On partial regularities and monomial preorders

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Introduction

The Castelnuovo-Mumford regularity and monomial orders are classical notions in commutative algebra and algebraic geometry. Recently, generalizations of these notions, namely on the one hand partial regularities and on the other hand monomial preorders, have attracted considerable attention due to their usefulness in many situations. The main purpose of this thesis is to study these generalizations and their applications.

The Castelnuovo-Mumford regularity was originally defined for sheaves on projective spaces by Mumford [**28**] based on a geometric idea of Castelnuovo (see [**14**, Chapter 20] for more details). This notion was then extended to graded modules by Eisenbud and Goto [**15**]. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K with maximal homogeneous ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, and let M be a finitely generated graded R-module. Then the Castelnuovo-Mumford regularity of M is defined to be

$$\operatorname{reg}(M) := \max\{j \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_{j-i} \neq 0 \ \text{ for some } \ i \geq 0\},$$

where $H^i_{\mathfrak{m}}(M)$ denotes the i-th local cohomology module of M with respect to $\mathfrak{m}$. This number can also be characterized in terms of Tor modules as follows

$$\operatorname{reg}(M) = \max\{j \in \mathbb{Z} \mid \operatorname{Tor}^R_i(M, K)_{j+i} \neq 0 \ \text{ for some } \ i \geq 0\}.$$

Thus, the Castelnuovo-Mumford regularity is an important invariant which measures the (homological) complexity of the given module. This invariant is, however, usually hard to compute. Therefore, it would be useful if there are invariants that are easier to handle with, but can be used to estimate the Castelnuovo-Mumford regularity. Partial regularities are such invariants.

Motivated by the above characterizations of the Castelnuovo-Mumford regularity, we define for each non-negative integer t the following partial regularities:

$$\begin{aligned}
\operatorname{reg}_{L,t}(M) &:= \max\{j \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_{j-i} \neq 0 \ \text{ for some } \ 0 \leq i \leq t\},\\
\operatorname{reg}^{L,t}(M) &:= \max\{j \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(M)_{j-i} \neq 0 \ \text{ for some } \ i \geq t\},\\
\operatorname{reg}_{T,t}(M) &:= \max\{j \in \mathbb{Z} \mid \operatorname{Tor}^R_i(M,K)_{j+i} \neq 0 \ \text{ for some } \ 0 \leq i \leq t\},\\
\operatorname{reg}^{T,t}(M) &:= \max\{j \in \mathbb{Z} \mid \operatorname{Tor}^R_i(M,K)_{j+i} \neq 0 \ \text{ for some } \ i \geq t\}.
\end{aligned}$$

Partial regularities can be seen as refinements of the Castelnuovo-Mumford regularity which carry more specific information on the graded structure of the module. Most of these partial regularities already appeared explicitly or implicitly in the literature (see [**13, 21, 42, 43**]), but have not been studied systematically together in a coherent way.

The first main goal of this thesis is to provide a systematic study of some interesting properties of partial regularities. Our results on this topic will be presented in Chapter 2. In particular, we estimate partial regularities by means of filter-regular sequences and investigate their behaviour with respect to short exact sequences. We also study the relationship between partial regularities and related invariants, such as the a-invariants or the Castelnuovo-Mumford regularity of syzygy modules. Generalizing a well-known result on the Castelnuovo-Mumford regularity, we are able to compute partial regularities of stable and squarefree stable monomial ideals. It can also be shown that partial regularities are upper-semicontinuous in flat families, providing bounds for partial regularities of a given ideal in terms of those of its initial ideals. Finally, we extend an upper bound for the Castelnuovo-Mumford regularity obtained in [**11, 20**] to partial regularities.

The second main goal of the thesis is to develop a theory of monomial preorders that generalizes the classical theory of monomial orders. A key feature of monomial orders is that they allow us to study an arbitrary ideal in a polynomial ring through its leading ideals. Many times these ideals can be studied by combinatorial methods: the leading ideals are monomial ideals and carry a lot of information about the original ideal, and the corresponding data often can be computed in a purely combinatorial way. Monomial preorders differ from monomial orders in the way that they are not necessarily total orders. The leading ideal of an ideal with respect to a monomial preorder is thus no longer a monomial ideal. So one may ask whether there is a theory for monomial preorders that is similar to that for monomial orders.

As we will see in Chapter 3, many interesting properties of monomial orders can be generalized to monomial preorders.

For instance, it is possible to characterize monomial preorders in terms of real matrices, extending a well-known characterization of monomial orders due to Robbiano [**31**]. We show that monomial preorders can also be approximated by integral weight orders, which implies that the leading ideal with respect to a monomial preorder is a flat deformation of the given ideal. Moreover, many important properties such as Cohen-Macaulayness, Gorensteinness, Koszulness are shown to descend from the leading ideal to the given ideal. Our results in this chapter have been published in [**24**].

In order to make the thesis as self-contained as possible, we will briefly recall in Chapter 1 several basic notions: minimal free resolutions, local cohomology, generic initial ideals, filter-regular sequences, etc. For unexplained terminology the reader is referred to [**5, 9, 14, 19**].

CHAPTER 1

Background

In this chapter we introduce and recall results from commutative algebra which are needed throughout this thesis. For further details and general references we refer in particular to [**8, 9, 14, 17, 19, 27, 30, 36, 37, 38**].

1.1. Minimal free resolutions

We assume that the reader is familiar with notations and results from homological algebra as discussed in [**5, 9, 14, 29, 34, 44**]. In this section we recall some special invariants related to free resolutions, such as Betti numbers or the Castelnuovo-Mumford regularity. Let us begin with the notion of graded rings and modules.

Definition 1.1.1. A ring R is called **graded** if it has a decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ as an abelian group such that $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$. When this is the case, an R-module M is called **graded** if it has a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ as an abelian group such that $R_iM_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$.

Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded R-module. Then M_i is called the i-th **homogeneous** (or **graded**) **component** of M. Each element $m \in M_i$ is called **homogeneous of degree** i; we denote the degree of m by $\deg m$. An arbitrary element $l \in M$ can be uniquely written as a finite sum $l = \sum_i l_i$ with $l_i \in M_i$; these l_i are called the **homogeneous components** of l.

For $p \in \mathbb{Z}$, denote by $M(p)$ the graded R-module whose graded components are given by $M(p)_i = M_{p+i}$. We say that $M(p)$ is the module M **shifted by** p **degrees**.

A submodule of M is called **graded** if it is generated by homogeneous elements. An ideal in R is **graded** or **homogeneous** if it is a graded submodule of R.

Let N be another graded R-module. We say that a homomorphism $\varphi : M \to N$ is **graded of degree** d if $\varphi(M_i) \subset N_{i+d}$ for all $i \in \mathbb{Z}$.

From now on, let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K.

Then for every integers $d_1, \dots, d_n$ there is a unique grading on R such that

$$\deg x_i = d_i \text{ and } \deg a = 0 \text{ for } a \in K.$$

Here the j-th graded component is the K-vector space generated by monomials $x_1^{u_1} \cdots x_n^{u_n}$ such that $\sum_{i=1}^n u_i d_i = j$. When $d_1 = \cdots = d_n = 1$, we say that R is **standard graded**. In this thesis, unless otherwise stated, we always assume that R is standard graded.

Definition 1.1.2. Let M be a finitely generated graded R-module. The function

$$H(M, -) : \mathbb{Z} \to \mathbb{Z}, \quad i \mapsto H(M, i) := \dim_K M_i$$

is called the **Hilbert function** of M. The **Hilbert series** of M is defined as the formal series

$$H_M(t) = \sum_{i \in \mathbb{Z}} H(M, i) t^i.$$

A famous result of Hilbert (see, e.g., [**19**, Theorem 6.1.3]) asserts that if M has dimension d, then the Hilbert function $H(M, i)$ is a polynomial of degree $d - 1$ for $i \gg 0$, and the Hilbert series can be written in the form

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d},$$

where $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) > 0$.

We now define some notions related to free resolutions. Let $\mathfrak{m} := (x_1, \dots, x_n)$ denote the maximal graded ideal in R. Let M be a finitely generated graded R-module. Then Hilbert's syzygy theorem (see, e.g., [**14**, Corollary 19.7]) tells us that M has a finite graded free resolution which is of length at most n. This means that there is an exact sequence

$$\mathcal{F} : 0 \to F_r \xrightarrow{\varphi_r} F_{r-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

with $r \le n$ such that

(i) the modules F_i are finitely generated graded free R-modules,
(ii) the maps φ_i are homogeneous of degree 0.

By abuse of notation, we will also denote by $\mathcal{F}$ the exact sequence of free modules:

$$0 \to F_r \xrightarrow{\varphi_r} F_{r-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0$$

and call either of them a **graded free resolution** of M. Note that if we fix a homogeneous basis for each free module F_i, then each map φ_i is given by a matrix whose entries are homogeneous elements in R.

Let $\mathcal{F}$ be a graded free resolution of M as above and let N be another finitely generated graded R-module. Consider the following complexes

$$\mathcal{F} \otimes_R N : 0 \to F_r \otimes_R N \xrightarrow{\varphi_r \otimes_R N} F_{r-1} \otimes_R N \to \cdots \to F_1 \otimes_R N \xrightarrow{\varphi_1 \otimes_R N} F_0 \otimes_R N \to 0,$$
$$\operatorname{Hom}_R(\mathcal{F}, N) : 0 \to \operatorname{Hom}_R(F_0, N) \xrightarrow{\operatorname{Hom}_R(\varphi_1, N)} \operatorname{Hom}_R(F_1, N) \to \cdots .$$

For $i \geq 0$ we define (quickly)

$$\operatorname{Tor}_i^R(M,N) := H_i(\mathcal{F} \otimes_R N) \text{ and } \quad \operatorname{Ext}_R^i(M,N) := H^i(\operatorname{Hom}_R(\mathcal{F}, N)).$$

Some basic properties of Tor and Ext modules that will be used later are summarized in the next result (see, e.g., [**30**, Section 38]).

Theorem 1.1.3. *Let M, N be finitely generated graded R-modules. Then:*

(i) $\operatorname{Tor}_i^R(M,N)$ *and* $\operatorname{Ext}_R^i(M,N)$ *do not depend on the choice of the free resolution, and therefore, are well-defined.*
(ii) $\operatorname{Tor}_i^R(M,N)$ *and* $\operatorname{Ext}_R^i(M,N)$ *are finitely generated graded R-modules for $i \geq 0$.*
(iii) $\operatorname{Tor}_0^R(M,N) = M \otimes_R N$ *and* $\operatorname{Ext}_R^0(M,N) = \operatorname{Hom}_R(M,N)$.
(iv) *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated graded R-modules, then we have the long exact sequence*

$$\begin{aligned} \cdots \to \operatorname{Tor}_i^R(M',N) &\to \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(M'',N) \\ &\to \operatorname{Tor}_{i-1}^R(M',N) \to \operatorname{Tor}_{i-1}^R(M,N) \to \operatorname{Tor}_{i-1}^R(M'',N) \to \\ \cdots \to \operatorname{Tor}_0^R(M',N) &\to \operatorname{Tor}_0^R(M,N) \to \operatorname{Tor}_0^R(M'',N) \to 0 \end{aligned}$$

and also the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Ext}_R^0(M'',N) &\to \operatorname{Ext}_R^0(M,N) \to \operatorname{Ext}_R^0(M',N) \to \\ \cdots \to \operatorname{Ext}_R^{i-1}(M'',N) &\to \operatorname{Ext}_R^{i-1}(M,N) \to \operatorname{Ext}_R^{i-1}(M',N) \\ \to \operatorname{Ext}_R^i(M'',N) &\to \operatorname{Ext}_R^i(M,N) \to \operatorname{Ext}_R^i(M',N) \to \cdots . \end{aligned}$$

The computation of Tor and Ext modules does not depend on the choice of the free resolution. Therefore, in practice, one usually uses a special free resolution to do such computation. In many cases, a minimal free resolution is the most convenient one to work with.

Definition 1.1.4. Let $\mathcal{F}$ be a graded free resolution of M. Then $\mathcal{F}$ is called **minimal** if

$$\varphi_i(F_{i+1}) \subseteq \mathfrak{m}F_i \quad \text{for all } i \geq 0.$$

This means that any matrix representing φ_i has all its entries in $\mathfrak{m}$, or equivalently, the maps of the following complex are all 0:

$$\mathcal{F} \otimes K : 0 \to F_r \otimes K \xrightarrow{\varphi_r \otimes K} F_{r-1} \otimes K \to \cdots \to F_1 \otimes K \xrightarrow{\varphi_1 \otimes K} F_0 \otimes K.$$

It is known that a minimal graded free resolution of M exists and is unique up to an isomorphism (see, e.g., [**14**, Theorem 20.2]). We can therefore also say "*the* minimal graded free resolution of M".

Definition 1.1.5. Let $\mathcal{F}$ be the minimal graded free resolution of M. For $i \geq 0$ the module

$$\mathrm{Im}\varphi_{i+1} = \mathrm{Ker}\varphi_i \subseteq F_i$$

is called the ***i*-th syzygy module** of M and denoted by $\mathrm{Syz}_i(M)$.

Observe that some authors prefer to shift the index of the i-th syzygy module by 1 and call $\mathrm{Syz}_i(M)$ the "$(i+1)$-st syzygy module". For our purposes we follow the convention of [**2**] given in the latter definition and this will be used in the following. From the definition it is clear that

$$0 \to F_r \xrightarrow{\varphi_r} F_{r-1} \to \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} \mathrm{Syz}_i(M) \to 0$$

is the minimal graded free resolution of $\mathrm{Syz}_i(M)$.

Definition 1.1.6. Let $\mathcal{F}$ be the minimal graded free resolution of M. Then each free module F_i is a direct sum of modules of the form $R(-j)$. We define

(i) the **(i,j)-th graded Betti number** of M as

$$\beta_{i,j}(M) := \text{ number of summands in } F_i \text{ of the form } R(-j);$$

(ii) the ***i*-th Betti number** of M as

$$\beta_i(M) := \mathrm{rank} F_i = \sum_{j \in \mathbb{Z}} \beta_{i,j}(M);$$

(iii) the **projective dimension** of M as

$$\mathrm{pd}_R(M) := \text{ length of } \mathcal{F} = \max\{i \mid \beta_i(M) \neq 0\}.$$

It is easy to see that Betti numbers have the following interpretations (see, e.g., [**30**, Sections 11, 12]):

$$\begin{aligned}\beta_{i,j}(M) &= \dim_K \operatorname{Tor}_i^R(M,K)_j = \dim_K \operatorname{Ext}_R^i(M,K)_j \\ &= \text{ number of minimal generators of } \operatorname{Syz}_{i-1}(M) \text{ of degree } j, \\ \beta_i(M) &= \dim_K \operatorname{Tor}_i^R(M,K) = \dim_K \operatorname{Ext}_R^i(M,K) \\ &= \text{ number of minimal generators of } \operatorname{Syz}_{i-1}(M).\end{aligned}$$

We conclude this section with a brief discussion of the Castelnuovo-Mumford regularity, which is a motivation for our study of partial regularities in Chapter 2.

Definition 1.1.7. Let $M \neq 0$ be a finitely generated graded R-module. For $i \geq 0$ we define the **i-th b-invariant** of M by

$$b_i(M) := \begin{cases} \max\{j \in \mathbb{Z} \mid \operatorname{Tor}_i^R(M,K)_j \neq 0\} & \text{if} \quad \operatorname{Tor}_i^R(M,K) \neq 0, \\ -\infty & \text{if} \quad \operatorname{Tor}_i^R(M,K) = 0. \end{cases}$$

The **Castelnuovo-Mumford regularity** of M is the number

$$\operatorname{reg}(M) := \max\{b_i(M) - i \mid i \geq 0\}.$$

The following characterization of the Castelnuovo-Mumford regularity is well-known (see [**14**, Proposition 20.16] or [**43**, Proposition 1.1]).

Proposition 1.1.8. *Let $M \neq 0$ be a finitely generated graded R-module. Then*

$$\begin{aligned}\operatorname{reg}(M) &= \max\{t \in \mathbb{Z} \mid \exists i \geq 0 \text{ such that } \operatorname{Tor}_i^R(M,K)_{t+i} \neq 0\} \\ &= \max\{t \in \mathbb{Z} \mid \exists i \geq 0 \text{ such that } \operatorname{Ext}_R^i(M,R)_{-t-i} \neq 0\}.\end{aligned}$$

Another characterization of the Castelnuovo-Mumford regularity in terms of local cohomology will be given in the next section.

1.2. Local cohomology

The main purpose of this section is to present an interpretation of the Castelnuovo-Mumford regularity in terms of local cohomology modules. We first briefly recall the construction of local cohomology. The reader is referred to [**5, 9**] for more details.

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K with maximal homogeneous ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Let M be a finitely generated graded R-module.

Denote by $\Gamma_\mathfrak{m}(M)$ the following graded submodule of M:

$$\Gamma_\mathfrak{m}(M) := \bigcup_{i\geq 1}(0 :_M \mathfrak{m}^i) = \{y \in M \mid \mathfrak{m}^i y = 0 \text{ for some } i \geq 1\}.$$

It can be checked that $\Gamma_\mathfrak{m}$ defines a left exact additive functor. For $i \geq 0$, the i-th right derived functor of $\Gamma_\mathfrak{m}$ is called the **i-th local cohomology functor** and denoted $H^i_\mathfrak{m}$.

We collect several well-known facts about local cohomology in the next result.

Theorem 1.2.1. *Let M be a finitely generated graded R-module with* $\operatorname{depth} M = t$ *and* $\dim M = d$. *Then the following statements hold.*

(i) $H^0_\mathfrak{m}(M) = H^0_\mathfrak{m}(\Gamma_\mathfrak{m}(M)) = \Gamma_\mathfrak{m}(M)$ *and* $H^i_\mathfrak{m}(\Gamma_\mathfrak{m}(M)) = 0$ *for* $i > 0$.

(ii) *For all* $i \geq 0$, $H^i_\mathfrak{m}(M)$ *is a graded R-module with* $H^i_\mathfrak{m}(M)_j = 0$ *for* $j \gg 0$.

(iii) **(Grothendieck)**

$$H^i_\mathfrak{m}(M) \begin{cases} \neq 0 & \text{for } i = t \text{ and } i = d, \\ = 0 & \text{for } i < t \text{ and } i > d. \end{cases}$$

(iv) *If* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of finitely generated graded R-modules, then we have the long exact sequence*

$$\begin{aligned} 0 \to H^0_\mathfrak{m}(M') \to H^0_\mathfrak{m}(M) &\to H^0_\mathfrak{m}(M'') \\ \to H^1_\mathfrak{m}(M') \to H^1_\mathfrak{m}(M) &\to H^1_\mathfrak{m}(M'') \to \\ \cdots \to H^i_\mathfrak{m}(M') \to H^i_\mathfrak{m}(M) &\to H^i_\mathfrak{m}(M'') \to \cdots . \end{aligned}$$

In order to characterize Castelnuovo-Mumford regularity in terms of local cohomology we will need Grothendieck's local duality theorem (see, e.g., [**5**, Theorem 14.4.1] or [**9**, Theorem 3.6.19]).

