Enriched Infinity Operads

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Hongyi Chu

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Erstgutachter: Prof. Dr. Markus Spitzweck

Zweitgutachter: Prof. Dr. Benoit Fresse

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Contents

Chapter 1

Introduction

1.1 Operads

The concept of *operads* was originally invented in the 70's by Boardman, Vogt and May in an effort to provide a general framework for defining and studying algebraic structures in topology. Since then many different generalizations and variations of the original definition have been developed. All these developments are not only indispensable in the modern category-theoretic study of algebraic topology, but also have inspired and influenced many other areas of mathematics such as category theory, homological algebra and algebraic geometry. One of the most notable generalizations of the original concept is the theory of *symmetric coloured operads*. A symmetric coloured operad $\mathcal{O}$ is determined by the following data:

- A set of objects $\mathrm{Obj}(\mathcal{O})$.
- A set of multimorphisms $\mathrm{Mul}_{\mathcal{O}}(x_1, \ldots, x_m; x)$ associated to every finite collection $x_1, \ldots, x_n, x$ of elements in $\mathrm{Obj}(\mathcal{O})$.
- Composition maps

$$\prod_{1 \leq i \leq n} \mathrm{Mul}_{\mathcal{O}}(x_{1_i}, \ldots x_{m_i}; x_i) \times \mathrm{Mul}_{\mathcal{O}}(x_1, \ldots, x_n; x) \to \mathrm{Mul}_{\mathcal{O}}(x_{1_1}, \ldots, x_{m_n}; x)$$

  which satisfy the obvious associativity and unitality conditions.

Although the term "operad" is usually reserved for a symmetric coloured operad with only one object, we will also use this term for a general symmetric coloured operad to ease notation.

Many interesting categories, such as the category of modules over a fixed ring, the category of chain complexes or the category of topological spaces, admit a symmetric monoidal structure. Every symmetric monoidal category in turn induces an operad in a natural way. To render this statement more precisely, for a given symmetric monoidal category $\mathcal{V}$ we can construct an operad $\mathcal{V}^{\otimes}$, where the objects of $\mathcal{V}^{\otimes}$ are those of $\mathcal{V}$ and the associated sets of multimorphisms are defined by

$$\mathrm{Mul}_{\mathcal{V}^{\otimes}}(x_1, \ldots, x_m; x) := \mathrm{Hom}_{\mathcal{V}}(x_1 \otimes \ldots \otimes x_m, x).$$

In general, given an operad $\mathcal{O}$, we can define an $\mathcal{O}$-algebra in $\mathcal{V}$ as a functor between the operads $\mathcal{O}$ and $\mathcal{V}^{\otimes}$, whose data consists of an assignment of objects and an assignment of multimorphisms which is identity preserving and compatible with compositions. This formalism allows us to identify topological monoids with Ass-algebras in the category of topological spaces, where Ass denotes the associative operad.

If we could go back to the 70's, then we would probably come to realize that the introduction of operads was greatly motivated by a desire to study iterated loop spaces. These spaces admit a monoidal structure where the algebraic identities, such as associativity, are required to hold only up to coherent homotopy. This was one of main reasons why Boardman and Vogt invented the little n-cube operad. For $n = 1$, the little 1-cube operad (or little interval operad) is a generalization of the associative operad Ass and has a space instead of a set of multimorphisms. Roughly speaking, in order to define algebraic structures in a homotopy coherent setting, we first need to introduce operads whose sets of mulitmorphisms are replaced by objects in either the symmetric monoidal category of topological spaces or in a homotopically equivalent category.

This observation leads us to the concept of enriched operads. If $\mathcal{V}$ is a symmetric monoidal category, then we obtain the definition of $\mathcal{V}$-enriched operads by replacing the sets of multimorphisms and products in the above definition of symmetric coloured operads by objects in the symmetric monoidal category $\mathcal{V}$ and the tensor products in $\mathcal{V}$, respectively. Operads which are enriched over topological spaces are then called topological operads and the little n-cube operad is one of their most prominent examples. Since we often find ourselves only caring about the weak homotopy types of spaces of multimorphisms, we want to consider the homotopy theory of topological operads under a suitable notion of weak equivalence. This can be done by imposing a model structure on the category of topological operads. By a recent result of Caviglia [Cav14], the category of $\mathcal{V}$-enriched operads admits a model structure provided $\mathcal{V}$ satisfies certain properties. Applied to the case of topological spaces, this result yields a nice model structure on the category of topological operads.

1.2 ∞-Operads

Topological operads can also be regarded as one of the simplest models for ∞-operads (or, more precisely, $(\infty, 1)$-operads). Roughly speaking, these objects are operads which additionally have homotopies between multimorphisms and homotopies between homotopies and so on. Although topological operads are easy to define, they have severe drawbacks and their structure is, in a sense, too rigid. This is, for example, indicated by the fact that the categories of algebras of topological operads behave rather badly, i.e. it is in general very hard to construct the homotopically correct topological category of algebras between two topological operads. A more subtle problem arises from a structural viewpoint when one takes into account that a desired model of ∞-operads should allow for compositions which are only associative up to a coherent choice of higher homotopies. For instance, in a correct model of ∞-operads, small symmetric monoidal categories should be the commutative monoids in the 2-category of small categories, but the associativity and symmetry conditions almost never hold strictly.

