: *ArXiv e-prints* (Nov. 2018). arXiv: 1811.03892 [math.CO].

[JK+18] M. Juhnke-Kubitzke, S. Murai, I. Novik, and C. Sawaske. "A generalized lower bound theorem for balanced manifolds". In: *Math. Z.* 289.3-4 (2018), pp. 921–942. ISSN: 0025-5874.

[Kal87] G. Kalai. "Rigidity and the lower bound theorem. I". In: *Invent. Math.* 88.1 (1987), pp. 125–151. ISSN: 0020-9910.

[Kal88] G. Kalai. "Many triangulated spheres". In: *Discrete Comput. Geom.* 3.1 (1988), pp. 1–14. ISSN: 0179-5376.

[KK79] B. Kind and P. Kleinschmidt. "Schälbare Cohen-Macauley-Komplexe und ihre Parametrisierung". In: *Math. Z.* 167.2 (1979), pp. 173–179. ISSN: 0025-5874.

[KMS19] S. Klee, S. Murai, and Y. Suzuki. "Exceptional Balanced Triangulations on Surfaces". In: *Graphs and Combinatorics* (2019). ISSN: 1435-5914.

[KN16a] S. Klee and I. Novik. "Face enumeration on simplicial complexes". In: *Recent trends in combinatorics*. Vol. 159. IMA Vol. Math. Appl. Springer, [Cham], 2016, pp. 653–686.

[KN16b] S. Klee and I. Novik. "Lower bound theorems and a generalized lower bound conjecture for balanced simplicial complexes". In: *Mathematika* 62.2 (2016), pp. 441–477. ISSN: 0025-5793.

[KK87] V. Klee and P. Kleinschmidt. "The d-step conjecture and its relatives". In: *Math. Oper. Res.* 12.4 (1987), pp. 718–755. ISSN: 0364-765X.

[Lic91] W. B. R. Lickorish. “Unshellable triangulations of spheres”. In: *European J. Combin.* 12.6 (1991), pp. 527–530. ISSN: 0195-6698.

[Lic99] W. B. R. Lickorish. “Simplicial moves on complexes and manifolds”. In: 2 (Nov. 1999), pp. 299–320.

[Lon13] M. de Longueville. *A course in topological combinatorics.* Universitext. Springer, New York, 2013, pp. xii+238. ISBN: 978-1-4419-7909-4; 978-1-4419-7910-0.

[LR06] M. Ludwig and M. Reitzner. “Elementary moves on triangulations”. In: *Discrete Comput. Geom.* 35 (2006), pp. 527–536.

[Lut] F. H. Lutz. The Manifold Page. `http://page.math.tu-berlin.de/~lutz/stellar/`.

[Lut99] F. H. Lutz. *Triangulated manifolds with few vertices and vertex-transitive group actions.* Berichte aus der Mathematik. [Reports from Mathematics]. Dissertation, Technischen Universität Berlin, Berlin, 1999. Verlag Shaker, Aachen, 1999, pp. vi+137. ISBN: 3-8265-6450-2.

[Lut08] F. H. Lutz. “Combinatorial 3-manifolds with 10 vertices”. In: *Beiträge Algebra Geom.* 49.1 (2008), pp. 97–106. ISSN: 0138-4821.

[LN16] F. H. Lutz and E. Nevo. “Stellar theory for flag complexes”. In: *Math. Scand.* 118.1 (2016), pp. 70–82.

[Mac27] F. S. Macaulay. “Some Properties of Enumeration in the Theory of Modular Systems”. In: *Proc. London Math. Soc. (2)* 26 (1927), pp. 531–555. ISSN: 0024-6115.

[Man16] C. Manolescu. “Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture”. In: *J. Amer. Math. Soc.* 29 (2016), pp. 147–176.

[Mat03] J. Matoušek. “Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry”. In: (2003).

[McM70] P. McMullen. “The maximum numbers of faces of a convex polytope”. In: *Mathematika* 17 (1970), pp. 179–184. ISSN: 0025-5793.

[MM11] J. Mermin and S. Murai. “The lex-plus-powers conjecture holds for pure powers”. In: *Adv. Math.* 226.4 (2011), pp. 3511–3539. ISSN: 0001-8708.

[MPS08] J. Mermin, I. Peeva, and M. Stillman. “Ideals containing the squares of the variables”. In: *Adv. Math.* 217.5 (2008), pp. 2206–2230. ISSN: 0001-8708.

[MN03] J. Migliore and U. Nagel. “Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers”. In: *Adv. Math.* 180.1 (2003), pp. 1–63. ISSN: 0001-8708.

[Mun84a] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984, pp. ix+454. ISBN: 0-201-04586-9.

[Mun84b] J. R. Munkres. “Topological results in combinatorics”. In: *Michigan Math. J.* 31.1 (1984), pp. 113–128. ISSN: 0026-2285.

[Mur08] S. Murai. “Betti numbers of strongly color-stable ideals and squarefree strongly color-stable ideals”. In: *J. Algebraic Combin.* 27.3 (2008), pp. 383–398. ISSN: 0925-9899.

[Mur15] S. Murai. “Tight combinatorial manifolds and graded Betti numbers”. In: *Collect. Math.* 66.3 (2015), pp. 367–386. ISSN: 0010-0757.

[MN13] S. Murai and E. Nevo. “On the generalized lower bound conjecture for polytopes and spheres”. In: *Acta Math.* 210.1 (2013), pp. 185–202. ISSN: 0001-5962.