Theorem 1.2.2 (The Local Duality Theorem)**.** *Let M be a finitely generated graded R-module. Then for every integer i there is a natural isomorphism*

$$\operatorname{Ext}^i_R(M, R(-n)) \cong \operatorname{Hom}_K(H^{n-i}_\mathfrak{m}(M), K).$$

In particular, $\operatorname{Ext}^i_R(M, R)_{-t-n} \cong H^{n-i}_\mathfrak{m}(M)_t$ *as K-vector spaces for every integers i and t.*

For a graded R-module N we set

$$a(N) := \begin{cases} \max\{j \mid N_j \neq 0\} & \text{if } N \neq 0, \\ -\infty & \text{if } N = 0. \end{cases}$$

Note that $H^i_\mathfrak{m}(M)_j = 0$ for $j \gg 0$ by Theorem 1.2.1(ii). So one can define the a-invariants and a^*-invariants of M as follows.

Definition 1.2.3. Let M be a finitely generated graded R-module. Let $i \geq 0$ be an integer. We define

(i) the **i-th a-invariant** of M as

$$a_i(M) := a(H^i_{\mathfrak{m}}(M));$$

(ii) the **i-th a^*-invariant** of M as

$$a_i^*(M) := \max\{a_j(M) \mid 0 \leq j \leq i\};$$

(iii) the **a^*-invariant** of M as

$$a^*(M) := \max\{a_i(M) \mid i \geq 0\}.$$

Now combining Proposition 1.1.8 and Theorem 1.2.2 we obtain the following formulas for Castelnuovo-Mumford regularity, which characterize this invariant in terms of Tor modules and local cohomology.

Theorem 1.2.4. *For any finitely generated graded module $M \neq 0$ over R, we have that:*

$$\begin{aligned} \operatorname{reg}(M) &= \max\{b_i(M) - i \mid i \geq 0\} \\ &= \max\{t \mid \exists i \geq 0 \text{ such that } \operatorname{Tor}_i^R(M, K)_{t+i} \neq 0\} \\ &= \max\{t \mid \exists i \geq 0 \text{ such that } H^i_{\mathfrak{m}}(M)_{t-i} \neq 0\} \\ &= \max\{a_i(M) + i \mid i \geq 0\}. \end{aligned}$$

From Proposition 1.1.8 and Theorem 1.2.2 it is also easy to check that the a^*-invariants and the b-invariants can be related as follows.

Proposition 1.2.5 ([**42**, Theorem 3.1(ii)] and [**43**, Theorem 4.1])**.** *Let M be a finitely generated graded R-module. Then*

(i) $a_i^*(M) = \max\{b_j(M) \mid j \geq n - t\} - n$ *for all* $i \geq 0$.
(ii) $a^*(M) = \max\{b_i(M) \mid i \geq 0\} - n$.

Remark. Theorem 1.2.4 presents two characterizations of the Castelnuovo-Mumford regularity. However, observe that such a statement is no longer true if R is not a polynomial ring. See, e.g., [**32**] for a discussion of regularities over positively graded algebras and possible generalizations of Theorem 1.2.4.

1.3. Monomial orders and initial ideals

As a motivation for Chapter 3, where a theory of preorders is developed, we summarize in this section some basic facts on monomial orders and initial ideals. For further information the reader is, e.g., referred to [**6, 7, 12, 14, 17, 26, 40, 41**].

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. Denote by $\mathcal{M}(R)$ the set of all monomials of R. For any vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ we write $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$.

Definition 1.3.1. A **monomial order** on R is a total order $<$ on $\mathcal{M}(R)$ such that

(i) if $m_1, m_2, m \in \mathcal{M}(R)$ and $m_1 < m_2$, then $mm_1 < mm_2$;
(ii) $1 < m$ for every $m \in \mathcal{M}(R) \setminus \{1\}$.

Remark. In order to have a more general theory (with more applications), some authors do not include the condition that $1 < m$ for every $m \in \mathcal{M}(R) \setminus \{1\}$ in the definition of monomial orders; see [**17**, Chapter 1]. We will discuss this extension in Chapter 3.

Example 1.3.2. Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be arbitrary vectors in $\mathbb{N}^n$.

(i) We define an order $<_{\text{lex}}$ on $\mathcal{M}(R)$ as follows: we set $\mathbf{x}^{\mathbf{u}} <_{\text{lex}} \mathbf{x}^{\mathbf{v}}$ if either

$$\sum_{i=1}^n a_i < \sum_{i=1}^n b_i,$$

or

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \text{ and the leftmost nonzero component of } \mathbf{u} - \mathbf{v} \text{ is negative.}$$

Then $<_{\text{lex}}$ is a monomial order on R, called the **lexicographic order** on S induced by the ordering $x_1 > x_2 > \cdots > x_n$.

(ii) Similarly as above, by setting $\mathbf{x}^{\mathbf{u}} <_{\text{rlex}} \mathbf{x}^{\mathbf{v}}$ if either

$$\sum_{i=1}^n a_i < \sum_{i=1}^n b_i,$$

or

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \text{ and the rightmost nonzero component of } \mathbf{u} - \mathbf{v} \text{ is positive,}$$

we obtain a monomial order $<_{\text{rlex}}$ on R, which is called the **reverse lexicographic order** on S induced by the ordering $x_1 > x_2 > \cdots > x_n$.

Let $<$ be a fixed monomial order on R. For a polynomial $f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ of R, the **initial monomial** of f with respect to $<$ is defined by

$$\text{in}_<(f) := c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \text{ if and only if } c_{\mathbf{u}} \neq 0 \text{ and } \mathbf{x}^{\mathbf{u}} > \mathbf{x}^{\mathbf{v}} \text{ for every } \mathbf{v} \neq \mathbf{u} \text{ such that } c_{\mathbf{v}} \neq 0.$$

Let I be an ideal in R. The **initial ideal** of I with respect to $<$ is the ideal in R generated by initial monomials of all polynomials of I. We denote this ideal by $\mathrm{in}_<(I)$. Thus

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f) \mid f \in I \rangle.$$

Definition 1.3.3. Let I be an ideal in R and $<$ a monomial order on R. A finite subset G of I is called a **Gröbner basis** for I with respect to $<$ if

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f) \mid f \in G \rangle.$$

By Hilbert's basis theorem (see, e.g., [**14**, Theorem 1.2]), Gröbner bases for ideals in R always exist. Moreover, one can compute Gröbner bases by using a simple and effective algorithm given by Buchberger (see, e.g., [**14**, Buchberger's Algorithm 15.9]).

A basic property of Gröbner bases is that they are generating sets of the ideal (see, e.g., [**14**, Lemma 15.5]):

Proposition 1.3.4. *Let $J \subseteq I$ be ideals in R and let $<$ be a monomial order on R. If $\mathrm{in}_<(I) = \mathrm{in}_<(J)$, then $I = J$. In particular, if G is a Gröbner basis for I, then G generates I.*

Given an ideal $I \subseteq R$ and a monomial order $<$ on R, the initial ideal $\mathrm{in}_<(I)$ is a monomial ideal and in general is much easier to study. So one would like to know the relationship between I and $\mathrm{in}_<(I)$, in particular, the properties that I can inherit from $\mathrm{in}_<(I)$. One of the earliest results in this direction is due to Macaulay (see, e.g., [**14**, Theorem 15.3]).

Theorem 1.3.5 (Macaulay)**.** *Let I be an ideal in R and $<$ a monomial order on R. Then the monomials not in $\mathrm{in}_<(I)$ form a basis for the K-vector space R/I. In particular, if I is homogeneous, then R/I and $R/\mathrm{in}_<(I)$ have the same Hilbert function.*

Further connections between I and $\mathrm{in}_<(I)$ can be deduced from the fact that there exists a flat family parameterized by the elements of K whose special fiber and general fiber are $\mathrm{in}_<(I)$ and I, respectively. Recall that a family of K-algebras $\{S_a\}$, parameterized by $a \in K$, is said to be a **one parameter flat family** of K-algebras if there exist a K-algebra S and a flat K-algebra homomorphism $\varphi : K[t] \to S$ such that the fibers $S/(\varphi(t) - a)S$ are isomorphic to S_a for all $a \in K$. In this case, the K-algebra S_0 is called the **special fiber**, and S_a for $a \neq 0$ a **general fiber** of the family.

In order to construct a flat family connecting I and $\mathrm{in}_<(I)$ we need to approximate the given monomial order $<$ by an integral weight order.

Definition 1.3.6. Let $\mathbf{w}$ be a vector in $\mathbb{Z}^n$. Define a partial order $<_{\mathbf{w}}$ on $\mathcal{M}(R)$ as follows:

$$\mathbf{x}^{\mathbf{u}} <_{\mathbf{w}} \mathbf{x}^{\mathbf{v}} \quad \text{if} \quad \mathbf{w} \cdot \mathbf{u} < \mathbf{w} \cdot \mathbf{v},$$

where the dot denotes the standard scalar product. We call $<_{\mathbf{w}}$ the **weight order** associated with $\mathbf{w}$.

Note that $<_{\mathbf{w}}$ is not necessarily a monomial order (but it is monomial preorder; see Chapter 3). However, one can still construct initial ideals with respect to $<_{\mathbf{w}}$. For a polynomial $f \in R$ let $\mathrm{in}_{<_{\mathbf{w}}}(f)$ denote the sum of all the terms of f that are maximal with respect to $<_{\mathbf{w}}$. If $I \subseteq R$ is an ideal we define

$$\mathrm{in}_{<_{\mathbf{w}}}(I) = \langle \mathrm{in}_{<_{\mathbf{w}}}(f) \mid f \in I \rangle.$$

The following result shows that for a given ideal in R, any monomial order can be approximated by an integer weight order (see, e.g., [**14**, Proposition 15.16]).

Theorem 1.3.7. *Let $I \subseteq R$ be an ideal and $<$ a monomial order on R. Then there exists $\mathbf{w} \in \mathbb{Z}^n$ such that*

$$\mathrm{in}_<(I) = \mathrm{in}_{<_{\mathbf{w}}}(I).$$

Working with an integral weight order has the advantage that we can link an ideal to its initial ideal via the homogenization with respect to the weighted degree.

Construction 1.3.8. Let $\mathbf{w} = (w_1, \dots, w_n)$ be an arbitrary vector in $\mathbb{Z}^n$ and t a new variable. For every polynomial $f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in R$ we set $\deg_{\mathbf{w}} f := \max\{\mathbf{w} \cdot \mathbf{u} | \ c_{\mathbf{u}} \neq 0\}$ and define

$$f^{\mathrm{hom}} := t^{\deg_{\mathbf{w}} f} f\big(t^{-w_1} x_1, \dots, t^{-w_n} x_n\big).$$

Then f^{hom} is a weighted homogeneous polynomial in $R[t] = K[x_1, \dots, x_n, t]$ with respect to the weighted degree $\deg x_i = w_i$ and $\deg t = 1$. We call f^{hom} the **homogenization** of f with respect to $\mathbf{w}$.

For an ideal I in R, the ideal

$$I^{\mathrm{hom}} := \langle f^{\mathrm{hom}} \mid f \in I \rangle \subseteq R[t]$$

is called the **homogenization** of I with respect to $\mathbf{w}$.

The above construction yields the desired flat family that connects I and $\mathrm{in}_<(I)$ (see, e.g., [**14**, Theorem 15.17]):

Theorem 1.3.9. *Let I be an ideal in R and $\mathbf{w} \in \mathbb{Z}^n$. Then the following statements hold:*

(i) $R[t]/I^{\mathrm{hom}}$ *is a flat extension of* $K[t]$.
(ii) $R[t]/(I^{\mathrm{hom}}, t) \cong R/\mathrm{in}_{<_{\mathbf{w}}}(I)$.
(iii) $(R[t]/I^{\mathrm{hom}})[t^{-1}] \cong (R/I)[t, t^{-1}]$.

Thus $R[t]/I^{\mathrm{hom}}$ is a flat family over $K[t]$ whose special fiber is isomorphic to $R/\mathrm{in}_{<_{\mathbf{w}}}(I)$ and whose general fibers are all isomorphic to R/I.

This result has many applications. For instance, one can deduce from it the following comparison between the Betti numbers of a homogeneous ideal and those of its initial ideal (see, e.g., [**19**, Theorem 3.3.1]).

Theorem 1.3.10. *Let $I \subseteq R$ be a homogeneous ideal and $\mathbf{w} \in \mathbb{Z}^n$. Then*

$$\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\mathrm{in}_{<_{\mathbf{w}}}(I)) \quad \text{for all integers } i \text{ and } j.$$

This result in turn leads to the following interesting conclusions (see, e.g., [**19**, Theorem 3.3.4, Corollaries 3.3.3, 3.3.5]).

Theorem 1.3.11. *Let $I \subseteq R$ be a homogeneous ideal and $<$ a monomial order on R. Then the following statements hold:*

(i) $\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\mathrm{in}_<(I))$ *for all integers i and j.*
(ii) $\mathrm{reg}(R/I) \leq \mathrm{reg}(R/\mathrm{in}_<(I))$.
(iii) $\mathrm{pd}(R/I) \leq \mathrm{pd}(R/\mathrm{in}_<(I))$.
(iv) $\mathrm{depth}(R/I) \geq \mathrm{depth}(R/\mathrm{in}_<(I))$.
(v) *If $R/\mathrm{in}_<(I)$ is Cohen-Macaulay (respectively, Gorenstein), then so is R/I.*

To conclude this section let us mention the following result of Sbarra [**35**, Theorem 2.4] which compares the Hilbert functions of local cohomology of the quotient rings R/I and $R/\mathrm{in}_<(I)$.

Theorem 1.3.12. *Let $I \subseteq R$ be a homogeneous ideal and $<$ a monomial order on R. Then*

$$\dim_K H^i_{\mathfrak{m}}(R/I)_j \leq \dim_K H^i_{\mathfrak{m}}(R/\mathrm{in}_<(I))_j \quad \text{for all integers } i \text{ and } j.$$

1.4. Generic initial ideals

The generic initial ideal essentially means an initial ideal with respect to a generic choice of coordinates. There are two important features that make the generic initial ideal very useful: first, it has a nice property, namely, Borel-fixed (which means strongly stable if the base field has characteristic 0); and second, it contains a lot of interesting information about the given ideal. This section provides a brief discussion on these features.

Throughout this section let K be an infinite field and $R = K[x_1, \dots, x_n]$ a polynomial ring over K. We always assume that any monomial order $<$ considered here satisfies the following conditions:

(i) $x_1 > x_2 > \cdots > x_n$;
(ii) $<$ refines the partial order by degree, i.e. $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ if $\deg \mathbf{x}^{\mathbf{u}} < \deg \mathbf{x}^{\mathbf{v}}$.

Recall that a subset V of the affine space K^m is said to be **Zariski closed** if there exist polynomials $f_1, \dots, f_k$ in m variables such that

$$V = \bigcap_{i=1}^{k} \{a \in K^m \mid f_i(a) = 0\}.$$

The complement of a Zariski closed subset is called **Zariski open**.

Let $M_n(K)$ be the set of all $n \times n$ matrices with entries in K. It can be identified with the affine space $K^{n\times n}$ in an obvious way. Let $\mathrm{GL}_n(K)$ denote the general linear group, i.e. the subset of $M_n(K)$ consists of invertible matrices. It is easy to see that $\mathrm{GL}_n(K)$ is a Zariski open subset of $M_n(K)$.

The group $\mathrm{GL}_n(K)$ acts on R as follows: any matrix $\alpha = (a_{ij})$ induces a K-algebra automorphism

$$\alpha : R \to R, \quad f(x_1, \dots, x_n) \mapsto f(\sum_{i=1}^{n} a_{i1}x_i, \dots, \sum_{i=1}^{n} a_{in}x_i).$$

The following interesting result shows the existence of the generic initial ideal (see, e.g., [**14**, Theorem 15.18]).

Theorem 1.4.1. *Let $I \subset R$ be a homogeneous ideal and $<$ a monomial order on R. Then there exists a nonempty Zariski open subset $U \subset \mathrm{GL}_n(K)$ and a monomial J such that $\mathrm{in}_<(\alpha I) = J$ for all $\alpha \in U$.*

Definition 1.4.2. The ideal J as in Theorem 1.4.1 is called the **generic initial ideal** of I with respect to the monomial order $<$. It is denoted by $\mathrm{Gin}_<(I)$.

Let $\mathcal{B}$ the **Borel subgroup** of $\mathrm{GL}_n(K)$, i.e. the subgroup consisting of all nonsingular upper triangular matrices. An ideal I in R is called **Borel-fixed** if it unchanged under the action of $\mathcal{B}$, i.e. $\alpha(I) = I$ for all $\alpha \in \mathcal{B}$. It is known that a Borel-fixed ideal is always a monomial ideal (see, e.g., [**14**, Theorem 15.23(a)]).

The fact that generic initial ideals are Borel-fixed was first shown in characteristic 0 by Galligo [**16**], and later in any characteristic by Bayer and Stillman [**4**] (see, e.g., [**14**, Theorem 15.20]).

Theorem 1.4.3. *If $I \subset R$ is a homogeneous ideal and $<$ is a monomial order on R, then $\mathrm{Gin}_<(I)$ is Borel-fixed.*

Borel-fixed ideals have a simple combinatorial characterization if the base field has characteristic 0. We say that a monomial ideal $I \subset R$ is **strongly stable** if $x_j(u/x_i) \in I$ for every monomial $u \in I$ and every $1 \leq j < i \leq n$ such that x_i divides u. Borel-fixed ideals and strongly stable ideals are related as follows (see [**19**, Proposition 4.2.4]).

Theorem 1.4.4. *Let $I \subset R$ be a homogeneous ideal. The following statements hold:*

(i) *If I is strongly stable, then I is Borel-fixed.*
(ii) *Assume that I is Borel-fixed. Let u be the largest exponent appearing among the monomial generators of I. Then I is strongly stable if either $\mathrm{char}K = 0$ or $\mathrm{char}K > u$.*

Thus in characteristic 0, Borel-fixed ideals are exactly strongly stable ideals. Note, however, that in positive characteristic, there exist Borel-fixed ideals which are not strongly stable. For example, if $\mathrm{char}K = p > 0$, then $I = (x_1^p, x_2^p) \subset K[x_1, x_2]$ is such an ideal.