The following list summarizes all models for ∞-operads:

- Lurie's ∞-operads: They were introduced in his fundamental work [Lur], where he provides a powerful machinery to study homotopy-coherent algebraic structures. At the moment this approach provides by far the best-developed theory of ∞-operads. Many homotopy-invariant constructions such as operadic Kan-extensions are only available in this model.

- Dendroidal sets and related models: The theory of dendroidal sets was introduced by Moerdijk and Weiss [MW07]. Later Cisinski and Moerdijk discovered related models such as dendroidal complete Segal spaces and Segal operads in [CM13a]. They showed in [CM13a] that the three provided dendroidal models for ∞-operads are equivalent. Furthermore, in [CM13b] they compared dendroidal sets with simplicial operads (which are equivalent to topological operads). The dendroidal models can be regarded as canonical generalizations of both the of quasi-categories of Joyal as well as complete Segal spaces of Rezk where the category Δ is replaced by the dendroidal category Ω whose objects are trees. The occurrence of tree structures in dendroidal models reflects the need to compose multimorphisms.

  One of the advantages of the dendroidal approach is that it is purely combinatorial. The ∞-operads are fibrant objects in the corresponding model structures and admit simple descriptions: They are presheaves on the dendroidal category satisfying certain Segal conditions. This is also a motivation for Chapter 4 and Chapter 7 where we introduce a Δ_Φ-presheaf model and an Ω-presheaf model for enriched ∞-operads. In the unenriched case the Ω-presheaves coincide with dendroidal Segal spaces of Cisinski and Moerdijk.

- Barwick's complete Segal operads: Barwick introduced the concept of operator categories in [Bar13] and developed the theory of Segal operads for these categories. He showed that, for the operator category of finite sets, the ∞-category of complete Segal operads is equivalent to that of Lurie's ∞-operads.

  The first advantage of his approach to ∞-operads is the vast generality that the theory allows. Since there are Segal Φ-operads for each operator category Φ, the theory of operator categories sheds light on the connections between different kinds of ∞-operads. For instance, in the dendroidal approach, the commutative ∞-operads, the non-commutative ∞-operads and quasi-categories are treated separately. From the operator categorical point of view, these objects are connected naturally by morphisms between the three corresponding operator categories. Apart from these categories, there is no other theory available to describe objects which behave like Segal Φ-operads for an arbitrary operator category Φ. The second advantage of this model is that the construction of Segal operads allows us to define the enrichment of ∞-operads in Chapter 3 in a broad setting.

As mentioned above, these different models for ∞-operads have been compared thorughout different publications.

- Cisinski and Moerdijk have proven the equivalence between the three dendroidal models and simplicial operads in [CM13a] and [CM13b].
- Barwick showed that, for the operator category of finite sets, the complete Segal operads are equivalent to Lurie's ∞-operads.

This already indicates that, as soon as one proves one of the dendroidal models of ∞-operads to be equivalent equivalent to Lurie's or Barwick's model, all available approaches to ∞-operads are equivalent. In [HHM13], Heuts, Hinich and Moerdijk provided a comparison between dendroidal sets and Lurie's model for operads without unit. In a recent joint work [CHH16] with Haugseng and Heuts, we managed to prove the equivalence between Barwick's model and dendroidal Segal spaces. Hence, all results about ∞-operads are independent of the chosen model.

1.3 Enrichment

It is often the case that interesting operads are not enriched over topological spaces but some other symmetric monoidal categories. For instance, the operads for Lie algebras, Poisson algebras and Gerstenhaber algebras are enriched over vector spaces or - more generally - over chain complexes. If we want to define and study the ∞-categorical version of these algebras, we need a theory of enriched ∞-operads in advance. Moreover, recent studies of chiral and factorization algebras such as [FG12] indicate a growing need for a good model for enriched ∞-operads. The aim of this thesis is to provide a suitable foundation for the theory of enriched ∞-operads. For this reason we provide six different models for enriched ∞-operads. Regardless of these cases, we will develop our theory of enriched ∞-operads entirely within the setting of ∞-categories and avoid the usage of strict models such as model categories for the following reasons:

- In this setting the theory developed by Lurie in [Lur09] and [Lur] allows us to work in great generality.
- The homotopy-coherent enrichment is the canonical notion of enrichment.
- In the model categorical setting it could be difficult to define the model structure on the category of enriched ∞-operads. Even if it is possible, a comparison of the different approaches to enriched ∞-operads would probably be hard to obtain in the setting of model categories.