[MS18] S. Murai and Y. Suzuki. “Balanced subdivisions and flips on surfaces”. In: *Proc. Amer. Math. Soc.* 146 (2018), pp. 939–951.

[New31] M. H. A. Newman. “A Theorem in Combinatory Topology”. In: *J. London Math. Soc.* S1-6.3 (1931), pp. 186–192.

[NS09a] I. Novik and E. Swartz. “Applications of Klee’s Dehn-Sommerville relations”. In: *Discrete Comput. Geom.* 42.2 (2009), pp. 261–276. ISSN: 0179-5376.

[NS09b] I. Novik and E. Swartz. “Socles of Buchsbaum modules, complexes and posets”. In: *Adv. Math.* 222.6 (2009), pp. 2059–2084. ISSN: 0001-8708.

[Pac78] U. Pachner. “Bistellare Äquivalenz kombinatorischer Mannigfaltigkeiten”. In: *Arch. Math. (Basel)* 30.1 (1978), pp. 89–98. ISSN: 0003-889X.

[Pac91] U. Pachner. “P.L. homeomorphic manifolds are equivalent by elementary shellings”. In: *European J. Combin.* 12.2 (1991), pp. 129–145. ISSN: 0195-6698.

[Par96] K. Pardue. “Deformation classes of graded modules and maximal Betti numbers”. In: *Illinois J. Math.* 40.4 (1996), pp. 564–585. ISSN: 0019-2082.

[PB80] J. S. Provan and L. J. Billera. “Decompositions of simplicial complexes related to diameters of convex polyhedra”. In: *Math. Oper. Res.* 5.4 (1980), pp. 576–594. ISSN: 0364-765X.

[Rei76] G. A. Reisner. “Cohen-Macaulay quotients of polynomial rings”. In: *Advances in Math.* 21.1 (1976), pp. 30–49. ISSN: 0001-8708.

[RV15] J. Roksvold and H. Verdure. “Betti numbers of skeletons”. In: *ArXiv e-prints* 1502.05670 (Feb. 2015). http://adsabs.harvard.edu/abs/2015arXiv150205670R.

[RS82] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology.* Springer Study Edition. Reprint. Springer-Verlag, Berlin-New York, 1982, pp. viii+123. ISBN: 3-540-11102-6.

[Rud58] M. E. Rudin. “An unshellable triangulation of a tetrahedron”. In: *Bull. Amer. Math. Soc.* 64 (1958), pp. 90–91. ISSN: 0002-9904.

[San12] F. Santos. “A counterexample to the Hirsch conjecture”. In: *Ann. of Math. (2)* 176.1 (2012), pp. 383–412. ISSN: 0003-486X.

[Slo] N.J.A. Sloane. The On-Line Encyclopedia of Integer Sequences. http://oeis.org.

[Sta77] R. P. Stanley. “Cohen-Macaulay complexes”. In: (1977), 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31.

[Sta79] R. P. Stanley. “Balanced Cohen-Macaulay complexes”. In: *Trans. Amer. Math. Soc.* 249.1 (1979), pp. 139–157. ISSN: 0002-9947.

[Sta80] R. P. Stanley. “The number of faces of a simplicial convex polytope”. In: *Adv. in Math.* 35.3 (1980), pp. 236–238. ISSN: 0001-8708.

[Sta96] R. P. Stanley. *Combinatorics and commutative algebra.* Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, pp. x+164. ISBN: 0-8176-3836-9.

[Swa14] E. Swartz. "Thirty-five years and counting". In: *ArXiv e-prints* (2014). arXiv: 1411.0987 [math.CO].

[Tay95] T.-S. Tay. "Lower-bound theorems for pseudomanifolds". In: *Discrete Comput. Geom.* 13.2 (1995), pp. 203–216. ISSN: 0179-5376.

[TH97] N. Terai and T. Hibi. "Computation of Betti numbers of monomial ideals associated with stacked polytopes". In: *Manuscripta Math.* 92.4 (1997), pp. 447–453. ISSN: 0025-2611.

[The17] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.1)*. http://www.sagemath.org. 2017.

[Ven] L. Venturello. Git Hub Repository. https://github.com/LorenzoVenturello/Cross-flips_and_balanced_library.

[Ven18] L. Venturello. "Balanced triangulations on few vertices and an implementation of cross-flips". In: *ArXiv e-prints* (Nov. 2018). arXiv: 1811.10271 [math.CO].

[Wac07] M. L. Wachs. "Poset topology: tools and applications". In: *Geometric combinatorics*. Vol. 13. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2007, pp. 497–615.

[Zhe16] H. Zheng. "Ear Decompostion and Balanced 2-neighborly Simplicial Manifolds". In: *ArXiv e-prints* (Dec. 2016). arXiv: 1612.03512 [math.CO].

[Zhe16] H. Zheng. "Minimal balanced triangulations of sphere bundles over the circle". In: *SIAM J. Discrete Math.* 30.2 (2016), pp. 1259–1268. ISSN: 0895-4801.

[Zie95] G. M. Ziegler. *Lectures on Polytopes*. Graduate texts in mathematics. Springer-Verlag, 1995. ISBN: 9780387943657.

[Zie98] G. M. Ziegler. "Shelling polyhedral 3-balls and 4-polytopes". In: *Discrete Comput. Geom.* 19.2 (1998), pp. 159–174. ISSN: 0179-5376.