Let us conclude this section with the following result of Bayer and Stillman [**3**] which says that several interesting properties of a homogeneous ideal are preserved in its generic initial ideal with respect to the reverse lexicographic order (see, e.g., [**19**, Corollary 4.3.18]).

Theorem 1.4.5. *Let $I \subset R$ be a homogeneous ideal. Then*

(i) $\mathrm{reg}(R/I) = \mathrm{reg}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(ii) $\mathrm{pd}(R/I) = \mathrm{pd}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(iii) $\mathrm{depth}(R/I) = \mathrm{depth}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(iv) *R/I is Cohen-Macaulay if and only if $R/\mathrm{Gin}_{\mathrm{rlex}}(I)$ is Cohen-Macaulay.*

1.5. Filter-regular sequences

Filter-regular sequences play an important role in the study of the Castelnuovo-Mumford regularity; see, e.g., [**3, 42, 43**]. We will see in Chapter 2 that they are also important for our study of partial regularities. In this section we discuss some basic properties of filter-regular sequences that are needed later. Further details can be found in [**19, 39, 43**].

Assume that K is an infinite field. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring and M a finitely generated graded R-module. Let $\mathfrak{m} = (x_1, \dots, x_n)$ denote the maximal homogeneous ideal in R.

Recall the following notation for a graded R-module N:

$$a(N) = \begin{cases} \max\{j \mid N_j \neq 0\} & \text{if } N \neq 0, \\ -\infty & \text{if } N = 0. \end{cases}$$

Definition 1.5.1. A homogeneous element $z \in R$ is called a **filter-regular element** on M if the following graded submodule of M has finite length:

$$(0 :_M z) := \{m \in M \mid zm = 0\}.$$

This means that $(0 :_M z)_j = 0$ for $j \gg 0$, or equivalently, $a((0 :_M z)) < \infty$.

We say that a sequence $z_1, \dots, z_s$ of homogeneous elements of R is a **filter-regular sequence** on M if

$$z_i \text{ is filter-regular element on } M/(z_1, \dots, z_{i-1})M \text{ for all } 1 \leq i \leq s.$$

Note that a filter-regular sequence of linear forms is also called an *almost regular sequence*; see [**1**] and [**19**, Section 4.3].

We have the following characterization of filter-regular sequences.

Proposition 1.5.2 ([**41**, Lemma 2.1])**.** *Let $M \neq 0$ be a finitely generated graded R-module and $z_1, \dots, z_s$ a sequence of homogeneous elements of R. Then the following statements are equivalent:*

(i) *$z_1, \dots, z_s$ is a filter-regular sequence on M.*
(ii) *$a((0 :_{M/(z_1,\dots,z_{i-1})M} z_i)) < \infty$ for $i = 1, \dots, s$.*
(iii) *For $i = 1, \dots, s$, we have $z_i \notin P$ for*

$$\textit{every associated prime ideal } P \textit{ of } M/(z_1, \dots, z_{i-1})M \textit{ with } P \neq \{\mathfrak{m}\}.$$

Example 1.5.3. (i) If M is a graded R-module with

$$\dim M = 0,$$

then every homogeneous element of $\mathfrak{m}$ is a filter-regular element on M.

(ii) Let $R = \mathbb{Q}[x_1, x_2, x_3]$ and

$$I = \langle -x_2^2 + x_1x_3, -x_2x_3, x_3^2 \rangle.$$

Then I has only one associated prime ideal, namely,

$$P = \langle x_2, x_3 \rangle.$$

So x_1 is a filter-regular element on R/I.

(iii) Let $R = \mathbb{Q}[x_1, x_2, x_3]$ and

$$I_1 = \langle x_2^2 - x_1x_3 \rangle.$$

Then I_1 is a prime ideal with $x_1 \in \mathfrak{m} \setminus I_1$, hence x_1 is a filter-regular on R/I_1. Evidently,

$$\langle I_1, x_1 \rangle = \langle x_1, x_2^2 \rangle$$

has only one associated prime ideal

$$P_1 = \langle x_1, x_2 \rangle.$$

So x_3 is a filter-regular element on $R/\langle I_1, x_1 \rangle$, and therefore, x_1, x_3 is a filter-regular sequence on R/I_1.

Since K is infinite, it follows easily from Proposition 1.5.2 that filter-regular elements on M of any degree always exist. Moreover, one has the following (see [**19**, Corollary 4.3.2, Theorem 4.3.6, Remark 4.3.10]):

Theorem 1.5.4. *Let M be a finitely generated graded R-module. Denote by R_1 the K-subspace of R generated by linear forms. Then there exists a K-basis for R_1 which is a filter-regular sequence on M.*

Moreover, there exists a Zariski open subset $U \subset \mathrm{GL}_n(K)$ such that $\alpha(x_1), \ldots, \alpha(x_n)$ is a filter-regular sequence on M for all $\alpha \in U$.

The role played by filter-regular sequences in the study of the Castelnuovo-Mumford regularity can be somewhat seen from the following result (see [**43**, Lemmas 2.1, 2.2, 4.3]). Recall that $a_i(M) = a(H^i_{\mathfrak{m}}(M))$ and $a_i^*(M) = \max\{a_j(M) \mid 0 \le j \le i\}$ for any integer $i \ge 0$.

Theorem 1.5.5. *Let $M \neq 0$ be a finitely generated graded R-module and let z be a filter-regular element on M of linear form. Then the following statements hold:*

(i) $a_0(M) = a((0 :_M z)) = a((0 :_M \mathfrak{m}))$.

(ii) $\operatorname{reg}(M) = \max\{a_0(M), \operatorname{reg}(M/zM)\}$.

(iii) $a_i(M) + i \leq a_{i-1}(M/zM) + (i-1) \leq \max\{a_{i-1}(M) + (i-1), a_i(M) + i\}$ *for every integer* $i \geq 1$.

(iv) $a_i^*(M) = \max\{a_0(M), a_{i-1}^*(M/zM) - 1\}$ *for every integer* $i \geq 1$.

CHAPTER 2

Partial regularities and related invariants

Concepts of partial regularities had been introduced by Trung in [**42**] and [**43**] as one refinement of the Castelnuovo-Mumford regularity. Many interesting questions related to these invariants are still open and these problems are parts of our study in this chapter. More precisely, we will consider: algebraic properties of partial regularities; comparisons between partial regularities and related invariants like the a-invariants or the regularity of the syzygy modules; relationships between initial ideals and the given ideals in terms of partial regularities; and generalization of known bounds for partial regularities.

2.1. Definition of partial regularities

Let K be a field and let M be a non-zero finitely generated graded module over the polynomial ring $R = K[x_1, \dots, x_n]$. Recall from Theorem 1.2.4 that the Castelnuovo-Mumford regularity of M can be characterized either via Tor modules or local cohomology modules as follows:

$$\operatorname{reg}(M) = \max\{b_i(M) - i \mid i \geq 0\} = \max\{a_i(M) + i \mid i \geq 0\},$$

where

$$b_i(M) = \begin{cases} \max\{j \in \mathbb{Z} \mid \operatorname{Tor}_i^R(M, K)_j \neq 0\} & \text{if} \quad \operatorname{Tor}_i^R(M, K) \neq 0, \\ -\infty & \text{if} \quad \operatorname{Tor}_i^R(M, K) = 0, \end{cases}$$

and

$$a_i(M) = \begin{cases} \max\{j \in \mathbb{Z} \mid H_{\mathfrak{m}}^i(M)_j \neq 0\} & \text{if} \quad H_{\mathfrak{m}}^i(M) \neq 0, \\ -\infty & \text{if} \quad H_{\mathfrak{m}}^i(M) = 0. \end{cases}$$

This result stimulates the study of the following refinements of the Castelnuovo-Mumford regularity, which will be the main objects of interest of this chapter.

Definition 2.1.1. For any integer $t \geq 0$, we define

$$\begin{aligned} \operatorname{reg}_{L,t}(M) &:= \max\{a_i(M) + i \mid i \leq t\}, \\ \operatorname{reg}^{L,t}(M) &:= \max\{a_i(M) + i \mid i \geq t\}, \\ \operatorname{reg}_{T,t}(M) &:= \max\{b_i(M) - i \mid i \leq t\}, \\ \operatorname{reg}^{T,t}(M) &:= \max\{b_i(M) - i \mid i \geq t\}. \end{aligned}$$

The invariants

$$\operatorname{reg}^{T,t}(M),\ \operatorname{reg}_{T,t}(M),\ \operatorname{reg}_{L,t}(M)$$

were studied in [**21, 42, 43**]. As far as we know, the invariant $\operatorname{reg}^{L,t}(M)$ has not been considered in the literature.

It is clear from the definition that

$$\begin{aligned} a_0(M) = \operatorname{reg}_{L,0}(M) \leq \operatorname{reg}_{L,1}(M) \leq \cdots \leq \operatorname{reg}_{L,n}(M) = \operatorname{reg}(M), \\ \operatorname{reg}(M) = \operatorname{reg}^{L,0}(M) \geq \operatorname{reg}^{L,1}(M) \geq \cdots \geq \operatorname{reg}^{L,n}(M) \geq \operatorname{reg}^{L,n+1}(M) = -\infty, \\ b_0(M) = \operatorname{reg}_{T,0}(M) \leq \operatorname{reg}_{T,1}(M) \leq \cdots \leq \operatorname{reg}_{T,n}(M) = \operatorname{reg}(M), \\ \operatorname{reg}(M) = \operatorname{reg}^{T,0}(M) \geq \operatorname{reg}^{T,1}(M) \geq \cdots \geq \operatorname{reg}^{T,n}(M) \geq \operatorname{reg}^{T,n+1}(M) = -\infty. \end{aligned}$$

2.2. Partial Regularities and Related Invariants

In this section, we study the relationship among partial regularities as well as the relationship between them and other invariants, such as the a-invariants or the regularity of syzygy modules.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field K and let M be a non-zero finitely generated graded R-module.

Let us first recall the following relationship between $\operatorname{reg}_{L,t}(M)$ and $\operatorname{reg}^{T,t}(M)$, which was shown by Trung [**42**, Theorem 3.1].

Theorem 2.2.1. *For $t \geq 0$ we have*

$$\operatorname{reg}^{T,t}(M) = \operatorname{reg}_{L,n-t}(M).$$

For the convenience of the reader and since this is a central observation used several times in the following, we will reproduce the proof of this result. We need the following simple but useful lemmas.

Lemma 2.2.2. *Let F be a finitely generated graded free R-module and $F^* := \operatorname{Hom}_R(F, R)$. If the generators of F have degrees less than or equal to r, then $(F^*)_{-j} = 0$ for all $j > r$.*

PROOF. By assumption, $F = \bigoplus_{i=1}^t Re_i$ with $\deg(e_i) = d_i \le r$ for all $1 \le i \le t$. Since $F^* = \bigoplus_{i=1}^t Re_i^*$ with $\deg(e_i^*) = -d_i$, it follows that $(F^*)_{-j} = 0$ for $j > r$. □

Lemma 2.2.3. *Let $\mathcal{F}$ be the minimal graded free resolution of the module M and let $\mathcal{F}^* = \operatorname{Hom}_R(\mathcal{F}, R)$. Let e^* be an element of a minimal system of generators of F_i^*. Then $e^* \notin \operatorname{Im}(F_{i-1}^* \to F_i^*)$.*

PROOF. Let $\{\varepsilon_j^*\}$ and $\{e_k^*\}$, respectively, be bases of F_{i-1}^* and F_i^* where $e^* = e_k^*$ for some k. Let $A = (a_{kj})$ be the corresponding matrix of the map $\varphi_i^* : F_{i-1}^* \to F_i^*$. Then $\varphi_i^*(\varepsilon_j^*) = \sum_k a_{kj} e_k^*$. Since $\mathcal{F}$ is minimal, $\deg(a_{kj}) > 0$ whenever $a_{kj} \neq 0$. It follows that e_k^* cannot belong to the image of φ_i^*. □

PROOF OF THEOREM 2.2.1. Assume that the minimal graded free resolution of M has the form

$$\mathcal{F} : 0 \to F_{\operatorname{pd}_R(M)} \to \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \to M \to 0.$$

We set

$$\mathcal{F}^* := \operatorname{Hom}_R(\mathcal{F}, R).$$

Then for every integer $i \ge 0$,

$$\operatorname{Ext}_R^i(M, R) = H^i(\mathcal{F}^*).$$

Consider

$$r = \operatorname{reg}^{T,t}(M)$$

and let i be the maximal number such that $i \ge t$ and $\operatorname{Tor}_i^R(M, K)_{i+r} \neq 0$.

Then the minimal generators of F_i and F_{i+1} have degree less than or equal to $(i + r)$, and F_i does have some minimal generator of degree $(i + r)$.

We will prove that

$$\text{for every } m \ge t,\ \operatorname{Ext}_R^m(M, R)_{-m-j} = 0 \text{ if } j > r, \text{ and when } m = i,\ \operatorname{Ext}_R^i(M, R)_{-i-r} \neq 0.$$

Indeed, for every $m \ge t$, the module F_m has no generator of degree greater than $(m + r)$. Hence, by Lemma 2.2.2, $(F_m^*)_{-m-j} = 0$ for every $j > r$.

This implies that

$$\operatorname{Ext}_R^m(M, R)_{-m-j} = 0 \text{ for every } j > r.$$

By the choice of i there is an element $e^* \in F_i^*$ of degree $-(i+r)$ which is part of a minimal system of generators of that module. By Lemma 2.2.3, the element e^* does not lie in $\mathrm{Im}(\varphi_i^*)$.

On the other hand, since the free resolution $\mathcal{F}$ is minimal and the generators of F_{i+1}^* have degree at least $-(i+r) = \deg(e^*)$, we must have $e^* \in \mathrm{Ker}(\varphi_{i+1}^*)$. Therefore, the image of e^* in $\mathrm{Ext}_R^i(M,R)$ is nonzero. This means that

$$\mathrm{Ext}_R^i(M,R)_{-i-r} \neq 0.$$

Consequently, together with Theorem 1.2.2 we get

$$\begin{aligned}\mathrm{reg}^{T,t}(M) &= \max\{j \mid \exists i \geq t \text{ such that } \mathrm{Ext}_R^i(M,R)_{-i-j} \neq 0\}\\ &= \max\{j \mid \exists i \leq n-t \text{ such that } H_{\mathfrak{m}}^i(M)_{j-i} \neq 0\}\\ &= \mathrm{reg}_{L,n-t}(M).\end{aligned}$$

□

In [**2**], Bayer, Charalambous and Popescu introduced the ***t*-regularity** of M, denoted t-reg(M), as the regularity of the t-th syzygy module of M. Thus, if M has the following minimal graded free resolution

$$0 \to F_r \to \cdots \to F_{t+1} \xrightarrow{\varphi_{t+1}} F_t \to \cdots \to F_0 \xrightarrow{\varepsilon} M \to 0,$$

then t-reg$(M) = \mathrm{reg}(\mathrm{Syz}_t(M))$, where $\mathrm{Syz}_t(M) = \mathrm{Ker}(\varphi_t)$. Recall that

$$0 \to F_r \to \cdots \to F_{t+1} \to \mathrm{Syz}_t(M) \to 0$$

is the minimal graded free resolution of $\mathrm{Syz}_t(M)$.

Thus

$$b_i(M) = b_{i-t-1}(\mathrm{Syz}_t(M)) \text{ for } i \geq t+1.$$

It follows that

$$\begin{aligned}\mathrm{reg}^{T,t+1}(M) &= \max\{b_i(M) - i \mid i \geq t+1\}\\ &= \max\{b_{i-t-1}(\mathrm{Syz}_t(M)) - (i-t-1) - t - 1 \mid i \geq t+1\}\\ &= \mathrm{reg}(\mathrm{Syz}_t(M)) - t - 1.\end{aligned}$$

Hence, together with Theorem 2.2.1 we obtain:

Theorem 2.2.4 (cf. [**42**, Corollary 3.2])**.** *For $t \geq 0$ we have*

$$t\text{-reg}(M) = \mathrm{reg}^{T,t+1}(M) + t + 1 = \mathrm{reg}_{L,n-t-1}(M) + t + 1.$$

The next result compares $\mathrm{reg}_{T,t}(M)$ and $\mathrm{reg}^{L,n-t}(M)$.

Theorem 2.2.5. *Let $t \geq 0$. Then the following statements hold:*

(i) $\operatorname{reg}^{L,n-t}(M) \leq \operatorname{reg}_{T,t}(M)$.

(ii) *Let i be the maximal number such that $i \leq t$ and $\operatorname{Tor}_i^R(M,K)_{i+\operatorname{reg}_{T,t}(M)} \neq 0$. If $i < t$, then $\operatorname{reg}^{L,n-t}(M) = \operatorname{reg}_{T,t}(M)$.*

Proof. (i) Set $r = \operatorname{reg}_{T,t}(M)$. Then $\operatorname{Tor}_m^R(M,K)_{m+j} = 0$ for all $m \leq t$ and $j > r$.

Let $\mathcal{F}$ be the minimal graded free resolution of M and $\mathcal{F}^* = \operatorname{Hom}_R(\mathcal{F}, R)$. Then for every $m \leq t$, the generators of F_m have degree less than or equal to $m + r$. By Lemma 2.2.2, $(F_m^*)_{-m-j} = 0$, and so $\operatorname{Ext}_R^m(M,R)_{-m-j} = 0$ for all $m \leq t$ and $j > r$. This implies that, as desired,

$$\operatorname{reg}^{L,n-t}(M) = \max\{j \mid \exists m \leq t \text{ such that } \operatorname{Ext}_R^m(M,R)_{-m-j} \neq 0\} \leq r.$$

(ii) By the assumption on i, the module F_{i+1} must have minimal generators of degree less than or equal to $i+r$. Thus, as in the proof of Theorem 2.2.1, any minimal generator of F_i^* of degree $i+r$ must belong to the kernel of $F_i^* \to F_{i+1}^*$. Also, such a generator cannot be in the image of $F_{i-1}^* \to F_i^*$, by Lemma 2.2.3. It follows that the image of such a generator in $\operatorname{Ext}_R^i(M,R)_{-i-r}$ is nonzero. Thus, together with (i) we get that

$$r = \max\{j \mid \exists m \leq t \text{ such that } \operatorname{Ext}_R^m(M,R)_{-m-j} \neq 0\},$$

and so $\operatorname{reg}^{L,n-t}(M) = \operatorname{reg}_{T,t}(M)$. □

In the remaining part of this section we discuss the relations between the partial regularities of M and those of $M/H_{\mathfrak{m}}^0(M)$.