1.4 Overview

In this thesis we are going to define and compare six different models for enriched ∞-operads:

1. After recalling and developing all the notions we need in Chapter 2, we define Φ-∞-operads in Chapter 3. The definition of these objects and the proof of the main theorem 3.24 are inspired by the work [GH15] of Gepner and Haugseng on enriched ∞-categories. The underlying idea is that, for a symmetric monoidal category $\mathcal{V}$, the $\mathcal{V}$-enriched operads can be interpreted as "algebraic objects" in $\mathcal{V}$. More precisely, for every set X, there is an operad OOp_X such that the OOp_X-algebras in $\mathcal{V}$ are exactly the $\mathcal{V}$-enriched operads with the fixed set X of colours. If $\mathcal{F}$ denotes the category of finite sets, then the theory of operator categories provides an ∞-category $\Delta^{\mathrm{op}}_{\mathcal{F},X}$ whose objects are trees together with a labelling of the edges by elements in X. We can regard this category as a model for the operad OOp_X. It is therefore not surprising that a functor from $\Delta^{\mathrm{op}}_{\mathcal{F},X}$ to a symmetric monoidal ∞-category can be thought of as an enriched ∞-operad. Here we should note that in the ∞-version, we only require X to be a space and not a set any more. This makes the homotopy theory easier. More generally, we will define, enriched Φ-∞-operads for every operator category Φ to be functors from the ∞-category $\Delta^{\mathrm{op}}_{\Phi,X}$ to a symmetric monoidal ∞-category. To justify this construction, we show that Φ-∞-operads which are enriched over spaces are equivalent to Segal Φ-operads of Barwick in Theorem 3.24. If Φ is the category

$\mathcal{F}$ of finite sets, then, by using results proven by Barwick, we prove in Corollary 3.32 that enriched $\mathcal{F}$-∞-operads which are complete are equivalent to Lurie's ∞-operads. Hence, our definition of enriched ∞-operads really generalizes the equivalent models of ∞-operads. Since the definition of enriched Φ-∞-operads is a natural generalization of the definition of an enriched ∞-category, we obtain the results of [GH15] by requiring Φ to be the trivial category.

2. In Chapter 4, we introduce our second model of enriched ∞-operads: The Φ-presheaf model. The enriched ∞-operads in this approach are called Segal Φ-presheaves and the associated ∞-category is denoted by $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$. Roughly speaking, the intuitive difference to the previous model is that we do not label the edges of trees with elements in a space, but instead label vertices of the trees with objects in a symmetric monoidal ∞-category. After proving that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is presentable if the symmetric monoidal ∞-category $\mathcal{V}$ is presentable, we use techniques developed in [GH15] and [GHN15] to prove in Theorem 4.22 that the Segal Φ-presheaves are equivalent to the Φ-∞-operads introduced in the first chapter. Apart from an easy description, the main advantage of the presheaf model is that it admits a tensor product
$$\otimes\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \times \mathrm{P}_{\mathrm{Seg}}(\Delta) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}),$$
which is proven in Theorem 5.5. This functor preserves colimits in each variable, so a standard adjunction argument yields a right adjoint functor $\mathrm{Alg}^{\Phi}_{\mathrm{Seg}}(-,-)$ which carries two Segal Φ-presheaves (or Φ-∞-operads) to the ∞-category of their algebras.

3. The dendroidal category Ω allows us to introduce the Ω-presheaf model in Chapter 7. The objects are called Ω-presheaves and they can be regarded as the enriched version of dendroidal Segal spaces. As in the case of Φ-presheaves, the vertices of the trees in Ω are labelled by objects in a symmetric monoidal ∞-category.

4. Since there is no dendroidal counterpart of an arbitrary operator category Φ, we can only provide a comparison between the Ω-presheaf model and the $\mathcal{F}$-presheaf model, which is done in Theorem 7.1. The proof of the theorem is based on the comparison of these two models with an intermediate presheaf model using $\Delta^1_{\mathcal{F}}$. This is our forth approach to enriched ∞-operads. The category $\Delta^1_{\mathcal{F}}$ lies between $\Delta_{\mathcal{F}}$ and Ω: On one hand, it is a full subcategory of $\Delta_{\mathcal{F}}$, hence it shares the same combinatorial description of objects and morphisms. On the other hand, its objects can be identified with objects in Ω. Altough the category Ω has much more morphisms than $\Delta^1_{\mathcal{F}}$ we can use monadic arguments to prove that the natural functor $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ introduced at the end of the second chapter induces an equivalence between the presheaf model of $\Delta^1_{\mathcal{F}}$ and Ω (see Theorem 7.1). The equivalence between the models induced by $\Delta^1_{\mathcal{F}}$ and $\Delta_{\mathcal{F}}$ is provided in Lemma 7.17. After replacing Ω by its planar (non-symmetric) variant, the same proof presented here also implies that there exists an enriched version of the planar dendroidal Segal spaces.

5. In the last chapter of this thesis we come back to the original inspiration of the thesis, namely to think of enriched operads as algebras in a symmetric monoidal category. We first introduce the ∞-operad OOp_X associated to the ordinary operad whose algebras are operads with a fixed set X of objects, then we define enriched ∞-operads with the set of

colours X as OOp_X-algebras in a symmetric monoidal ∞-category. Thus, this approach is an ∞-categorical reinterpretation of the classical definition of enriched operads as algebras. By letting the set X vary, we can then define the ∞-category of all enriched ∞-operads.

6. To compare the previous model with other approaches, we first introduce Ω-∞-operads in Definition 8.27. Then we realize in Proposition 8.29 that Ω-∞-operads are equivalent to Segal Ω-presheaves. Finally, Theorem 8.28 proves the equivalence between the approach using OOp_X and the one using Ω-∞-operads.

To navigate the reader through the thesis we provide a diagram of some the main results of the thesis below. Note that $\mathcal{V}$ is presentable symmetric monoidal for all the presheaf models.

$$
\begin{tikzcd}
\boxed{\mathrm{Alg}_{\mathcal{F}}(\mathcal{V})[\mathrm{FFES}^{-1}],\ \text{Def. 3.9, Def. 6.7, Def. 6.10}} \arrow[r, Leftrightarrow, "\text{Th. 4.22}"'] \arrow[d, Leftrightarrow, "\text{Th. 8.32}"'] & \boxed{\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})[\mathrm{FFES}^{-1}],\ \text{Def. 4.13}} \arrow[d, Leftrightarrow, "\text{Lem. 7.17}"] \\
\boxed{\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V})[\mathrm{FFES}^{-1}],\ \text{Def. 8.27, Def. 8.31}} \arrow[d, Leftrightarrow, "\text{Th. 8.28, Th. 8.30}"'] & \boxed{\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})[\mathrm{FFES}^{-1}],\ \text{Def. 7.5}} \arrow[d, Leftrightarrow, "\text{proof of Th. 7.1}"] \\
\boxed{\mathrm{Alg}_{\Omega}(\mathcal{V})[\mathrm{FFES}^{-1}],\ \text{Def. 8.27, Def. 8.31}} \arrow[r, Leftrightarrow, "\text{Prop. 8.29}"] & \boxed{\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})[\mathrm{FFES}^{-1}],\ \text{Def. 7.6}}
\end{tikzcd}
$$

1.5 Terminology and Notation

1. We assume the existence of three nexted Grothendieck universes, the sets contained in them are called *small*,*large* and *very large*, respectively

2. In general we use the same notation as Lurie in [Lur09] and [Lur].

3. Whenever an object it allows, we regard it as an ∞-category without mentioning the specific implementation as quasicategories. In particular, we do not distinguish notationally between an (simplicial) category and its nerve. However, we do write the nerve functor N in front of a (simplicial) category if we regard it as a simplicial set.

4. For the whole thesis we write $\mathcal{S}$ for the ∞-category of spaces (or ∞-groupoids).

5. If $\mathcal{C}$ is an ∞-category, we write $\mathrm{P}(\mathcal{C})$ for the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves of spaces on $\mathcal{C}$.

6. Δ denotes the usual simplicial indexing category.

7. For every an integer $k \geq 0$, we write $\mathbf{k}$ for the set $\{1, 2, \ldots, k\}$ with k elements.

Chapter 2

Preliminaries

In this chapter we introduce objects and notations which are used in the main part of the thesis.

We first recall the definition of operator categories Φ and their associated categories Δ_Φ, then we introduce corollas in Δ_Φ and study their properties in the next section. At the end of Section 2.2 the category Δ_Φ is used to define the category $\Delta^{\mathrm{op}}_{\Phi,X}$ for an ∞-groupoid $X \in \mathcal{S}$. These categories are important for the definition of enriched ∞-operads provided in the next chapter. The category patterns on the categories $\Delta^{\mathrm{op}}_\Phi$ are then defined in Section 2.3. The theory of categorical patterns allows us to prove the existence of the ∞-categories $\mathrm{coCart}^{\Phi}_{\mathrm{Seg}}$. In Section 2.4 and Section 2.5 we recall the definition of the dendroidal categories Ω using polynomial endofunctors and provide categorical patterns on Ω^{op}. In the last section of this chapter we compare a full subcategory $\Delta^1_{\mathcal{F}}$ of $\Delta_{\mathcal{F}}$ with the dendroidal category Ω. This will be used for the comparison of two different approaches of enriched ∞-operads as shown in Chapter 7.

2.1 Operator Categories

We know that a finite sequence of morphisms in an arbitrary category $\mathcal{C}$ is composable if and only if the target object of each morphism in this sequence coincides with the source object of the next morphism. Therefore, we can think of a composable finite sequence of morphisms in $\mathcal{C}$ as a linear tree (a tree with only one leaf) whose edges are the objects of the morphisms. This is - in a sense - dual to the way that most people would visualize a morphism (i.e. as an "edge" connecting two objects), but it serves us well since our focus in on the morphisms themselves. This also means that we have a slightly more general idea of what a "graph" is meant to be: In this picture two adjacent edges corresponds to the source and target object of one of the morphisms in this sequence, and the edge corresponding to the source object of the "initial" morphism in this sequence is left "dangling", i.e. is just "connected" to a single vertex. In the case of symmetric operads, a multimorphism f can be composed with a collection of multimorphisms f_i if the collection of their target objects coincides with the source objects of f. Thus, each set of composable multimorphisms again admits a tree-like structure.

Suppose the indexing sets of multimorphisms admit the additional structure of a linear ordering. Then $\mathcal{O}$ is called a *non-symmetric* or *planar* operad if the composition is associative and respects the identity morphism for the sets $\mathrm{Mul}_{\mathcal{O}}((x_i)_{i\in I}, x)$, where we have replaced the set of objects $\{x_i\}_{i\in I}$ by an ordered finite sequence $(x_i)_{i\in I}$ associated to the linear ordering of I. In particular, defining multimorphisms in a non-symmetric operad requires finite index set endowed with an additional linear ordering.