Proposition 2.2.6. *Let M be a nonzero finitely generated R-module. Then the following statements hold:*

(i) $\operatorname{reg}^{L,0}(M) = \max\{a_0(M), \operatorname{reg}^{L,0}(M/H_{\mathfrak{m}}^0(M))\}$;
$\operatorname{reg}^{L,t}(M) = \operatorname{reg}^{L,t}(M/H_{\mathfrak{m}}^0(M))$ *for* $t > 0$.

(ii) $\operatorname{reg}_{L,0}(M) = a_0(M)$ *and* $\operatorname{reg}_{L,0}(M/H_{\mathfrak{m}}^0(M)) = -\infty$;
$\operatorname{reg}_{L,t}(M) = \max\{a_0(M); \operatorname{reg}_{L,t}(M/H_{\mathfrak{m}}^0(M))\}$ *for* $t > 0$.

(iii) $\operatorname{reg}^{T,t}(M) = \max\{a_0(M), \operatorname{reg}^{T,t}(M/H_{\mathfrak{m}}^0(M))\}$ *for* $t < n$;
$\operatorname{reg}^{T,n}(M) = a_0(M)$ *and* $\operatorname{reg}^{T,n}(M/H_{\mathfrak{m}}^0(M)) = -\infty$.

(iv) $\operatorname{reg}_{T,n}(M) = \max\{a_0(M), \operatorname{reg}_{T,n}(M/H_{\mathfrak{m}}^0(M))\}$;
$\operatorname{reg}_{T,t}(M) \leq \max\{\operatorname{reg}_{T,t}(H_{\mathfrak{m}}^0(M)), \operatorname{reg}_{T,t}(M/H_{\mathfrak{m}}^0(M))\}$ *for* $t < n$.

PROOF. Consider the following exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to M/H^0_{\mathfrak{m}}(M) \to 0.$$

Using Theorem 1.2.1(i), (iv) one easily gets

$$H^0_{\mathfrak{m}}(M/H^0_{\mathfrak{m}}(M)) = 0 \text{ and } H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M/H^0_{\mathfrak{m}}(M)) \text{ for } i \geq 1.$$

It follows that

$$a_0(M/H^0_{\mathfrak{m}}(M)) = -\infty \text{ and } a_i(M) = a_i(M/H^0_{\mathfrak{m}}(M)) \text{ for } i \geq 1.$$

Thus,

$$\begin{aligned}
\operatorname{reg}(M/H^0_{\mathfrak{m}}(M)) &= \max\{a_i(M/H^0_{\mathfrak{m}}(M)) + i \mid i \geq 1\} = \max\{a_i(M) + i \mid i \geq 1\},\\
\operatorname{reg}(M) &= \max\{a_i(M) + i \mid i \geq 0\} = \max\{a_0(M), \operatorname{reg}(M/H^0_{\mathfrak{m}}(M))\}.
\end{aligned}$$

(i) By definition, $\operatorname{reg}^{L,0}(M) = \operatorname{reg}(M)$. So the observation above gives

$$\operatorname{reg}^{L,0}(M) = \max\{a_0(M), \operatorname{reg}^{L,0}(M/H^0_{\mathfrak{m}}(M))\}.$$

Moreover, for $t > 0$,

$$\begin{aligned}
\operatorname{reg}^{L,t}(M) = \max\{a_i(M) + i \mid i \geq t\} &= \max\{a_i(M/H^0_{\mathfrak{m}}(M)) + i \mid i \geq t\}\\
&= \operatorname{reg}^{L,t}(M/H^0_{\mathfrak{m}}(M)).
\end{aligned}$$

(ii) By definition,

$$\operatorname{reg}_{L,0}(M) = a_0(M) \text{ and } \operatorname{reg}_{L,0}(M/H^0_{\mathfrak{m}}(M)) = a_0(M/H^0_{\mathfrak{m}}(M)) = -\infty.$$

For $t > 0$ we have

$$\begin{aligned}
\operatorname{reg}_{L,t}(M/H^0_{\mathfrak{m}}(M)) &= \max\{a_i(M/H^0_{\mathfrak{m}}(M)) + i \mid 1 \leq i \leq t\}\\
&= \max\{a_i(M) + i \mid 1 \leq i \leq t\}.
\end{aligned}$$

Thus,

$$\operatorname{reg}_{L,t}(M) = \max\{a_i(M) + i \mid 0 \leq i \leq t\} = \max\{a_0(M), \operatorname{reg}_{L,t}(M/H^0_{\mathfrak{m}}(M))\}.$$

(iii) This statement follows by combining (ii) and Theorem 2.2.1.

(iv) Since $\operatorname{reg}_{T,n}(M) = \operatorname{reg}(M)$, it follows from the above observation that

$$\operatorname{reg}_{T,n}(M) = \max\{a_0(M), \operatorname{reg}_{T,n}(M/H^0_{\mathfrak{m}}(M))\}.$$

For $t < n$, Proposition 2.3.7(ii) gives

$$\operatorname{reg}_{T,t}(M) \leq \max\{\operatorname{reg}_{T,t}(H^0_{\mathfrak{m}}(M)), \operatorname{reg}_{T,t}(M/H^0_{\mathfrak{m}}(M))\}.$$

$\square$

2.3. Algebraic properties of partial regularities

In this section we use filter-regular sequences to study partial regularities. We also discuss the behaviour of partial regularities with respect to short exact sequences.

Throughout the section let M be a non-zero finitely generated graded module over the polynomial ring $R = K[x_1, \ldots, x_n]$. We first compare partial regularities of M and those of M/zM, where z is a linear filter-regular element on M. Such an element exists by Theorem 1.5.4.

Recall for a graded R-module N we denote

$$a(N) = \begin{cases} \max\{j \mid N_j \neq 0\} & \text{if } N \neq 0, \\ -\infty & \text{if } N = 0. \end{cases}$$

Proposition 2.3.1. *Let z be a linear filter-regular element on M. Set $\bar{M} = M/zM$ and $N = (0 :_M z)$. Then the following statements hold:*

(i) $b_{i-1}(M) - (i-1) \leq \max\{b_i(\bar{M}) - i, a(N)\}$ *for all* $i \geq 1$.
(ii) $b_i(\bar{M}) - i \leq \max\{b_{i-1}(M) - (i-1), b_i(M) - i, a(N)\}$ *for all* $i \geq 1$.
(iii) $\operatorname{reg}_{T,t-1}(M) \leq \max\{\operatorname{reg}_{T,t}(\bar{M}), a(N)\} \leq \max\{\operatorname{reg}_{T,t}(M), a(N)\}$.
(iv) $\max\{\operatorname{reg}_{T,t}(M), a(N)\} = \max\{b_t(M) - t, \operatorname{reg}_{T,t}(\bar{M}), a(N)\}$.

Proof. Set $M_1 := M/N$. Consider the following short exact sequences

$$0 \to N \to M \to M_1 \to 0, \tag{1}$$

$$0 \to M_1(-1) \xrightarrow{\cdot z} M \to \bar{M} \to 0. \tag{2}$$

Since z is a filter-regular element, N has finite length. So $\operatorname{reg}(N) = a_0(N) = a(N)$. By Theorem 1.2.4, $\operatorname{Tor}_i^R(N,K)_{i+s} = 0$ for all i and $s > a(N)$. From (1) we get the following exact sequence

$$\operatorname{Tor}_i^R(N,K) \to \operatorname{Tor}_i^R(M,K) \to \operatorname{Tor}_i^R(M_1,K) \to \operatorname{Tor}_{i-1}^R(N,K) \quad \text{for} \quad i \geq 1.$$

This yields

$$\operatorname{Tor}_i^R(M,K)_{i+s} \cong \operatorname{Tor}_i^R(M_1,K)_{i+s} \quad \text{for all } i \text{ and } s > a(N), \tag{3}$$

because $\operatorname{Tor}_i^R(N,K)_{i+s} = \operatorname{Tor}_{i-1}^R(N,K)_{(i-1)+(s+1)} = 0$ for all $s > a(N)$. On the other hand, (2) gives the following exact sequence

$$\operatorname{Tor}_i^R(M,K) \to \operatorname{Tor}_i^R(\bar{M},K) \to \operatorname{Tor}_{i-1}^R(M_1(-1),K) \to \operatorname{Tor}_{i-1}^R(M,K) \quad \text{for} \quad i \geq 1. \tag{4}$$

(i) If $s > \max\{b_i(\bar{M}) - i, a(N)\}$, then from (4) we obtain

$$0 = \operatorname{Tor}_i^R(\bar{M}, K)_{i+s} \to \operatorname{Tor}_{i-1}^R(M_1, K)_{(i-1)+s} \to \operatorname{Tor}_{i-1}^R(M, K)_{i+s}.$$

Together with (3) this yields $\operatorname{Tor}_{i-1}^R(M, K)_{(i-1)+s} \hookrightarrow \operatorname{Tor}_{i-1}^R(M, K)_{i+s}$. By induction we get

$$\operatorname{Tor}_{i-1}^R(M, K)_{(i-1)+s} \hookrightarrow \operatorname{Tor}_{i-1}^R(M, K)_{i+s} \hookrightarrow \cdots \hookrightarrow \operatorname{Tor}_{i-1}^R(M, K)_{i+s+h} = 0$$

for some $h > 0$ large enough. It follows that

$$\operatorname{Tor}_{i-1}^R(M, K)_{(i-1)+s} = 0 \text{ for all } s > \max\{b_i(\bar{M}) - i, a(N)\}.$$

Hence, $b_{i-1}(M) - (i-1) \leq \max\{b_i(\bar{M}) - i, a(N)\}$.

(ii) Let $s > \max\{b_{i-1}(M) - (i-1), b_i(M) - i, a(N)\}$. Consider the following exact sequence which is obtained from (4):

$$\operatorname{Tor}_i^R(M, K)_{i+s} \to \operatorname{Tor}_i^R(\bar{M}, K)_{i+s} \to \operatorname{Tor}_{i-1}^R(M_1, K)_{(i-1)+s}.$$

In this sequence,

$$\operatorname{Tor}_i^R(M, K)_{i+s} = 0 \text{ since } s > b_i(M) - i.$$

Also, since $s > a(N)$ and $s > b_{i-1}(M) - (i-1)$, it follows from (3) that

$$\operatorname{Tor}_{i-1}^R(M_1, K)_{(i-1)+s} \cong \operatorname{Tor}_{i-1}^R(M, K)_{(i-1)+s} = 0.$$

Thus, $\operatorname{Tor}_i^R(\bar{M}, K)_{i+s} = 0$ and we get

$$b_i(\bar{M}) - i \leq \max\{b_{i-1}(M) - (i-1), b_i(M) - i, a(N)\},$$

as desired.

(iii) Taking the maximum over $1 \leq i \leq t$ of the both sides of the equality obtained in (i) yields

$$\operatorname{reg}_{T,t-1}(M) \leq \max\{\operatorname{reg}_{T,t}(\bar{M}), a(N)\}.$$

Similarly, it follows from (ii) that

$$\max\{b_i(\bar{M}) - i \mid 1 \leq i \leq t\} \leq \max\{\operatorname{reg}_{T,t}(M), a(N)\}.$$

But also $b_0(\bar{M}) \leq b_0(M)$, hence

$$\operatorname{reg}_{T,t}(\bar{M}) \leq \max\{\operatorname{reg}_{T,t}(M), a(N)\}.$$

(iv) This statement is a consequence of (iii). Indeed, we have

$$\max\{\operatorname{reg}_{T,t}(M), a(N)\} = \max\{b_t(M) - t, \operatorname{reg}_{T,t-1}(M), a(N)\}.$$

But from (iii) we know that

$$\operatorname{reg}_{T,t-1}(M) \leq \max\{\operatorname{reg}_{T,t}(\bar{M}), a(N)\}.$$

Thus,

$$\max\{\operatorname{reg}_{T,t}(M), a(N)\} = \max\{b_t(M) - t, \operatorname{reg}_{T,t}(\bar{M}), a(N)\}.$$

$\square$

Applying Theorem 2.2.1, Theorem 1.5.5(iii) and a similar argument as in the proof of Proposition 2.3.1 we obtain:

Corollary 2.3.2. *Let z be a linear filter-regular element on M. Set $\bar{M} = M/zM$. Then the following statements hold:*

(i) $\operatorname{reg}^{L,t}(M) \leq \operatorname{reg}^{L,t-1}(\bar{M}) \leq \operatorname{reg}^{L,t-1}(M)$,
$\operatorname{reg}^{L,t-1}(M) = \max\{a_{t-1}(M) + (t-1), \operatorname{reg}^{L,t-1}(\bar{M})\}$.
(ii) $\operatorname{reg}_{L,t}(M) = \max\{a_0(M), \operatorname{reg}_{L,t-1}(\bar{M})\}$.
(iii) $\operatorname{reg}^{T,t}(M) = \max\{a_0(M), \operatorname{reg}^{T,t+1}(\bar{M})\}$.

In particular, if z is a regular element on M, then $N = (0 :_M z) = 0$ and $H^0_{\mathfrak{m}}(M) = 0$, yielding $a_0(N) = a_0(M) = -\infty$. So from Proposition 2.3.1 and Corollary 2.3.2 we get:

Corollary 2.3.3. *Let z be a linear M-regular element and $\bar{M} = M/zM$. Then the following statements hold:*

(i) $\operatorname{reg}_{T,t}(M) = \max\{b_t(M) - t, \operatorname{reg}_{T,t}(\bar{M})\}$.
(ii) $\operatorname{reg}_{L,t}(M) = \operatorname{reg}_{L,t-1}(\bar{M})$.
(iii) $\operatorname{reg}^{T,t}(M) = \operatorname{reg}^{T,t+1}(\bar{M})$.

We now examine how partial regularities change in a short exact sequence. The following propositions generalize a basic property of Castelnuovo-Mumford regularity (see, e.g., [**14**, Corollary 20.19]).

Proposition 2.3.4. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated R-modules. Then the following statements hold:*

(i) $\operatorname{reg}_{L,t}(L) \leq \max\{\operatorname{reg}_{L,t}(M), \operatorname{reg}_{L,t-1}(N) + 1\}$.
(ii) $\operatorname{reg}_{L,t}(M) \leq \max\{\operatorname{reg}_{L,t}(L), \operatorname{reg}_{L,t}(N)\}$.
(iii) $\operatorname{reg}_{L,t}(N) \leq \max\{\operatorname{reg}_{L,t+1}(L) - 1, \operatorname{reg}_{L,t}(M)\}$.
(iv) *If L has finite length, then* $\operatorname{reg}_{L,t}(M) = \max\{\operatorname{reg}_{L,t}(L), \operatorname{reg}_{L,t}(N)\}$.

Proposition 2.3.5. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated R-modules. Then the following statements hold:*

(i) $\operatorname{reg}^{L,t}(L) \leq \max\{\operatorname{reg}^{L,t}(M), \operatorname{reg}^{L,t-1}(N)+1\}$.
(ii) $\operatorname{reg}^{L,t}(M) \leq \max\{\operatorname{reg}^{L,t}(L), \operatorname{reg}^{L,t}(N)\}$.
(iii) $\operatorname{reg}^{L,t}(N) \leq \max\{\operatorname{reg}^{L,t+1}(L)-1, \operatorname{reg}^{L,t}(M)\}$.
(iv) *If L has finite length, then* $\operatorname{reg}^{L,t}(M) = \max\{\operatorname{reg}^{L,t}(L), \operatorname{reg}^{L,t}(N)\}$.

Proposition 2.3.6. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated R-modules. Then the following statements hold:*

(i) $\operatorname{reg}^{T,t}(L) \leq \max\{\operatorname{reg}^{T,t}(M), \operatorname{reg}^{T,t+1}(N)+1\}$.
(ii) $\operatorname{reg}^{T,t}(M) \leq \max\{\operatorname{reg}^{T,t}(L), \operatorname{reg}^{T,t}(N)\}$.
(iii) $\operatorname{reg}^{T,t}(N) \leq \max\{\operatorname{reg}^{T,t-1}(L)-1, \operatorname{reg}^{T,t}(M)\}$.
(iv) *If L has finite length, then* $\operatorname{reg}^{T,t}(M) = \max\{\operatorname{reg}^{T,t}(L), \operatorname{reg}^{T,t}(N)\}$.

Proposition 2.3.7. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated R-modules. Then the following statements hold:*

(i) $\operatorname{reg}_{T,t}(L) \leq \max\{\operatorname{reg}_{T,t}(M), \operatorname{reg}_{T,t+1}(N)+1\}$.
(ii) $\operatorname{reg}_{T,t}(M) \leq \max\{\operatorname{reg}_{T,t}(L), \operatorname{reg}_{T,t}(N)\}$.
(iii) $\operatorname{reg}_{T,t}(N) \leq \max\{\operatorname{reg}_{T,t-1}(L)-1, \operatorname{reg}_{T,t}(M)\}$.

The proofs of Proposition 2.3.4 and Proposition 2.3.5 are similar. Moreover, Proposition 2.3.5 and Proposition 2.3.6 are equivalent by Theorem 2.2.1. Therefore, we will only prove Proposition 2.3.4 and Proposition 2.3.7.

Proof of Proposition 2.3.4. We use a similar argument as in the proof of [**14**, Corollary 20.19]. At first, recall that Theorem 1.2.2 gives

$$\begin{aligned}\operatorname{reg}_{L,t}(M) &= \max\{s \mid \exists i \leq t \text{ such that } H^i_{\mathfrak{m}}(M)_{s-i} \neq 0\}\\ &= \max\{s \mid \exists i \leq t \text{ such that } \operatorname{Ext}^{n-i}_R(M,R)_{-n-s+i} \neq 0\}\\ &= \max\{s \mid \exists j \geq n-t \text{ such that } \operatorname{Ext}^{j}_R(M,R)_{-j-s} \neq 0\}.\end{aligned}$$

Now the given short exact sequence yields the following exact sequence

$$\begin{aligned}\ldots &\to \operatorname{Ext}^{j-1}_R(L,R)_{-(j-1)-(s+1)} \to \operatorname{Ext}^{j}_R(N,R)_{-j-s}\\ &\to \operatorname{Ext}^{j}_R(M,R)_{-j-s} \to \operatorname{Ext}^{j}_R(L,R)_{-j-s} \to \operatorname{Ext}^{j+1}_R(N,R)_{-(j+1)-(s-1)} \to \ldots,\end{aligned}$$

from which we easily get (i)–(iii).

To prove (iv), recall that if L has finite length, then $\operatorname{Ext}^j_R(L,R)=0$ for every $j<n$. This implies that

$$\operatorname{Ext}^j_R(N,R)\cong \operatorname{Ext}^j_R(M,R) \text{ for } j<n$$

and that

$$0\to \operatorname{Ext}^n_R(N,R)\to \operatorname{Ext}^n_R(M,R)\to \operatorname{Ext}^n_R(L,R)\to 0$$

is a short exact sequence. Therefore, $\operatorname{reg}_{L,t}(M)$ is greater or equal to both $\operatorname{reg}_{L,t}(L)$ and $\operatorname{reg}_{L,t}(N)$. Together with (ii), we get

$$\operatorname{reg}_{L,t}(M)=\max\{\operatorname{reg}_{L,t}(L),\operatorname{reg}_{L,t}(N)\}.$$

This concludes the proof. □

PROOF OF PROPOSITION 2.3.7. One argues similarly as in the proof of Proposition 2.3.4, using the following long exact sequence of Tor modules:

$$\cdots\to \operatorname{Tor}^R_{i+1}(N,K)\to \operatorname{Tor}^R_i(L,K)\to \operatorname{Tor}^R_i(M,K)\to \operatorname{Tor}^R_i(N,K)\to \operatorname{Tor}^R_{i-1}(L,K)\to\cdots$$

□

2.4. Partial regularities of stable monomial ideals

This section is devoted to the study of partial regularities of stable monomial ideals. We characterize partial regularities of this kind of ideals in terms of their minimal generators. Our result generalizes several previous results of Bayer and Stillman [**3**] and Trung [**42**].

Let $R=K[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field K.

Definition 2.4.1. A monomial ideal I in R is called **stable** if for every monomial $u\in I$ and every $j<m(u):=\max\{i\mid x_i \text{ divides } u\}$, we have that

$$x_j(u/x_{m(u)})\in I.$$

It is clear that this notion extends the notion of strongly stable ideals defined in Section 1.4. So by Theorem 1.4.4, if a given monomial ideal $I\subset R$ is Borel-fixed, then I is stable if $\operatorname{char}(K)=0$ or $\operatorname{char}(K)$ large enough. In particular, this is true for the generic initial ideal of any homogeneous ideal in R provided that the monomial order chosen refines the partial order by degree and satisfies $x_1>x_2>\cdots>x_n$, by Theorem 1.4.3.

In order to compute partial regularities of stable ideals we will need the following result.

Lemma 2.4.2. *Let $0 \neq I$ be a stable monomial ideal in R. Denote by $\operatorname{Min}(I)$ the set of minimal monomial generators of I, and by $\operatorname{Min}(I)_j$ the set of elements of $\operatorname{Min}(I)$ of degree j. For a monomial $u \in R$ let $m(u)$ be the largest index i such that x_i divides u. Then:*

(i) (Eliahou–Kervaire) *For integers $i, j \geq 0$ we have:*

$$\beta_{i,i+j}(R/I) = \sum_{u \in \operatorname{Min}(I)_{j+1}} \binom{m(u)-1}{i-1}.$$

(ii) *For $1 \leq i \leq n$,*

$$a\left(\frac{(I, x_n, \ldots, x_{n+1-i}) : x_{n-i}}{(I, x_n, \ldots, x_{n+1-i})}\right) = \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I), m(u) = n - i\}.$$

(iii) *$x_n, x_{n-1}, \ldots, x_1$ is a filter-regular sequence on R/I.*

Proof. For (i) see, e.g., [**19**, Corollary 7.2.3]. We will prove (ii), which implies (iii). We argue similarly as in the proof of [**42**, Lemma 2.3]. Set

$$J = (I, x_n, \ldots, x_{n+1-i}) : x_{n-i} \ \text{ and } \ J_1 = (I, x_n, \ldots, x_{n+1-i}).$$

Recall that

$$a(J/J_1) = \max\{d \mid (J/J_1)_d \neq 0\}.$$

Let $u \in \operatorname{Min}(I)$ with $m(u) = n - i$. Write $u = v x_{n-i}$. Then $v \in J$. On the other hand, $v \notin J_1$ since $v \notin I$ and v is not divisible by the variables $x_n, \ldots, x_{n+1-i}$. It follows that

$$(J/J_1)_{\deg(u)-1} \neq 0$$

and hence

$$a(J/J_1) \geq \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I), m(u) = n - i\}.$$

To prove the reverse inequality it suffices to show that for any monomial v of J which does not belong to J_1, there exists $u \in \operatorname{Min}(I)$ with $m(u) = n-i$ and $\deg(u) - 1 = \deg(v)$.

Indeed, for such a monomial v we have that

$$m(v) \leq n - i, \ x_{n-i} v \in J_1, \text{ and } v \notin I.$$

Let $u = x_{n-i} v$. Then

$$m(u) = n - i \text{ and so } u \in I.$$

It remains to show that $u \in \operatorname{Min}(I)$. Let $u' \in \operatorname{Min}(I)$ be such that $u' | u$. Assume

$$u = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-i}^{\alpha_{n-i}} \text{ and } u' = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n-i}^{\beta_{n-i}}.$$

As $u'|u$ but $u' \nmid v$, we have $\alpha_j \geq \beta_j$ for $1 \leq j \leq n-i-1$, and $\alpha_{n-i} = \beta_{n-i}$. If there exists $j < n-i$ such that $\alpha_j > \beta_j$, then $x_j(u'/x_{n-i})$ divides v. But $x_j(u'/x_{n-i}) \in I$ since I is stable. It follows that $v \in I$, which is impossible. Thus,

$$\alpha_j = \beta_j \text{ for } j = 1, \ldots, n-i$$

and

$$u = u' \in \mathrm{Min}(I).$$

□

Now we are able to compute partial regularities of stable monomial ideals. Our result generalizes [**42**, Theorem 2.4].

Theorem 2.4.3. *Let $0 \neq I \subset R$ be a stable monomial ideal with the set of minimal monomial generators* $\mathrm{Min}(I)$. *For a monomial $u \in R$ let $m(u)$ be the largest index i such that x_i divides u. Then for any integer $t \geq 0$:*

(i) $\mathrm{reg}^{T,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I), m(u) \geq t\}$.
(ii) $\mathrm{reg}_{L,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I), m(u) \geq n-t\}$.
(iii) $\mathrm{reg}^{L,n-t}(R/I) \leq \mathrm{reg}_{T,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I)\}$.
(iv) $a_t^*(R/I) = \max\{\deg(u) + m(u) - n - 1 \mid u \in \mathrm{Min}(I), m(u) \geq n-t\}$.

Proof. By Lemma 2.4.2(i), we have

$$\begin{aligned}
\mathrm{reg}^{T,t}(R/I) &= \max\{j \mid \exists\ i \geq t \text{ s.t. } \beta_{i,i+j}(R/I) \neq 0\} \\
&= \max\{j \mid \exists\ i \geq t, u \in \mathrm{Min}(I) \text{ s.t. } \deg(u) = j+1, i-1 \leq m(u) - 1\} \\
&= \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I), m(u) \geq t\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{reg}_{T,t}(R/I) &= \max\{j \mid \exists\ 0 \leq i \leq t \text{ s.t. } \beta_{i,i+j}(R/I) \neq 0\} \\
&= \max\{j \mid \exists\ 0 \leq i \leq t, u \in \mathrm{Min}(I) \text{ s.t. } \deg(u) = j+1, i-1 \leq m(u) - 1\} \\
&= \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I)\}.
\end{aligned}$$

Now by Theorem 2.2.1 and Theorem 2.2.5 we get the desired formulas for $\mathrm{reg}_{L,t}(R/I)$ and $\mathrm{reg}^{L,n-t}(R/I)$.

It remains to compute $a_t^*(R/I)$. By Lemma 2.4.2(iii),

$$x_n, x_{n-1}, \ldots, x_1$$

is a filter-regular sequence on R/I.

So iteratively applying Theorem 1.5.5(iv) yields

$$\begin{aligned}
&a_t^*(R/I)\\
&= \max\{a_0(R/I), a_{t-1}^*(R/(I,x_n)) - 1\}\\
&= \max\{a(\frac{I:x_n}{I}), a(\frac{(I,x_n):x_{n-1}}{(I,x_n)}) - 1, \ldots, a_0^*(\frac{R}{(I,x_n,\ldots,x_{n+1-t})}) - t\}\\
&= \max\{a(\frac{I:x_n}{I}), a(\frac{(I,x_n):x_{n-1}}{(I,x_n)}) - 1, \ldots, a(\frac{(I,x_n,\ldots,x_{n+1-t}):x_{n-t}}{(I,x_n,\ldots,x_{n+1-t})}) - t\}\\
&= \max\{a(\frac{(I,x_n,\ldots,x_{n+1-i}):x_{n-i}}{(I,x_n,\ldots,x_{n+1-i})}) - i \mid 0 \le i \le t\}\\
&= \max\{\max\{\deg(u) - 1 - i \mid u \in \operatorname{Min}(I), m(u) = n - i\} \mid 0 \le i \le t\}\\
&= \max\{\deg(u) + m(u) - n - 1 \mid u \in \operatorname{Min}(I), m(u) \ge n - t\}.
\end{aligned}$$

Note that the second-last equality follows from Lemma 2.4.2(ii). □

Remark. In the previous proof one can alternatively compute $\operatorname{reg}_{L,t}(R/I)$ by using the filter-regular sequence $x_n, \ldots, x_1$.

Indeed, from Corollary 2.3.2(ii) and Lemma 2.4.2(ii) one gets

$$\begin{aligned}
\operatorname{reg}_{L,t}(R/I) &= \max\{a_0(R/I), \operatorname{reg}_{L,t-1}(R/(I,x_n))\}\\
&= \max\{a(\frac{I:x_n}{I}), a(\frac{(I,x_n):x_{n-1}}{(I,x_n)}), \ldots, \operatorname{reg}_{L,0}(\frac{R}{(I,x_n,\ldots,x_{n+1-t})})\}\\
&= \max\{a(\frac{I:x_n}{I}), a(\frac{(I,x_n):x_{n-1}}{(I,x_n)}), \ldots, a(\frac{(I,x_n,\ldots,x_{n+1-t}):x_{n-t}}{(I,x_n,\ldots,x_{n+1-t})})\}.\\
&= \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I), m(u) \ge n - t\}.
\end{aligned}$$

Since

$$\operatorname{reg}(R/I) = \operatorname{reg}_{L,n}(R/I) \text{ and } a^*(R/I) = a_n^*(R/I),$$

by the above theorem we immediately get the following consequence:

Corollary 2.4.4 (cf. [**43**, Corollary 6.9])**.** *Let $0 \ne I \subseteq R$ be a stable monomial ideal. Then*

(i) $\operatorname{reg}(R/I) = \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I)\}$.
(ii) $a^*(R/I) = \max\{\deg(u) + m(u) - n - 1 \mid u \in \operatorname{Min}(I)\}$.

As another corollary of Theorem 2.4.3, we obtain formulas for partial regularities of generic initial ideals, which recover [**3**, Proposition 2.11] and [**42**, Corollary 2.5].

Corollary 2.4.5. *Let $0 \neq I \subset R$ be a homogeneous ideal. Let $<$ be a monomial order on R that refines the partial order by degree and satisfies $x_1 > x_2 > \cdots > x_n$. Assume that* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 0$ *large enough. Denote by $G(I)$ the set of minimal monomial generators of* $\mathrm{Gin}_<(I)$*. Then:*

(i) $\mathrm{reg}(R/\mathrm{Gin}_<(I)) = \max\{\deg(u) - 1 \mid u \in G(I)\}$.
(ii) $\mathrm{reg}^{T,t}(R/\mathrm{Gin}_<(I)) = \max\{\deg(u) - 1 \mid u \in G(I), m(u) \geq t\}$.
(iii) $\mathrm{reg}_{L,t}(R/\mathrm{Gin}_<(I)) = \max\{\deg(u) - 1 \mid u \in G(I), m(u) \geq n - t\}$.
(iv) $\mathrm{reg}^{L,n-t}(R/\mathrm{Gin}_<(I)) \leq \mathrm{reg}_{T,t}(R/\mathrm{Gin}_<(I)) = \max\{\deg(u) - 1 \mid u \in G(I)\}$.
(v) $a^*(R/\mathrm{Gin}_<(I)) = \max\{\deg(u) + m(u) - n - 1 \mid u \in G(I)\}$.
(vi) $a^*_t(R/\mathrm{Gin}_<(I)) = \max\{\deg(u) + m(u) - n - 1 \mid u \in G(I), m(u) \geq n - t\}$.

Proof. It follows from Theorem 1.4.3 and Theorem 1.4.4 that $\mathrm{Gin}_<(I)$ is a strongly stable, and hence, a stable monomial ideal. We now obtain the corollary by applying Theorem 2.4.3 and Corollary 2.4.4. □

We conclude this section with:

Proposition 2.4.6. *Let $0 \neq I$ be a stable monomial ideal. Then for every integers $t \geq 0$ and $0 \leq s \leq n$, we have*

$$\mathrm{reg}_{T,t}(R/I) = \max\{\mathrm{reg}_{T,t+s}(R_{[n-s]}/I_{[n-s]}), \mathrm{reg}^{T,n+1-s}(R/I)\},$$

where $R_{[n-s]} = K[x_1, \ldots, x_{n-s}]$ and $I_{[n-s]} = I \cap R_{[n-s]}$.

Proof. It is easy to see that $I_{[n-s]}$ is a stable monomial ideal in $R_{[n-s]}$ and

$$\mathrm{Min}(I_{[n-s]}) = \{u \in \mathrm{Min}(u) \mid m(u) \leq n - s\}.$$

So by Theorem 2.4.3,

$$\mathrm{reg}_{T,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I)\},$$
$$\mathrm{reg}_{T,t+s}(R_{[n-s]}/I_{[n-s]}) = \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I), m(u) \leq n - s\},$$
$$\mathrm{reg}^{T,n+1-s}(R/I) = \max\{\deg(u) - 1 \mid u \in \mathrm{Min}(I), m(u) \geq n + 1 - s\},$$

from which the desired equality follows. □

2.5. Partial regularities of squarefree stable monomial ideals

In this section we discuss partial regularities of squarefree stable monomial ideals. As before, let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, and for a monomial $u \in R$ denote by $m(u)$ the largest index i such that x_i divides u.

Definition 2.5.1. Let $I \subset R$ be a monomial ideal. Then I is called:

(i) **squarefree** if I is generated by squarefree monomials, i.e. monomials of the form

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ with } \alpha_1, \dots, \alpha_n \in \{0, 1\}.$$

(ii) **squarefree stable** if I is squarefree and for every squarefree monomial $u \in I$ and for every $i < m(u)$ such that x_i does not divide u, we have that

$$x_i u / x_{m(u)} \in I.$$

For squarefree stable ideals we have the following analogue of Lemma 2.4.2(i).

Lemma 2.5.2 ([**19**, Corollary 7.4.2])**.** *Let $I \subset R$ be a squarefree stable monomial ideal. Denote by $\operatorname{Min}(I)$ the set of minimal monomial generators of I, and by $\operatorname{Min}(I)_j$ the set of elements of $\operatorname{Min}(I)$ of degree j. Then*

$$\beta_{i,i+j}(R/I) = \sum_{u \in \operatorname{Min}(I)_{j+1}} \binom{m(u) - \deg(u)}{i-1} \text{ for every integers } i, j \geq 0.$$

Remark. Analog statements of Lemma 2.4.2(ii)-(iii) for squarefree stable ideals are not true in general. For example, consider the ideal

$$I = (x_1 x_2, x_1 x_3) \subset R = K[x_1, x_2, x_3].$$

Clearly, I is squarefree stable. However, x_3, x_2, x_1 is not a filter-regular sequence on R/I. Indeed, the module

$$(I : x_3)/I = (x_1)/(x_1 x_2, x_1 x_3)$$

contains (the image of) x_1^n for $n \geq 1$, and therefore, has infinite length.

As an analogue of Theorem 2.4.3 we obtain the following formulas for partial regularities of squarefree stable ideals.

Theorem 2.5.3. *Let $0 \neq I \subset R$ be a squarefree stable monomial ideal with the set of minimal monomial generators $\operatorname{Min}(I)$. Then for any integer $t \geq 0$:*

(i) $\operatorname{reg}^{T,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I), m(u) - \deg(u) \geq t - 1\}$.

(ii) $\operatorname{reg}_{L,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I), m(u) - \deg(u) \geq n - t - 1\}$.

(iii) $\operatorname{reg}^{L,n-t}(R/I) \leq \operatorname{reg}_{T,t}(R/I) = \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I)\}$.

Proof. The argument is almost the same as the one used in the proof of Theorem 2.4.3. So we will only prove the formula for $\operatorname{reg}^{T,t}(R/I)$.

This can be done by using Lemma 2.5.2 as follows

$$\begin{aligned}\operatorname{reg}^{T,t}(R/I) &= \max\{j \mid \exists\, i \geq t \text{ s.t. } \beta_{i,i+j}(R/I) \neq 0\}\\ &= \max\{j \mid \exists\, i \geq t, u \in \operatorname{Min}(I) \text{ s.t. } \deg(u) = j+1, i-1 \leq m(u) - \deg(u)\}\\ &= \max\{\deg(u) - 1 \mid u \in \operatorname{Min}(I), m(u) - \deg(u) \geq t-1\}.\end{aligned}$$

$\square$

Remark. Recall that every squarefree monomial ideal I in R corresponds bijectively to an abstract simplicial complexes Δ on the vertex set $\{1, \ldots, n\}$. Then we write $I = I_\Delta$ and call it the *Stanley-Reisner* ideal of Δ. See [**9, 38, 33**] for further details.

Kalai introduced algebraic shifted complexes. See for example [**18**] and [**22**] for surveys on this topic. To a simplicial complex Δ one can always associate a shifted simplicial complex Δ^s which shares many properties with Δ. For example, they have the same f-vector. The transition from Δ to Δ^s corresponds in a certain sense to the consideration of the generic initial ideal to an arbitrary homogeneous ideal in I.

It turns out that for a shifted simplicial complex Δ^s, its Stanley-Reisner ideal I_{Δ^s} is always squarefree (strongly) stable and this gives interesting ideals to which one can apply Theorem 2.5.3. It is left as an exercise to the reader to write down an analogue of Corollary 2.4.5 for the Stanley-Reisner ideals of shifted simplicial complexes using Theorem 2.5.3.

2.6. Partial regularities and initial ideals

In this section we discuss briefly the relationship between partial regularities of an ideal and those ones of one of its initial ideals. Let $<$ be a monomial order on the polynomial ring $R = K[x_1, \ldots, x_n]$.