This observation indicates that we can generalize the concept of symmetric operads by imposing more structures on the finite indexing sets. To this end, Barwick introduced the concept of operator categories in [Bar13], where he defines a new category Δ_Φ associated to an operator category Φ in order to study compositions of multmorphisms. He also showed that the category $\mathcal{F}$ of finite sets is an operator category and that the objects in $\Delta_{\mathcal{F}}$ encode the tree-like structures of composable multimorphisms of symmetric operads.

To start off, we will first recall the definition of operator categories. As for notation, if an ordinary category Φ has a terminal object $*$, we will write $|x|:=\mathrm{Hom}_\Phi(*,x)$ for any object x in Φ.

Definition 2.1. ([Bar13, 1.2]) An ordinary category Φ is called an *operator category* if it satisfies the following conditions:

1. The category Φ is essentially small.
2. The category Φ has a terminal object, denoted by $*$.
3. For any morphism $x\to y$ in Φ and any element $i\in|y|$, the pullback $*\times_y x=: x_i$ exists in Φ.
4. For any two objects $x,y\in\Phi$, the set $\mathrm{Hom}_\Phi(x,y)$ is finite.

One should think of the objects of an operator category as finite sets equipped with some additional structure.

Example 2.2. *The following categories are operator categories (see [Bar13, 1.4]):*

- *The category $\mathcal{F}$ of finite sets.*
- *The category category* Ord *of ordered finite sets.*
- *The trivial category $*$.*

Definition 2.3. Let Φ be an operator category and let $\Delta(\Phi)\colon \Delta^{\mathrm{op}}\to \mathrm{Cat}$ denote the functor which assigns every $[n]\in\Delta^{\mathrm{op}}$ the category $\mathrm{Fun}([n],\Phi)$, where $[n]$ is regarded as a category with respect to the structure induced by the canonical linear ordering of $[n]$. The category Δ'_Φ denotes the total category of the Grothendieck fibration $\Delta'_\Phi\to\Delta$ associated to $\Delta(\Phi)$.

Notation 2.4. *An object in Δ'_Φ is said to* be of length *m, if it has the form $([m],I)$. We write $\mathfrak{c}$ for the object $([0],J)$ if J carries $[0]$ to the terminal object $*\in\Phi$.*

It follows from the definition that an object in Δ'_Φ is of the form $([n],J)$, where $[n]\in\Delta$ and $J\in\Delta(\Phi)([n])$. Therefore, we can think of the object $([n],J)$ as a sequence of morphisms $J(\{0\})\to J(\{1\})\to\ldots\to J(\{n\})$ in Φ.

The description of the Grothendieck fibration implies that a morphism $(\alpha,\phi)\colon([n],J)\to([m],I)$ in Δ'_Φ is given by a morphism $\alpha\colon[n]\to[m]$ in Δ and a natural transformation $\phi\colon J\to I\circ\alpha$ in $\Delta(\Phi)([n])$. The morphism $(\alpha,\phi)\colon([n],J)\to([m],I)$ is then a Cartesian lift of α at the object $([m],I)$ if and only if the natural transformation $\phi\colon J\to I\circ\alpha$ is a natural equivalence.

Definition 2.5. ([Bar13, 2.1]) A morphism $f\colon x \to y$ in an operator category Φ is called a *fibre inclusion* if there exist morphisms $g\colon y \to z$ and $i \in |z|$ such that the diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d, "g"] \\ * \arrow[r, "i"'] & z. \end{tikzcd}$$

is a pullback square. We call a morphism f an *interval inclusion* and denote it by a hooked arrow $f\colon x \hookrightarrow y$ if it can be written as the composite of finitely many fibre inclusions.

Definition 2.6. ([Bar13, 2.4]) Let Δ_Φ be the subcategory of Δ'_Φ (see Definition 2.3) consisting of the same objects, but containing only the morphisms (α, ϕ) which satisfy the following conditions:

1. For every $k\colon 0 \leq k \leq n$ the morphism $\phi_k\colon J(k) \hookrightarrow I(\alpha(k))$ is an interval inclusion.
2. For $k, l\colon 0 \leq k \leq l \leq n$, the induced square

$$\begin{tikzcd} J(k) \arrow[r, hook, "\phi_k"] \arrow[d] & I(\alpha(k)) \arrow[d] \\ J(l) \arrow[r, hook, "\phi_l"] & I(\alpha(l)) \end{tikzcd} \tag{2.1}$$

   is a pullback square in Φ.

Notation 2.7. *For an object $([m], I) \in \Delta_\Phi$ and $0 \leq i \leq j \leq m$, let $I^{(i,j)}\colon I(i) \to I(j)$ denote the image of the morphism $i \to j$ in $[m]$ under the functor $I\colon [m] \to \Phi$. For an object $x \in |I(j)|$, let $I(i)_x$ denote the induced fibre product $I(i) \times_{I(j)} \{*\}$.*

For us, the most important operator category is the category of finite sets $\mathcal{F}$. To get a feeling for what the objects and morphisms in $\Delta_\mathcal{F}$ look like, we should unwind Definition 2.6:

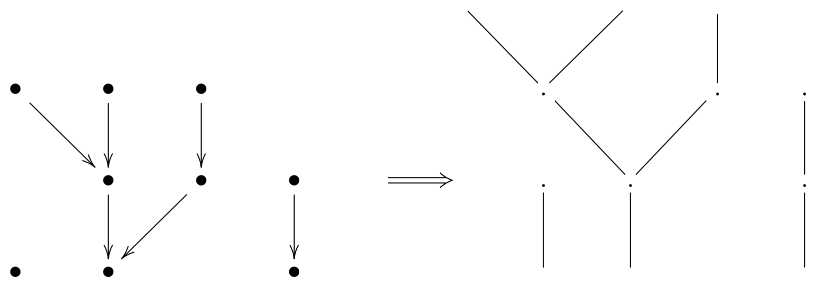
Remark 2.8. *An object $([m], I)$ in $\Delta_\mathcal{F}$ is given by a sequence of morphisms*

$$\mathbf{k_0} \to \mathbf{k_1} \to \ldots \to \mathbf{k_m}$$

in $\mathcal{F}$, where $\mathbf{k_j} = I(j)$, for $0 \leq j \leq m$. Here, we have a typical object in $\Delta_\mathcal{F}$:

$$\begin{tikzcd} \bullet \arrow[dr] & \bullet \arrow[d] & \bullet \arrow[d] & & \mathbf{3} \arrow[d] \\ & \bullet \arrow[d] & \bullet \arrow[dl] & \bullet \arrow[d] & \mathbf{3} \arrow[d] \\ \bullet & \bullet & & \bullet & \mathbf{3} \end{tikzcd}$$

For our considerations, it is more convenient to think of objects in $\Delta_\mathcal{F}$ as graphs obtained by replacing the sets $\mathbf{k}_i \in \mathcal{F}$ in an object $\mathbf{k_0} \to \mathbf{k_1} \to \ldots \to \mathbf{k_m}$ by a finite set of edges "$|$" and each *morphism* $\mathbf{k_i} \to \mathbf{k_{i+1}}$ *by* k_{i+1}*-many vertices such that each of the vertex* $x \in \mathbf{k_{i+1}}$ *has* $(\mathbf{k_i})_x$*-many incoming edges. In picture, this means:*



More precisely, each object $([m], I)$ *in* $\Delta_{\mathcal{F}}$ *should be interpreted as a finite graph, whose sets of vertices and edges are given by the sets* $\coprod_{j=1}^{m} \mathbf{k_j}$ *and* $\coprod_{j=0}^{m} \mathbf{k_j}$*, respectively. Given a vertex* $v \in \mathbf{k_j} \subseteq \coprod_{j=1}^{m} \mathbf{k_j}$*, we say that an edge* $e \in \coprod_{j=0}^{m} \mathbf{k_j}$ *is an* incoming *edge of* v *if and only if* $e \in I(j-1)_j \subseteq \mathbf{k_{j-1}}$ *and* e *is the unique* outgoing *edge of* v *if and only if* e *and* v *correspond to the same element in* $\mathbf{k_j}$*. The object* $([m], I)$ *can be regarded as a symmetric (non-planar) tree if* $\mathbf{k_m} = \mathbf{1}$*. For this reason, we can think of an arbitrary object* $([m], I)$ *in* $\Delta_{\mathcal{F}}$ *as a finite forest containing* $I(m)$*-many trees.*

As already mentioned at the beginning of this section, the objects in $\Delta_{\mathcal{F}}$ *encode the tree-like structure of the composable multimorphisms. In this picture a vertex* v *corresponds to a multimorphism* f*. The incoming edges and the unique outgoing edge of* v *correspond to the source objects and the unique target object of* f*.*

Definition 2.9. [Bar13, 1.10]

1. We call a functor $f\colon \Phi \to \Psi$ between operator categories *admissible* if it preserves terminal objects and fibre inclusions.

2. We write Adm for the strict 2-category whose objects are small operator categories, whose 1-morphisms are admissible functors, and whose 2-morphisms are natural isomorphisms between functors.

3. We call an admissible functor $f\colon \Phi \to \Psi$ an *operator morphism* if, for every object $x \in \Phi$, the induced morphism $|x| \to |f(x)|$ of sets is an isomorphism.

4. We write OpCat for the sub-2-category of Adm consisting of all objects of Adm but containing only 1-morphisms which are operator morphisms as well as all 2-morphisms between these operator morphisms.

Remark 2.10. *The original definition of an operator morphism in [Bar13, 1.10] only requires surjectivity of the morphisms* $|x| \to |f(x)|$, $x \in \Phi$*. It is later shown in [Bar13, Proposition 1.12] that surjectivity suffices for these morphisms to be isomorphisms.*

We would like to construct an ∞-categeory starting from the strict 2-category Adm. To this end, we apply the nerve functor to the groupoids of morphisms in Adm and we obtain a fibrant simplicial category, whose associated ∞-category will again be denoted by Adm. Similarly, by applying the same construction to the strict 2-category OpCat we obtain another ∞-category which will also be denoted by OpCat. It follows from the definition above that the ∞-categories Adm and OpCat are actually $(2,1)$-categories.