Proposition 2.6.1. *Let $I \subset R$ be a homogeneous ideal. Denote by $\operatorname{in}_<(I)$ the initial ideal of I with respect to $<$. Let $t \geq 0$ be an integer. Then*

(i) $\operatorname{reg}(R/I) \leq \operatorname{reg}(R/\operatorname{in}_<(I))$.
(ii) $\operatorname{reg}^{T,t}(R/I) \leq \operatorname{reg}^{T,t}(R/\operatorname{in}_<(I))$.
(iii) $\operatorname{reg}_{T,t}(R/I) \leq \operatorname{reg}_{T,t}(R/\operatorname{in}_<(I))$.
(iv) $\operatorname{reg}_{L,t}(R/I) \leq \operatorname{reg}_{L,t}(R/\operatorname{in}_<(I))$.
(v) $\operatorname{reg}^{L,t}(R/I) \leq \operatorname{reg}^{L,t}(R/\operatorname{in}_<(I))$.
(vi) $a^*(R/I) \leq a^*(R/\operatorname{in}_<(I))$.
(vii) $a^*_t(R/I) \leq a^*_t(R/\operatorname{in}_<(I))$.

Proof. By Theorem 1.3.11(i) and Theorem 1.3.12 we know that

$$\beta_{i,j}(I) \le \beta_{i,j}(\mathrm{in}_<(I)) \text{ and } \dim_K H^i_{\mathfrak{m}}(R/I)_j \le \dim_K H^i_{\mathfrak{m}}(R/\mathrm{in}_<(I))_j$$

for every integers i and j. The proposition follows easily from these inequalities. □

The inequalities in the previous proposition are in general strict. However, some of them become equalities when the monomial order chosen is rlex, the reverse lexicographic order. For this, we recall:

Lemma 2.6.2 (cf. [**3**, Lemma 2.3], [**43**, Lemma 6.1])**.** *Let $I \subset R$ be a homogeneous ideal. Then for $i = n, \dots, 1$,*

$$a\left(\frac{(I, x_n, \dots, x_{i+1}) : x_i}{(I, x_n, \dots, x_{i+1})}\right) = a\left(\frac{(\mathrm{in}_{\mathrm{rlex}}(I), x_n, \dots, x_{i+1}) : x_i}{(\mathrm{in}_{\mathrm{rlex}}(I), x_n, \dots, x_{i+1})}\right).$$

Thus, in particular, $x_n, \dots, x_i$ is a filter-regular sequence on R/I if and only if it is a filter-regular sequence on $R/\mathrm{in}_{\mathrm{rlex}}(I)$.

From this lemma, Corollary 2.3.2(ii) and Theorem 2.2.1 one can easily deduce:

Theorem 2.6.3 ([**3**, Theorem 2.4], [**42**, Theorem 1.3])**.** *Let $I \subset R$ be a homogeneous ideal. Assume that $x_n, x_{n-1}, \dots, x_1$ is a filter-regular sequence on $R/\mathrm{in}_{\mathrm{rlex}}(I)$. Let $t \ge 0$ be an integer. Then*

(i) $\mathrm{reg}(R/I) = \mathrm{reg}(R/\mathrm{in}_{\mathrm{rlex}}(I))$.
(ii) $\mathrm{reg}^{T,t}(R/I) = \mathrm{reg}^{T,t}(R/\mathrm{in}_{\mathrm{rlex}}(I))$.
(iii) $\mathrm{reg}_{L,t}(R/I) = \mathrm{reg}_{L,t}(R/\mathrm{in}_{\mathrm{rlex}}(I))$.
(iv) $a^*(R/I) = a^*(R/\mathrm{in}_{\mathrm{rlex}}(I))$.
(v) $a^*_t(R/I) = a^*_t(R/\mathrm{in}_{\mathrm{rlex}}(I))$.

Corollary 2.6.4 ([**3**, Proposition 2.11], [**42**, Corollary 1.4])**.** *Let $I \subset R$ be a homogeneous ideal. Denote by $\mathrm{Gin}_{\mathrm{rlex}}(I)$ the generic initial of I with respect to the reverse lexicographic order. Assume that the field K is infinite. Let $t \ge 0$ be an integer. Then*

(i) $\mathrm{reg}(R/I) = \mathrm{reg}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(ii) $\mathrm{reg}^{T,t}(R/I) = \mathrm{reg}^{T,t}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(iii) $\mathrm{reg}_{T,t}(R/I) \le \mathrm{reg}_{T,t}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(iv) $\mathrm{reg}_{L,t}(R/I) = \mathrm{reg}_{L,t}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(v) $\mathrm{reg}^{L,t}(R/I) \le \mathrm{reg}^{L,t}(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(vi) $a^*(R/I) = a^*(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.
(vii) $a^*_t(R/I) = a^*_t(R/\mathrm{Gin}_{\mathrm{rlex}}(I))$.

PROOF. By Theorem 1.5.4, $x_n, \dots, x_1$ is a filter-regular sequence on R/I after a generic choice of coordinates. Since $\mathrm{Gin}_{\mathrm{rlex}}(I)$ is the initial ideal of I for a generic choice of coordinates, the corollary now follows from Proposition 2.6.1 and Theorem 2.6.3. □

2.7. Upper bounds for partial regularities

In this short section we study generalizations of upper bounds for the Castelnuovo-Mumford regularity considered in [**11**] and [**20**]. Assume that K is an infinite field and $R = K[x_1, \dots, x_n]$ is a polynomial ring over K. Let $<$ be a monomial order on R that refines the partial order by degree and satisfies

$$x_1 > x_2 > \cdots > x_n.$$

Theorem 2.7.1. *Let $0 \neq I \subset R$ be a homogeneous ideal. Let $J = \mathrm{Gin}_<(I)$ be the generic initial ideal of I with respect to $<$. Assume that* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 0$ *large enough. Then for any integer $0 \leq t \leq n$, we have that*

$$\mathrm{reg}^{L,t}(R/I) \leq \mathrm{reg}(R_{[n-t]}/J_{[n-t]}),$$

where $R_{[n-t]} = K[x_1, \dots, x_{n-t}]$ and $J_{[n-t]} = J \cap R_{[n-t]}$.

PROOF. By Proposition 2.6.1(v), it suffices to show that

$$\mathrm{reg}^{L,t}(R/J) \leq \mathrm{reg}(R_{[n-t]}/J_{[n-t]}).$$

By the assumption on the base field, J is a stable monomial ideal and $x_n, \dots, x_1$ is a filter-regular sequence on R/J. By iteratively applying Corollary 2.3.2(i) we get

$$\mathrm{reg}^{L,t}(R/J) \leq \mathrm{reg}^{L,t-1}(R/(J, x_n)) \leq \cdots \leq \mathrm{reg}^{L,0}(R/(J, x_n, \dots, x_{n-t+1})).$$

Since

$$\mathrm{reg}^{L,0}(R/(J, x_n, \dots, x_{n-t+1})) = \mathrm{reg}(R/(J, x_n, \dots, x_{n-t+1})) = \mathrm{reg}(R_{[n-t]}/J_{[n-t]}),$$

the assertion follows. □

The following result extends a well-known upper bound for the Castelnuovo-Mumford regularity (see, e.g., [**11**, Corollary 1.9]) to $\mathrm{reg}^{L,t}(I)$.

Corollary 2.7.2. *Let $0 \neq I \subset R$ be a homogeneous ideal and let d be the largest degree of a minimal generator of I. Assume that* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > 0$ *large enough. Then*

$$\mathrm{reg}^{L,t}(I) \leq (2d)^{2^{n-t-2}}.$$

PROOF. Let $J = \mathrm{Gin}_{\mathrm{rlex}}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. Next define $I_{[n-t]}$ to be the image of I in the polynomial ring

$$R/(x_{n-t+1}, \ldots, x_n) \cong K[x_1, \ldots, x_{n-t}].$$

Note that $I_{[n-t]}$ is also generated in degrees smaller or equal to d. Define $J_{[n-t]}$ analogously to $I_{[n-t]}$ using the ideal J. Observe that the monomial ideal $J_{[n-t]}$ can be identified with $J \cap R_{[n-t]}$, so we can apply Theorem 2.7.1 to obtain

$$\mathrm{reg}^{L,t}(R/I) \leq \mathrm{reg}(R_{[n-t]}/J_{[n-t]}).$$

In [**11**] it was noted that $J_{[n-t]} = \mathrm{Gin}_{\mathrm{rlex}}(I_{[n-t]})$ in $K[x_1, \ldots, x_{n-t}]$ (see the remark before Corollary 1.10). In particular,

$$\mathrm{reg}(R_{[n-t]}/J_{[n-t]}) = \mathrm{reg}(R_{[n-t]}/I_{[n-t]}).$$

The desired upper bound follows by applying [**11**, Corollary 1.11] to the ideal $I_{[n-t]}$. $\square$

CHAPTER 3

Monomial preorders and leading ideals

Motivated by an earlier result of Robbiano on characterizing monomial orders (cf. [**31**]), we are interested in the question that whether we can find a similar characterization for so-called monomial preorders. By this initial question, we develop in this chapter a theory on monomial preorders, based on the joint work of Kemper, Trung and the author [**24**]. Throughout the chapter let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K.

3.1. Definition and characterization by matrices

Monomial preorders were introduced by Kemper and Trung [**23**], and then further studied in their joint work with the author [**24**]. The main goal of this section is to characterize monomial preorders in terms of real matrices. Our result generalizes a well-known characterization of monomial orders due to Robbiano [**31**].

Let $\mathcal{M}$ be the set of monomials of $K[\mathbf{x}]$. For any vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ we write $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$.

Definition 3.1.1 (Monomial preorders)**.** A strict partial order $<$ on the set $\mathcal{M}$ is called a **monomial preorder** if it satisfies the following conditions:

(i) $<$ is a **weak order**, i.e. the negation $\not<$ of $<$ is transitive.
(ii) Compatible with multiplication: for every $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{w}} \in \mathcal{M}$, $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ implies $\mathbf{x}^{\mathbf{u}}\mathbf{x}^{\mathbf{w}} < \mathbf{x}^{\mathbf{v}}\mathbf{x}^{\mathbf{w}}$.
(iii) Cancellative with multiplication: for every $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{w}} \in \mathcal{M}$, $\mathbf{x}^{\mathbf{u}}\mathbf{x}^{\mathbf{w}} < \mathbf{x}^{\mathbf{v}}\mathbf{x}^{\mathbf{w}}$ implies $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$.

The above notion of monomial preorders differs from that of monomial orders (see Definition 1.3.1) in two respects: first, a monomial preorder is not necessarily a total order; and second, it is not required that $1 < \mathbf{x}^{\mathbf{u}}$ for all $\mathbf{x}^{\mathbf{u}} \in \mathcal{M}\backslash\{1\}$.

It should be noted that for a total monomial order, the cancellative property can be deduced from the compatibility with the multiplication. However, this is no longer true for a weak order.

For example, define $\mathbf{x^u} < \mathbf{x^v}$ if

$$\deg \mathbf{x^u} < \deg \mathbf{x^v},\ \text{or}\ \deg \mathbf{x^u} = \deg \mathbf{x^v} > 1 \text{ and } \mathbf{x^u} <_{\text{lex}} \mathbf{x^v},$$

where $<_{\text{lex}}$ is the usual lexicographic order. This weak order is compatible with the multiplication but not cancellative, because

$$x_1x_2 < x_1^2 \text{ but } x_2 \not< x_1.$$

Example 3.1.2. (i) Similarly as in Definition 1.3.6, one can define the weight order $<_{\mathbf{w}}$ for an arbitrary real vector $\mathbf{w} \in \mathbb{R}^n$:

$$\mathbf{x^u} <_{\mathbf{w}} \mathbf{x^v} \quad \text{if} \quad \mathbf{w} \cdot \mathbf{u} < \mathbf{w} \cdot \mathbf{v},$$

with the dot denoting the standard scalar product. It is easily seen that $<_{\mathbf{w}}$ is a monomial preorder. For instance, the **degree order** (respectively, **reverse degree order**) defined by $\mathbf{x^u} < \mathbf{x^v}$ if $\deg \mathbf{x^u} < \deg \mathbf{x^v}$ (respectively, $\deg \mathbf{x^u} > \deg \mathbf{x^v}$) is the weight order of the vector $(1, \dots, 1)$ (respectively, $(-1, \dots, -1)$).

(ii) More generally, we can associate with every real $(m \times n)$-matrix M a monomial preorder $<$ by defining $\mathbf{x^u} < \mathbf{x^v}$ if $M \cdot \mathbf{u} <_{\text{lex}} M \cdot \mathbf{v}$, where $<_{\text{lex}}$ denotes the lexicographic order on $\mathbb{R}^m$. As we will see below, every monomial preorder on $K[\mathbf{x}]$ arises in this way.

We may identify the multiplicative monoid $\mathcal{M}$ with the additive monoid $\mathbb{N}^n$ via the isomorphism $\varphi : \mathcal{M} \to \mathbb{N}^n$ which associates to every monomial $\mathbf{x^u}$ its exponent $\mathbf{u}$. After this identification, we can speak of preorder on $\mathbb{N}^n$. Thus, a preorder $<$ on the additive monoid $\mathbb{N}^n$ is a weak order that is *compatible* and *cancellative* with the addition, i.e. $\mathbf{u} < \mathbf{v}$ if and only if $\mathbf{u}+\mathbf{w} < \mathbf{v}+\mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{N}^n$. In the same manner, one can also define preorder for additive groups, such as $\mathbb{Z}^n$ or $\mathbb{Q}^n$.

The main result of this section is the following characterization of monomial preorders.

Theorem 3.1.3 ([**24**, Theorem 1.2])**.** *For every monomial preorder $<$ on $K[\mathbf{x}]$, there is a real $(m \times n)$-matrix M for some $m > 0$ such that $\mathbf{x^u} < \mathbf{x^v}$ if and only if $M \cdot \mathbf{u} <_{\text{lex}} M \cdot \mathbf{v}$.*

Theorem 3.1.3 is actually about preorders on $\mathbb{N}^n$. For total orders on $\mathbb{Q}^n$, it was first shown by Robbiano [**31**, Theorem 4]. We will deduce Theorem 3.1.3 from Robbiano's result by using the following simple observations.

Lemma 3.1.4 (cf. [**24**, Lemma 1.3])**.** *Every preorder on $\mathbb{N}^n$ can be uniquely extended to a preorder on $\mathbb{Q}^n$.*

PROOF. Let $<$ be a preorder on $\mathbb{N}^n$. Every vector $\mathbf{u} \in \mathbb{Z}^n$ can be uniquely written as $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$ with $\mathbf{u}_+, \mathbf{u}_- \in \mathbb{N}^n$ having disjoint supports. For arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ we define $\mathbf{u} < \mathbf{v}$ if $\mathbf{u}_+ + \mathbf{v}_- < \mathbf{u}_- + \mathbf{v}_+$. One can easily show that $<$ is a preorder on $\mathbb{Z}^n$ extending the preorder $<$ on $\mathbb{N}^n$. Now, for arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^n$, we can always find a positive integer p such that $p\mathbf{u}, p\mathbf{v} \in \mathbb{Z}^n$. We define $\mathbf{u} < \mathbf{v}$ if $p\mathbf{u} < p\mathbf{v}$. It is easy to see that $<$ is a well-defined preorder on $\mathbb{Q}^n$. $\square$

Lemma 3.1.5 (cf. [**24**, Lemma 1.4])**.** *Let $<$ be a preorder on $\mathbb{Q}^n$. Denote by E the set of the elements which are incomparable to* 0*. Then the following statements hold:*

(i) *E is a linear subspace of $\mathbb{Q}^n$.*
(ii) *If we define $\mathbf{u} + E < \mathbf{v} + E$ if $\mathbf{u} < \mathbf{v}$ for arbitrary $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^n$, then $<$ is a total order on $\mathbb{Q}^n/E$.*

PROOF. It is clear that two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^n$ are incomparable if and only if $\mathbf{u}-\mathbf{v} \not< 0$ and $0 \not< \mathbf{u} - \mathbf{v}$, which means $\mathbf{u} - \mathbf{v} \in E$. Since the incomparability is transitive, $\mathbf{u}, \mathbf{v} \in E$ implies $\mathbf{u}, \mathbf{v}$ are incomparable and, therefore, $\mathbf{u}-\mathbf{v} \in E$. As a consequence, $\mathbf{u} \in E$ implies $p\mathbf{u} \in E$ for any $p \in \mathbb{N}$. From this it follows that $(p/q)\mathbf{u} = p\mathbf{u}/q \in E$ for any $q \in \mathbb{Z}$, $q \neq 0$. Therefore, E is a linear subspace of $\mathbb{Q}^n$ and $\mathbf{u} + E$ is the set of the elements which are incomparable to $\mathbf{u}$. Now, it is easy to see that the induced relation $<$ on $\mathbb{Q}^n/E$ is a total order on $\mathbb{Q}^n/E$. $\square$

We are now ready to prove Theorem 3.1.3.

PROOF OF THEOREM 3.1.3. As explained before, we may regard the given monomial preorder $<$ on $K[\mathbf{x}]$ as a preorder on the additive monoid $\mathbb{N}^n$. By Lemma 3.1.4, $<$ can be extended to a preorder on $\mathbb{Q}^n$. Let E be the set of the incomparable elements to 0 in $\mathbb{Q}^n$. By Lemma 3.1.5, E is a linear subspace of $\mathbb{Q}^n$ and $<$ induces a total order $<$ on $\mathbb{Q}^n/E$. By [**31**, Theorem 4], there is an injective $\mathbb{Q}$-linear map ϕ from $\mathbb{Q}^n/E$ to $\mathbb{R}^m$ such that $\mathbf{u} + E < \mathbf{v} + E$ if and only if $\phi(\mathbf{u} + E) <_{\text{lex}} \phi(\mathbf{v} + E)$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^n$. The composition of the natural map from $\mathbb{Q}^n$ to $\mathbb{Q}^n/E$ with ϕ is a linear map ψ from $\mathbb{Q}^n$ to $\mathbb{R}^m$ such that $\mathbf{u} < \mathbf{v}$ if and only if $\psi(\mathbf{u}) <_{\text{lex}} \psi(\mathbf{v})$. Since ψ is a linear map, we can find a real $(m \times n)$-matrix M such that $\psi(\mathbf{u}) = M \cdot \mathbf{u}$ for all $\mathbf{u} \in \mathbb{Q}^n$. Therefore, $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ if and only if $M \cdot \mathbf{u} <_{\text{lex}} M \cdot \mathbf{v}$. $\square$

Based on Theorem 3.1.3 one can characterize the following types of preorders, which play an important role in this chapter. Note that the analogous notions for monomial orders were introduced by Greuel and Pfister [**17**].

Definition 3.1.6. Let $<$ be a monomial preorder on $K[\mathbf{x}]$. Then $<$ is called:

(i) a **global monomial preorder** if $1 < x_i$ or 1 and x_i are incomparable for all i.
(ii) a **local monomial preorder** if $x_i < 1$ for all i.

Proposition 3.1.7. *Let $<$ be a monomial preorder on $K[\mathbf{x}]$ and let M be a real matrix represented $<$ as in Theorem 3.1.3. Then*

(i) *$<$ is a global monomial preorder if and only if for every nonzero column of M, the first nonzero entry is positive.*
(ii) *$<$ is a local monomial preorder if and only if every column of M is nonzero and the first nonzero entry in each column is negative.*

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the unit vectors of $\mathbb{N}^n$. The statements follow from the following easy observations:

- $M \cdot \mathbf{e}_i$ is the i-th column of M.
- 1 and x_i are incomparable if and only if $M \cdot \mathbf{e}_i = 0$.
- $1 < x_i$ (respectively, $x_i < 1$) if and only if $M \cdot \mathbf{e}_i \neq 0$ and its first nonzero entry is positive (respectively, negative).

□

3.2. Monomial preorders and leading term ideals

In this section we study leading term ideals of ideals in $K[\mathbf{x}]$ with respect to a monomial preorder. At first glance, the computation of such ideals seems to be difficult, because there is no division algorithm for monomial preorders in general. However, one can overcome this obstacle by refining the given monomial preorder to a monomial order. Our exposition is based on [**24**, Section 2].

Let us begin by recalling some basic constructions. Let $<$ be an arbitrary monomial preorder on $K[\mathbf{x}]$.

Definition 3.2.1. Let G be a subset of $K[\mathbf{x}]$ and $f = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} \in K[\mathbf{x}]$. Let

$$\max\nolimits_<(f) := \{\mathbf{x}^{\mathbf{u}} \mid c_{\mathbf{u}} \neq 0 \text{ and } \nexists c_{\mathbf{v}} \neq 0 \text{ with } \mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}\}$$

be the set of maximal monomials of f with respect to $<$.

Then we define:

(i) The **leading part** of f is

$$L_<(f) := \sum_{\mathbf{x}^{\mathbf{u}} \in \max_<(f)} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}.$$

(ii) The **leading ideal** of G is the ideal $L_<(G)$ in $K[\mathbf{x}]$ generated by $L_<(f)$, $f \in G$.

As we will see shortly, in order to study leading ideals with respect to a monomial preorder, it is more convenient to work with a certain localization of the ring $K[\mathbf{x}]$. The construction of this ring, originally due to Greuel and Pfister [**17**], is as follows.

Construction 3.2.2. Let $S_< := \{f \in K[\mathbf{x}] \mid L_<(f) = 1\}$. Since $<$ is compatible with the multiplication, we have $L_<(fg) = L_<(f)L_<(g)$ for $f, g \in K[\mathbf{x}]$. It follows that $S_<$ is a multiplicatively closed set, and we can form the localization $K[\mathbf{x}]_< := S_<^{-1}K[\mathbf{x}]$.

Example 3.2.3. If $<$ is a global monomial preorder, then $S_< = \{1\}$ and $K[\mathbf{x}]_< = K[\mathbf{x}]$. If $<$ is a local monomial preorder, then $S_< = K[\mathbf{x}] \backslash (\mathbf{x})$ and $K[\mathbf{x}]_< = K[\mathbf{x}]_{(\mathbf{x})}$.

One can define leading ideals for ideals in $K[\mathbf{x}]_<$:

Definition 3.2.4. Let $f \in K[\mathbf{x}]_<$ and $G \subseteq K[\mathbf{x}]_<$. We define:

(i) $L_<(f) := L_<(gf) \in K[\mathbf{x}]$, where g is an element in $S_<$ such that $gf \in K[\mathbf{x}]$.
(ii) $L_<(G) := \langle L_<(f) \mid f \in G \rangle \subseteq K[\mathbf{x}]$

Note that if there are two elements $g, h \in S_<$ such that $gf, hf \in K[\mathbf{x}]$, then

$$L_<(gf) = L_<(ghf) = L_<(hf),$$

because $L_<(g) = L_<(h) = 1$. Thus the notion of leading ideals in Definition 3.2.4 is well-defined. It should be emphasized that these leading ideals are ideals in the ring $K[\mathbf{x}]$, not the localization $K[\mathbf{x}]_<$.

Definition 3.2.1 and Definition 3.2.4 allow us to work in both rings $K[\mathbf{x}]$ and $K[\mathbf{x}]_<$. Actually, we can move from one ring to the other by the following lemma.

Lemma 3.2.5 ([**24**, Lemma 2.1])**.** *Let Q be an ideal in $K[\mathbf{x}]$ and I an ideal in $K[\mathbf{x}]_<$. Then*

(i) $L_<(QK[\mathbf{x}]_<) = L_<(Q)$.
(ii) $L_<(I \cap K[\mathbf{x}]) = L_<(I)$.

Proof. For (i), it can be seen that $L_<(Q) \subseteq L_<(QK[\mathbf{x}]_<)$. To prove the other direction, let $f \in QK[\mathbf{x}]_<$. Then there exists $u \in S$ such that $uf \in Q$. This implies

$L_<(f) = L_<(uf) \in L_<(Q)$. Thus, $L_<(QK[\mathbf{x}]) = L_<(Q)$. For (ii), let $g \in I$. Then there exists $u' \in S$ such that $u'g \in I \cap K[\mathbf{x}]$. Hence, $L_<(g) = L_<(u'g) \in L_<(I \cap K[\mathbf{x}])$. From this, it follows that $L_<(I) \subseteq L_<(I \cap K[\mathbf{x}])$. The other direction is obvious. This completes the proof of the lemma. $\square$

From Lemma 3.2.5(i) we see that two different ideals in $K[\mathbf{x}]$ have the same leading ideal if they have the same extension in $K[\mathbf{x}]_<$. This explains why one should work with ideals in $K[\mathbf{x}]_<$.

In the next part of this section we consider the problem of computing leading ideals of ideals in $K[\mathbf{x}]_<$. For convenience, let us first recall:

Definition 3.2.6. Let I be an ideal in $K[\mathbf{x}]_<$. A finite subset G of I is called a **standard basis** for I if $L_<(G) = L_<(I)$.

For monomial orders, this definition coincides with [**17**, Definition 1.6.1]. Note that if $<$ is a global monomial order, then $K[\mathbf{x}]_< = K[\mathbf{x}]$ and a standard basis is nothing else than a Gröbner basis (see Definition 1.3.3).

Let I be an ideal in $K[\mathbf{x}]_<$ generated by a finite set G. In order to compute the leading ideal $L_<(I)$ it suffices to compute a standard basis for I. When $<$ is a monomial order, this can be done by using the division algorithm [**17**]. Recall that the division algorithm gives a remainder h of the division of an element $f \in K[\mathbf{x}]_<$ by the elements of G such that if $h \neq 0$, then $L_<(h) \notin L_<(G)$. Unfortunately, we do not have a division algorithm for monomial preorders in general. For instance, if $<$ is the monomial preorder without comparable monomials, then $L_<(f) = f$ for all $f \in K[\mathbf{x}]$. In this case, there are no ways to construct such an algorithm. However, we can overcome this obstacle by refining the monomial preorder $<$.

Definition 3.2.7. Let $<$ and $<^*$ be monomial preorders on $K[\mathbf{x}]$. We say that $<^*$ is a **refinement** of $<$ if $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ implies $\mathbf{x}^{\mathbf{u}} <^* \mathbf{x}^{\mathbf{v}}$.

One usually refines a monomial preorder by taking its product with another monomial preorder.

Definition 3.2.8. Let $<$ and $<'$ be monomial preorders on $K[\mathbf{x}]$. The **product** of $<$ with $<'$ is a monomial preorder $<^*$ defined as follows: $\mathbf{x}^{\mathbf{u}} <^* \mathbf{x}^{\mathbf{v}}$ if either $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$, or $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}$ are incomparable with respect to $<$ and $\mathbf{x}^{\mathbf{u}} <' \mathbf{x}^{\mathbf{v}}$.

The following observations are immediate from the definitions:

- If $<^*$ is a refinement of $<$, then $S_< \subseteq S_{<^*}$ and $K[\mathbf{x}]_< \subseteq K[\mathbf{x}]_{<^*}$.
- The product of $<$ with another monomial preorder $<'$ is a refinement of $<$. Conversely, every refinement $<^*$ of $<$ is the product of $<$ with $<^*$.
- The product of a monomial preorder with a monomial order is a monomial order.

The leading ideal behaves nicely with the product of monomial preorders, as shown in the following result.

Lemma 3.2.9 ([**24**, Lemma 2.2])**.** *Let $<^*$ be the product of $<$ with a monomial preorder $<'$. Then*

(i) $L_{<^*}(G) \subseteq L_{<'}\big(L_<(G)\big)$ *for every subset* $G \subseteq K[\mathbf{x}]_<$.
(ii) $L_{<^*}(I) = L_{<'}\big(L_<(I)\big)$ *for every ideal* $I \subseteq K[\mathbf{x}]_<$.
(iii) *if $<'$ is global, then* $K[\mathbf{x}]_{<^*} = K[\mathbf{x}]_<$.

In particular, when I is a homogeneous ideal, one can always replace a monomial preorder by a global monomial preorder in the computation of its leading ideal.

Lemma 3.2.10 ([**24**, Lemma 2.8])**.** *Let I be a homogeneous ideal in $K[\mathbf{x}]$. Let $<^*$ be the product of the degree order with $<$. Then $1 <^* x_i$ for all i and $L_{<^*}(I) = L_<(I)$.*

By Lemma 3.2.9(ii), I and $L_<(I)$ share the same leading ideal with respect to $<^*$. If we choose $<'$ to be a monomial order, then $<^*$ is also a monomial order. Therefore, we can use results on the relationship between ideals and their leading ideals in the case of monomial orders to study this relationship in the case of monomial preorders. For instance, we have the following criterion for the equality of ideals by means of their leading ideals.

Theorem 3.2.11 ([**24**, Theorem 2.4, Corollary 2.5])**.** *Let $J \subseteq I$ be ideals in $K[\mathbf{x}]_<$ such that $L_<(J) = L_<(I)$. Then $J = I$. In particular, if G is a standard basis for I, then G generates I.*

It should be noted that a standard basis for an ideal in $K[\mathbf{x}]$ needs not generate the ideal. Indeed, for every ideal Q in $K[\mathbf{x}]$ we define

$$Q^* := QK[\mathbf{x}]_< \cap K[\mathbf{x}].$$

Then $Q \subseteq Q^*$. By Lemma 3.2.5, $L_<(Q) = L_<(Q^*)$. Therefore, a standard basis for Q is also a standard basis for Q^*. Now one can easily construct ideals Q such that $Q^* \neq Q$. For instance, if $Q = (gf)$ with $1 \neq g \in S_<$ and $0 \neq f \in K[\mathbf{x}]$, then $f \in Q^* \setminus Q$.

We now come to a solution for the problem of computing leading ideals with respect to monomial preorders. The following result shows, as mentioned before, that one can pass to the familiar case of monomial orders.

Theorem 3.2.12 ([**24**, Theorem 2.6]). *Let $<^*$ be the product of $<$ with a global monomial order. Let I be an ideal in $K[\mathbf{x}]_<$ (which by Lemma 3.2.9(iii) equals $K[\mathbf{x}]_{<^*}$). Then every standard basis G for I with respect to $<^*$ is also a standard basis for I with respect to $<$.*

Proof. Let $<^*$ be the product of $<$ with a global monomial order $<'$. Let G be a standard basis for I with respect to $<^*$. By Lemma 3.2.9(i)-(ii), we have

$$L_{<'}\big(L_<(I)\big) = L_{<^*}(I) = L_{<^*}(G) \subseteq L_{<'}\big(L_<(G)\big) \subseteq L_{<'}\big(L_<(I)\big).$$

This implies $L_{<'}(L_<(G)) = L_{<'}(L_<(I))$. Therefore, applying Theorem 3.2.11 to $<'$, we obtain $L_<(G) = L_<(I)$. $\square$

In the remainder of this chapter, we will investigate the relationship between an ideal and its leading ideal with respect to a monomial preorder. Let us begin with the case of homogeneous ideals. Further connections between the two ideals will be studied in the next sections.

Theorem 3.2.13. ([**24**, Theorem 2.10, Corollaries 2.9, 2.11]) *Let I be a homogeneous ideal in $K[\mathbf{x}]$. Then*

(i) *$L_<(I)$ is also a homogeneous ideal.*
(ii) *The quotient rings $K[\mathbf{x}]/I$ and $K[\mathbf{x}]/L_<(I)$ have the same Hilbert function. In particular, they have the same dimension.*

3.3. Approximation by integral weight orders

An important feature of the initial ideal with respect to a monomial order is that it is a flat deformation of the given ideal (see Theorem 1.3.9). In this section we show a similar result for monomial preorders. For that we need to approximate a monomial preorder by an *integral* weight order, i.e. a weight order $<_{\mathbf{w}}$ with $\mathbf{w} \in \mathbb{Z}^n$. We know from Chapter 1 that this approximation is possible for monomial orders (see Theorem 1.3.7). Compared to the case of monomial orders, the approximation for a monomial preorder is more complicated because of the existence of incomparable monomials, which must be given the same weight.

Lemma 3.3.1 ([**24**, Lemma 3.1])**.** *For any finite set S of monomials in $K[\mathbf{x}]$ we can find $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ if and only if $\mathbf{x}^{\mathbf{u}} <_{\mathbf{w}} \mathbf{x}^{\mathbf{v}}$ for all $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in S$.*

Proof. Let $<$ denote the preorder on $\mathbb{N}^n$ induced by the monomial preorder $<$ on $K[\mathbf{x}]$.

By Lemma 3.1.4, $<$ can be extended to a preorder on $\mathbb{Q}^n$. By Lemma 3.1.5, the set E of the elements incomparable to 0 is a linear subspace of $\mathbb{Q}^n$. Let $s = \dim \mathbb{Q}^n/E$. Let $\phi : \mathbb{Q}^n \to \mathbb{Q}^s$ be a surjective map such that $\ker \phi = E$.

Set

$$S' = \{\phi(\mathbf{u}) - \phi(\mathbf{v}) \mid \mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in S, \mathbf{u} < \mathbf{v}\}.$$

If $\phi(\mathbf{u}) - \phi(\mathbf{v}) = -(\phi(\mathbf{u}') - \phi(\mathbf{v}'))$ for some $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}, \mathbf{x}^{\mathbf{u}'}, \mathbf{x}^{\mathbf{v}'} \in S$ with $\mathbf{u} < \mathbf{v}$, $\mathbf{u}' < \mathbf{v}'$, then $\phi(\mathbf{u} + \mathbf{u}') = \phi(\mathbf{v} + \mathbf{v}')$. By Lemma 3.1.5, this implies that $\mathbf{u} + \mathbf{u}'$ and $\mathbf{v} + \mathbf{v}'$ are incomparable, which is a contradiction to the fact that $\mathbf{u} + \mathbf{u}' < \mathbf{v} + \mathbf{v}'$. Thus, if $\mathbf{a} \in S'$, then $-\mathbf{a} \notin S'$.

Now, we can find an integral vector $\mathbf{w}' \in \mathbb{Z}^s$ such that $\mathbf{w}' \cdot \mathbf{a} < 0$ for all $\mathbf{a} \in S'$. Thus, $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ if and only if $\mathbf{w}' \cdot \phi(\mathbf{u}) < \mathbf{w}' \cdot \phi(\mathbf{v})$ for all $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in S$. We can extend $\mathbf{w}'$ to an integral vector $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{w} \cdot \mathbf{u} = \mathbf{w}' \cdot \phi(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{Q}^n$. From this it follows that $\mathbf{u} < \mathbf{v}$ if and only if $\mathbf{w} \cdot \mathbf{u} < \mathbf{w} \cdot \mathbf{v}$ for all $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in S$. Hence $\mathbf{x}^{\mathbf{u}} < \mathbf{x}^{\mathbf{v}}$ if and only if $\mathbf{x}^{\mathbf{u}} <_{\mathbf{w}} \mathbf{x}^{\mathbf{v}}$ for all $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}} \in S$. □

Using Lemma 3.3.1 we can show that on a finite set of ideals, any monomial preorder $<$ can be replaced by an integral weight order. A complicated proof for global monomial preorders was given by Kemper and Trung in [**23**, Lemma 3.3].

Theorem 3.3.2 ([**24**, Theorem 3.2])**.** *Let $I_1, \ldots, I_r$ be ideals in $K[\mathbf{x}]$. Then there exists an integral vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ such that $L_<(I_i) = L_{<_{\mathbf{w}}}(I_i)$ for $i = 1, \ldots, r$.*

Proof. Let $<^*$ be the product of $<$ with a global monomial order $<'$. Then $<^*$ is a monomial order. Moreover, $K[\mathbf{x}]_{<^*} = K[\mathbf{x}]_<$ by Lemma 3.2.9(iii). For each i, let $G_i \subset K[\mathbf{x}]$ be a standard basis for $I_i K[\mathbf{x}]_<$ with respect to $<^*$. Then by Lemma 3.2.5(i) and Theorem 3.2.12,

$$L_<(I_i) = L_<(I_i K[\mathbf{x}]_<) = L_<(G_i).$$

Since $<^*$ is a monomial order, there exists a finite set S_i of monomials such that G_i is a standard basis for I_i with respect to any monomial order coinciding with $<^*$ on S_i; see [**17**, Corollary 1.7.9].

Let S be the union of the set of all monomials of the polynomials in the G_i with $\cup_{i=1}^r S_i$. By Lemma 3.3.1, there is an integral vector $\mathbf{w} \in \mathbb{Z}^n$ such that $L_{<_\mathbf{w}}(f) = L_<(f)$ for all $f \in S$. This implies

$$L_<(G_i) = L_{<_\mathbf{w}}(G_i) \ \text{ for } \ i = 1, \ldots, r.$$

Let $<_\mathbf{w}^*$ be the product of $<_\mathbf{w}$ with $<'$. For every $f \in S$, it follows from the definition of the product of monomial orders that

$$L_{<_\mathbf{w}^*}(f) = L_{<'}(L_{<_\mathbf{w}}(f)) = L_{<'}(L_<(f)) = L_{<^*}(f).$$

So $<_\mathbf{w}^*$ coincides with $<^*$ on S_i. Therefore, G_i is a standard basis for I_i with respect to $<_\mathbf{w}^*$. By Theorem 3.2.12, this implies $L_{<_\mathbf{w}}(G_i) = L_{<_\mathbf{w}}(I_i)$. Summing up we get

$$L_<(I_i) = L_<(G_i) = L_{<_\mathbf{w}}(G_i) = L_{<_\mathbf{w}}(I_i) \ \text{ for } \ i = 1, \ldots, r.$$

□

Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a vector in $\mathbb{Z}^n$, I an ideal in $K[\mathbf{x}]$, and t a new variable. Recall from Construction 1.3.8 that the homogenization of I with respect to $\mathbf{w}$ is defined as follows. For a polynomial $f = \sum c_\mathbf{u} \mathbf{x}^\mathbf{u} \in K[\mathbf{x}]$, we set $\deg_\mathbf{w} f = \max\{\mathbf{w} \cdot \mathbf{u} | \ c_\mathbf{u} \neq 0\}$ and

$$f^{\text{hom}} = t^{\deg_\mathbf{w} f} f\big(t^{-w_1}x_1, \ldots, t^{-w_n}x_n\big).$$

Then

$$I^{\text{hom}} = \langle f^{\text{hom}} \mid f \in I \rangle.$$

Note that I^{hom} is a weighted homogeneous ideal in $R := K[\mathbf{x}, t]$ with respect to the weighted degree $\deg x_i = w_i$ and $\deg t = 1$.