Since each admissible functor $f\colon \Phi \to \Psi$ induces a functor $\Delta_f\colon \Delta_\Phi \to \Delta_\Psi$ which takes the object $([m], I) \in \Delta_\Phi$ to $([m], f \circ I) \in \Delta_\Psi$, we can define a functor $\Delta_{(-)}$ as follows:

Definition 2.11. We define $\Delta_{(-)}\colon \mathrm{Adm} \to \mathrm{Cat}_\infty$ to be the functor induced by the strict 2-functor, which carries each operator category Φ to Δ_Φ and each admissible functor f to Δ_f.

Example 2.12. *[Bar13, 1.11]*

- *For any operator category Φ, the unique functor $\Phi \to \{*\}$ is an admissible functor, but in general not an operator morphism.*
- *By [Bar13, Proposition 1.14], the trivial operator category $*$ is terminal in* Adm*.*
- *Every equivalence between operator categories is an operator morphism.*
- *For any operator category Φ, the inclusion $\{*\} \hookrightarrow \Phi$ of the terminal object is an operator morphism.*
- *By [Bar13, Proposition 1.14], the operator category $\mathcal{F}$ is terminal in* OpCat *and the functor $|-|\colon \Phi \to \mathcal{F}$ given by $I \mapsto |I|$ is the unique operator morphism between Φ and $\mathcal{F}$.*

Definition 2.13. Let $\Delta_\pi\colon \Delta_\Phi \to \Delta_* \simeq \Delta$ be the functor induced by the unique map $\pi\colon \Phi \to *$. In more explicit terms, we have $\pi([m], I) = [m]$ and $\pi((\alpha, \phi)) = \alpha$.

1. We say a map $(\alpha, \phi)\colon ([n], J) \to ([m], I)$ in Δ_Φ is
   (a) *injective* if $\alpha\colon [n] \to [m]$ is injective,
   (b) *surjective* if α is surjective and (α, ϕ) is a Cartesian morphism, i.e. the morphisms $\phi_k\colon J(k) \to I(\alpha(k))$ are isomorphism for all k (see Notation 2.4),
   (c) *inert* if α is inert in Δ,
   (d) *active* if α is active in Δ and (α, ϕ) is a Cartesian morphism.
2. We let $\Delta_\Phi^{\mathrm{inj}}$ and $\Delta_\Phi^{\mathrm{in}}$ denote the subcategories of Δ_Φ consisting of all objects of Δ_Φ but containing only injective or inert morphisms in Δ_Φ, respectively.

Remark 2.14. *By the above definition, a morphism in Δ_Φ is inert or injective if and only if its image under the canonical projection is inert or injective in Δ. This implies that for a given admissible functor of operator categories $f\colon \Phi \to \Psi$, the functor Δ_f preserves and reflects inert as well as injective morphisms. The operator morphism $\Delta_{|-|}$ induces equivalences $\Delta_\Phi^{\mathrm{inj}} \simeq \Delta_{\mathcal{F}}^{\mathrm{inj}} \times_{\Delta_{\mathcal{F}}} \Delta_\Phi$ and $\Delta_\Phi^{\mathrm{in}} \simeq \Delta_{\mathcal{F}}^{\mathrm{in}} \times_{\Delta_{\mathcal{F}}} \Delta_\Phi$.*

2.2 Corollas

In this section we introduce and study corollas. They are the simplest non-trivial examples of objects in Δ_Φ. In the next chapter we will realize that the concept of corollas plays an essential role in the theory of (enriched) ∞-operads. They are, roughly speaking, the building blocks out of which each (enriched) ∞-operad is built of.

Definition 2.15. 1. We call an object $([1], J) \in \Delta_\Phi$ a *corolla* if $J(1) = * \in \Phi$. In this case we write $\mathfrak{c}_J$ for $([1], J)$ and $\mathfrak{c}$ if there is no need to emphasize J. An inert morphism $\mathfrak{c}_J \to ([m], I)$ in Δ_Φ will be called a *corolla in* $([m], I)$.

2. We will write $\mathfrak{c}_n$ instead of $\mathfrak{c}_J$ if $\mathfrak{c}_J$ is a corolla in $\Delta_{\mathcal{F}}$ and $\mathbf{n} = J(0)$

3. Given an object $([m], I) \in \Delta_\Phi$, the set of corollas in $([m], I)$ is defined to be $\coprod_{k\in\mathbf{m}} |I(k)|$.

4. A morphism $\mathfrak{e} \to ([m], I)$ is called an *edge in* $([m], I)$ and we write $\Delta_\Phi^{\mathrm{el}}$ for the full subcategory of $\Delta_\Phi^{\mathrm{in}}$ spanned by corollas and the object $\mathfrak{e}$.

Remark 2.16. *Given an object $([m], I) \in \Delta_\Phi$, it follows from the definition above that each element i in the set $\coprod_{k\in\mathbf{m}} |I(k)|$ of corollas in $([m], I)$ corresponds to an isomorphisms class of objects in $(\Delta_\Phi)_{/([m],I)}$ given by inert morphisms $f\colon \mathfrak{c}_J \to ([m], I)$ such that $f(J(1)) = i$.*

For the operator category $\mathcal{F}$, the notation of corollas coincides with the usual definition, i.e. a corolla is a tree with one vertex. Moreover, for an object $([m], I)$ in $\Delta_{\mathcal{F}}$ given by $\mathbf{k_0} \to \mathbf{k_1} \to \ldots \to \mathbf{k_m} = \mathbf{1}$, the set of corollas $\coprod_{j=1}^m \mathbf{k_j}$ coincides with the set of vertices of the tree corresponding to $([m], I)$.