To conclude this section, we show that the leading ideal $L_<(I)$ of an arbitrary ideal I in $K[\mathbf{x}]$ with respect to a monomial preorder $<$ is a flat deformation of I. By Theorem 3.3.2, it suffices to show this for the case $<$ is an integral weight order.

Theorem 3.3.3 ([**24**, Lemma 3.3 and Proposition 3.4])**.** *Let* $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ *and* $R = K[\mathbf{x}, t]$. *For any ideal* I *in* $K[\mathbf{x}]$*, the following statements hold:*

(i) R/I^{hom} *is a flat extension of* $K[t]$.

(ii) $R/(I^{\text{hom}}, t) \cong K[\mathbf{x}]/L_{<_\mathbf{w}}(I)$.

(iii) $(R/I^{\text{hom}})[t^{-1}] \cong (K[\mathbf{x}]/I)[t, t^{-1}]$.

Thus $\{R/(I^{\text{hom}}, t-a)\}_{a \in K}$ *is a flat family over* $K[t]$ *whose special fiber is isomorphic to* $K[\mathbf{x}]/L_{<_\mathbf{w}}(I)$ *and whose general fibers are all isomorphic to* $K[\mathbf{x}]/I$.

This result was already stated for an arbitrary integral order $<_{\mathbf{w}}$ by Eisenbud [**14**, Theorem 15.17]. However, the proof there required that all w_i are positive. This case was also proved by Kreuzer and Robbiano in [**25**, Theorem 4.3.22]. For the case that $w_i \neq 0$ for all i, it was proved by Greuel and Pfister [**17**, Exercise 7.3.19 and Theorem 7.5.1].

PROOF. It is clear that if we write f^{hom} as a polynomial in t, then $L_{<_{\mathbf{w}}}(f)$ is just the constant coefficient of f^{hom}. Thus,

$$L_{<_{\mathbf{w}}}(I) \cong (I^{\text{hom}}, t)/(t).$$

From this one easily gets the isomorphism stated in (ii).

Now consider the automorphism of $R[t^{-1}]$ defined by

$$\Phi_{\mathbf{w}}(x_i) = t^{-w_i} x_i \quad \text{for} \quad i = 1, \ldots, n.$$

Then $\Phi_{\mathbf{w}}(f) = t^{-\deg_{\mathbf{w}}} f^{\text{hom}}$ for every $f \in K[\mathbf{x}]$. Therefore,

$$\Phi_{\mathbf{w}}(IR[t^{-1}]) = I^{\text{hom}} R[t^{-1}].$$

It follows that $\Phi_{\mathbf{w}}$ induces an isomorphism between $(R/I^{\text{hom}})[t^{-1}]$ and $(K[\mathbf{x}]/I)[t, t^{-1}]$. This proves (iii).

Finally, we prove (i). It is known that a module over a principal ideal domain is flat if and only if it is torsion-free (see Eisenbud [**14**, Corollary 6.3]). Therefore, we only need to show that R/I^{hom} is torsion-free. Let $g \in K[t] \setminus \{0\}$ and $F \in R \setminus I^{\text{hom}}$. Then we have to show that $gF \notin I^{\text{hom}}$. Assume that $gF \in I^{\text{hom}}$. Since I^{hom} is weighted homogeneous, we may assume that g and F are weighted homogeneous polynomials. Then $g = \lambda t^d$ for some $\lambda \in K$, $\lambda \neq 0$, and $d \geq 0$. Note that t is a non-zerodivisor on R/I^{hom} [**25**, Proposition 4.3.5(e)]. So from $gF \in I^{\text{hom}}$ it follows that $F \in I^{\text{hom}}$, a contradiction. $\square$

Remark. From Theorem 3.2.13, we have seen that in the case I is an homogeneous ideal of $K[\mathbf{x}]$, the dimensions of the rings $K[\mathbf{x}]/I$ and $K[\mathbf{x}]/L_<(I)$ are the same. In general, for an arbitrary ideal I in $K[\mathbf{x}]$ (respectively, $K[\mathbf{x}]_<$), the dimension of the ring $K[\mathbf{x}]/L_<(I)$ can be different from that of $K[\mathbf{x}]/I$ (respectively, $K[\mathbf{x}]_</I$) [**24**, Remarks 3.8, 3.11]. However, it can be shown that:

(i) If I is an ideal in $K[\mathbf{x}]$, then

$$\dim K[\mathbf{x}]/I \geq \dim K[\mathbf{x}]/L_<(I),$$

with equality if $<$ is a global monomial preorder [**24**, Theorem 3.6, Corollary 3.7].

(ii) If I is an ideal in $K[\mathbf{x}]_<$, then

$$\dim K[\mathbf{x}]_</I \le \dim K[\mathbf{x}]/L_<(I),$$

with equality if and only if $1 \notin (P, \mathbf{x}_-)$ for at least one prime P of I with height$P$ = height$I$, where $\mathbf{x}_- := \{x_i \mid x_i < 1\}$ [**24**, Theorem 3.10].

3.4. Comparison of invariants

Let $<$ be an arbitrary monomial preorder on $K[\mathbf{x}]$. In this section, we briefly discuss the relationship between several invariants of a homogeneous ideal in $K[\mathbf{x}]$ with those of its leading ideal. These include Betti numbers, Hilbert functions of local cohomology of quotient rings, and partial regularities.

Let us begin with the following result which is essentially due to Caviglia in his proof of Sturmfels' conjecture on the Koszul property of the pinched Veronese [**10**].

Proposition 3.4.1 ([**24**, Proposition 4.10])**.** *Let I, J, Q be homogeneous ideals in $K[\mathbf{x}]$. Then*

$$\dim_K \operatorname{Tor}_i^{K[\mathbf{x}]/I}(K[\mathbf{x}]/J, K[\mathbf{x}]/Q)_j \le \dim_K \operatorname{Tor}_i^{K[\mathbf{x}]/L_<(I)}(K[\mathbf{x}]/L_<(J), K[\mathbf{x}]/L_<(Q))_j$$

for every integers i and j.

Proof. By Lemma 3.2.10 we may assume that $<$ is a monomial preorder with $1 < x_i$ for all i. Applying Theorem 3.3.2 to I, J, Q we can find $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ with $w_i > 0$ for all i such that

$$L_<(I) = L_{<_\mathbf{w}}(I),\ L_<(J) = L_{<_\mathbf{w}}(J),\ \text{and } L_<(Q) = L_{<_\mathbf{w}}(Q).$$

For a positive weight vector $\mathbf{w}$, Caviglia [**10**, Lemma 2.1] already showed that

$$\dim_K \operatorname{Tor}_i^{K[\mathbf{x}]/I}(K[\mathbf{x}]/J, K[\mathbf{x}]/Q)_j \le \dim_K \operatorname{Tor}_i^{K[\mathbf{x}]/L_{<_\mathbf{w}}(I)}(K[\mathbf{x}]/L_{<_\mathbf{w}}(J), K[\mathbf{x}]/L_{<_\mathbf{w}}(Q))_j$$

for every integers i and j. $\square$

This result leads to some interesting consequences. First, as in the case of monomial orders, one can show that the Koszul property descents from $L_<(I)$ to I. Recall that a K-algebra R is said to be Koszul if K has a linear free resolution as an R-module or, equivalently, if $\operatorname{Tor}_i^R(K, K)_j = 0$ for every $j \ne i$.

Corollary 3.4.2 ([**24**, Corollary 4.11])**.** *Let I be a homogeneous ideal in $K[\mathbf{x}]$. Assume that $K[\mathbf{x}]/L_<(I))$ is a Koszul algebra. Then $K[\mathbf{x}]/I$ is also Koszul.*

Proof. We apply Proposition 3.4.1 to the case $J = Q = (X)$. From this it follows that if $\operatorname{Tor}_i^{K[\mathbf{x}]/L_<(I)}(K,K)_j = 0$, then $\operatorname{Tor}_i^{K[\mathbf{x}]/I}(K,K)_j = 0$ for every $j \neq i$. □

As another consequence of Proposition 3.4.1, we have the following comparison for Betti numbers.

Corollary 3.4.3 ([**24**, Proposition 4.12])**.** *Let I be a homogeneous ideal in $K[\mathbf{x}]$. Then*

$$\beta_{i,j}(K[\mathbf{x}]/I) \leq \beta_{i,j}(K[\mathbf{x}]/L_<(I)) \text{ for every integers } i \text{ and } j.$$

Proof. We apply Proposition 3.4.1 to the case $I = 0$, $Q = (X)$ and replace J by I. Then

$$\dim_K \operatorname{Tor}_i^{K[\mathbf{x}]}(K[\mathbf{x}]/I, K)_j \leq \dim_K \operatorname{Tor}_i^{K[\mathbf{x}]}(K[\mathbf{x}]/L_<(I), K)_j$$

which implies $\beta_{i,j}(K[\mathbf{x}]/I) \leq \beta_{i,j}(K[\mathbf{x}]/L_<(I))$ for every integers i and j. □

We now turn to the comparison of Hilbert functions of local cohomology of the quotient rings $K[\mathbf{x}]/I$ and $K[\mathbf{x}]/L_<(I)$. As already mentioned in Section 1.3, the first result of this type was first shown by Sbarra [**35**] for monomial orders.

Proposition 3.4.4. *Let I be a homogeneous ideal in $K[\mathbf{x}]$. Then*

$$\dim_K H^i_{\mathfrak{m}}(K[\mathbf{x}]/I)_j \leq \dim_K H^i_{\mathfrak{m}}(K[\mathbf{x}]/L_<(I))_j \text{ for every integers } i \text{ and } j.$$

Proof. Sbarra [**35**, Theorem 2.4] already proved the above inequality for an arbitrary global monomial order. Actually, his proof shows that for every integral weight order $<_{\mathbf{w}}$ and every integers i and j,

$$\dim_K H^i_{\mathfrak{m}}(K[\mathbf{x}]/I)_j \leq \dim_K H^i_{\mathfrak{m}}(K[\mathbf{x}]/L_{<_{\mathbf{w}}}(I))_j.$$

By Theorem 3.3.2, there exists $\mathbf{w} \in \mathbb{Z}^n$ such that $L_<(I) = L_{<_{\mathbf{w}}}(I)$. Therefore, Sbarra's result implies the conclusion. □

From Corollary 3.4.3, Proposition 3.4.4 and Theorem 1.2.2, we have the following comparisons for projective dimensions, depths and types of the rings $K[\mathbf{x}]/I$ and $K[\mathbf{x}]/L_<(I)$.

Corollary 3.4.5. *Let I be a homogeneous ideal in $K[\mathbf{x}]$ and $<$ a monomial preorder. Then we have that*

(i) $\operatorname{pd}(K[\mathbf{x}]/I) \leq \operatorname{pd}(K[\mathbf{x}]/L_<(I))$.
(ii) $\operatorname{depth}(K[\mathbf{x}]/I) \geq \operatorname{depth}(K[\mathbf{x}]/L_<(I))$.
(iii) *If* $\operatorname{depth}(K[\mathbf{x}]/I) = \operatorname{depth}(K[\mathbf{x}]/L_<(I))$, *then* $r(K[\mathbf{x}]/I) \leq r(K[\mathbf{x}]/L_<(I))$.

PROOF. (i) follows directly from Corollary 3.4.3. To prove (ii), denote by d the number $\operatorname{depth}(K[\mathbf{x}]/L_<(I))$. By Proposition 3.4.4, we get

$$\dim_K H^i_{\mathfrak{m}}(K[\mathbf{x}]/I) \leq \dim_K H^i_{\mathfrak{m}}(K[\mathbf{x}]/L_<(I)), \text{ for all } i < d.$$

This implies $H^i_{\mathfrak{m}}(K[\mathbf{x}]/I) = 0$ for every $i < d$. Thus, $\operatorname{depth}(K[\mathbf{x}]/I) \geq d$. We now prove (iii), for which we also denote by d the common depth of $K[\mathbf{x}]/I$ and $K[\mathbf{x}]/L_<(I)$. By using Proposition 3.4.4 and Theorem 1.2.2, we get for every j,

$$\begin{aligned}\dim_K \operatorname{Ext}^d_{K[\mathbf{x}]}(K, K[\mathbf{x}]/I)_j &= \dim_K H^{n-d}_{\mathfrak{m}}(K[\mathbf{x}]/I)_{-n-j} \\ &\leq \dim_K H^{n-d}_{\mathfrak{m}}(K[\mathbf{x}]/L_<(I))_{-n-j} = \dim_K \operatorname{Ext}^d_{K[\mathbf{x}]}(K, K[\mathbf{x}]/L_<(I))_j.\end{aligned}$$

Hence, from the definition of the type of a module we have that

$$r(K[\mathbf{x}]/I) = \dim_K \operatorname{Ext}^d_{K[\mathbf{x}]}(K, K[\mathbf{x}]/I) \leq \dim_K \operatorname{Ext}^d_{K[\mathbf{x}]}(K, K[\mathbf{x}]/L_<(I)) = r(K[\mathbf{x}]/L_<(I)),$$

as desired. □

We will prove in the following proposition that the properties of being Cohen-Macaulay or Gorenstein are transferred from the ring $K[\mathbf{x}]/L_<(I)$ to $K[\mathbf{x}]/I$.

Proposition 3.4.6. *If $K[\mathbf{x}]/L_<(I)$ satisfies one of the following properties, then so does the ring $K[\mathbf{x}]/I$: (i) Cohen-Macaulay; (ii) Gorenstein.*

The proof of this proposition needs the following theorem which is a well-known result.

Theorem 3.4.7. *For any homogeneous ideal I in $K[\mathbf{x}]$, we have the following:*

(i) *(*[**14**, Section 18.2]*) $K[\mathbf{x}]/I$ is Cohen-Macaulay if and only if*

$$\operatorname{codim}(I) = \operatorname{pd}(K[\mathbf{x}]/I).$$

(ii) (**Auslander-Buchsbaum Formula**)

$$\operatorname{pd}(K[\mathbf{x}]/I) + \operatorname{depth}(K[\mathbf{x}]/I) = n.$$

PROOF OF PROPOSITION 3.4.6. Our proof of this proposition is based on Theorem 3.2.13, Corollary 3.4.5 and Theorem 3.4.7. First assume that $K[\mathbf{x}]/L_<(I)$ is Cohen-Macaulay. By Theorem 3.2.13, the Krull dimensions of $K[\mathbf{x}]/I$ and $K[\mathbf{x}]/L_<(I)$ are equal, hence the codimensions of I and $L_<(I)$ are equal. This fact together with Theorem 3.4.7 (i) and Corollary 3.4.5 implies that

$$\operatorname{codim}(I) = \operatorname{codim}(L_<(I)) = \operatorname{pd}(K[\mathbf{x}]/L_<(I)) \geq \operatorname{pd}(K[\mathbf{x}]/I).$$

So by Theorem 3.4.7 (ii), $\operatorname{codim}(I) \geq \operatorname{pd}(K[\mathbf{x}]/I) = n - \operatorname{depth}(K[\mathbf{x}]/I)$. Hence,

$$\operatorname{depth}(K[\mathbf{x}]/I) \geq n - \operatorname{codim}(I) = \dim(K[\mathbf{x}]/I).$$

It follows that $\operatorname{depth}(K[\mathbf{x}]/I) = \dim(K[\mathbf{x}]/I)$, and so $K[\mathbf{x}]/I$ is Cohen-Macaulay.

We now assume that $K[\mathbf{x}]/L_<(I)$ is Gorenstein, then $K[\mathbf{x}]/L_<(I)$ is Cohen-Macaulay and $\mathrm{r}(K[\mathbf{x}]/L_<(I)) = 1$. As we have seen above, $K[\mathbf{x}]/I$ is also Cohen-Macaulay. Therefore, $K[\mathbf{x}]/L_<(I)$ and $K[\mathbf{x}]/I$ have the same dimension and depth. By Corollary 3.4.5, we have that $0 \neq \mathrm{r}(K[\mathbf{x}]/I) \leq \mathrm{r}(K[\mathbf{x}]/L_<(I)) = 1$. Thus, $\mathrm{r}(K[\mathbf{x}]/I) = 1$ and $K[\mathbf{x}]/I$ is also Gorenstein. $\square$

Finally, from Corollary 3.4.3 and Proposition 3.4.4 we immediately get the following comparisons for partial regularities, which generalizes Proposition 2.6.1.

Proposition 3.4.8. *Let I be a homogeneous ideal in $K[\mathbf{x}]$. Let $t \geq 0$. Then*

(i) $\operatorname{reg}(K[\mathbf{x}]/I) \leq \operatorname{reg}(K[\mathbf{x}]/L_<(I))$.
(ii) $\operatorname{reg}^{T,t}(K[\mathbf{x}]/I) \leq \operatorname{reg}^{T,t}(K[\mathbf{x}]/L_<(I))$.
(iii) $\operatorname{reg}_{T,t}(K[\mathbf{x}]/I) \leq \operatorname{reg}_{T,t}(K[\mathbf{x}]/L_<(I))$.
(iv) $\operatorname{reg}_{L,t}(K[\mathbf{x}]/I) \leq \operatorname{reg}_{L,t}(K[\mathbf{x}]/L_<(I))$.
(v) $\operatorname{reg}^{L,t}(K[\mathbf{x}]/I) \leq \operatorname{reg}^{L,t}(K[\mathbf{x}]/L_<(I))$.
(vi) $a^*(K[\mathbf{x}]/I) \leq a^*(K[\mathbf{x}]/L_<(I)$.
(vii) $a^*_t(K[\mathbf{x}]/I) \leq a^*_t(K[\mathbf{x}]/L_<(I))$.

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