A corolla $\mathfrak{c}_n$ should be interpreted as a multimorphism with n-many source objects. For an arbitrary operator category Φ, a corolla $\mathfrak{c} \in \Delta_\Phi$ can be regarded as a multimorphism whose collection of source objects admits additional structures, e.g. a linear ordering if $\Phi = \mathrm{Ord}$.

Definition 2.17. 1. For a finite set K, let K_+ denote the pointed set $K \coprod \{*\}$ given by adjoining a distinguished base point $*$ to K. For every $n \geq 0$, we write $\langle n\rangle^\circ$ for the set $\{1, \ldots, n\}$ and $\langle n\rangle$ for $\langle n\rangle^\circ_+$, i.e. $\langle n\rangle^\circ_+$ is the set obtained from the set $\langle n\rangle^\circ$ by adjoining a base point $*$.

2. Let $\mathcal{F}_*$ denote the category whose objects are pointed finite sets of the form $\langle n\rangle$ with $n \geq 0$ and whose morphisms are base point preserving maps of finite sets.

3. A morphism $f\colon \langle m\rangle \to \langle n\rangle$ is called *inert* if the preimage $f^{-1}(i)$ of i has exactly one element for every $i \in \langle n\rangle^\circ$.

Remark 2.18. *The category $\mathcal{F}_*$ is equivalent to the category of* all *pointed finite sets. Therefore, we will identify a pointed finite set K_* with the object $\langle n\rangle$, where n is the cardinality of K and refer to $\mathcal{F}_*$ as the* category of pointed finite sets.

Definition 2.19. Let $\mathrm{Cr}_\Phi\colon \Delta_\Phi^{\mathrm{op}} \to \mathcal{F}_*$ denote the assigment given as follows.

1. An object $([m], I) \in \Delta_\Phi$ is mapped to the pointed set $\langle\coprod_{k\in\mathbf{m}} |I(k)|\rangle \in \mathcal{F}_*$.

2. A morphism $(\alpha, \phi)\colon ([n], J) \to ([m], I)$ in Δ_Φ is mapped to $\mathrm{Cr}_\Phi(\alpha, \phi)\colon \langle\coprod_{k\in\mathbf{m}} |I(k)|\rangle \to \langle\coprod_{l\geq 1} |J_l|\rangle$ given by

$$\mathrm{Cr}_\Phi(\alpha, \phi)(x) = \begin{cases} y & \text{if } x \in |I_k|,\ y \in |J_l|,\ \alpha(l-1) < k \leq \alpha(l) \text{ and } I^{(k,\alpha(l))}(x) = \phi_l(y) \\ * & \text{otherwise.} \end{cases}$$

By Remark 2.8, an object $([m], I) \in \Delta_{\mathcal{F}}$ can be regarded as a forest. In this case, $\mathrm{Cr}_\Phi(\alpha, \phi)$ carries the corolla $x \in \langle \coprod_{k \in \mathbf{m}} |I(k)| \rangle$ to the corolla $y \in \langle \coprod_{l \geq 1} |J_l| \rangle$ if and only if y lies in the subtree of $([n], J)$ lying between the image of the leaves and the root of x under the map $(\alpha, \phi)\colon ([n], J) \to ([m], I)$.

It follows directly from the definition that $\mathrm{Cr}_\Phi(\alpha, \phi)$ is base point preserving and it is easy to see that the assignment $\mathrm{Cr}_\Phi\colon \Delta_\Phi^{\mathrm{op}} \to \mathcal{F}_*$ is compatible with compositions and can thus be extended to a functor.

Lemma 2.20. *The assigment* $\mathrm{Cr}_\Phi\colon \Delta_\Phi^{\mathrm{op}} \to \mathcal{F}_*$ *is a functor.*

The following lemma is obvious from the definition.

Lemma 2.21. *If* $f\colon \Phi \to \Psi$ *is a map in* OpCat *and* $\Delta_f^{\mathrm{op}}\colon \Delta_\Phi^{\mathrm{op}} \to \Delta_\Psi^{\mathrm{op}}$ *is as in Definition 2.11, then* $\mathrm{Cr}_\Psi = \mathrm{Cr}_\Phi \circ \Delta_f^{\mathrm{op}}$*. In particular, we have* $\mathrm{Cr}_\Phi = \mathrm{Cr}_{\mathcal{F}} \circ \Delta_{|-|}$*.*

Remark 2.22. *Let* $p\colon \mathrm{Ass}^{\otimes} \to \mathcal{F}_*$ *denote the associative* ∞*-operad defined in [Lur, 4.1.1.3]. For the trivial operator category* $*$*, it is easy to see that* $\Delta_* = \Delta$ *and that the functor* Cr_* *can be identified with* $p \circ \mathrm{Cut}$*, where* $\mathrm{Cut}\colon \Delta^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}$ *is the functor defined in the construction presented in [Lur, 4.1.2.5]. Since* $\mathrm{Cr}_* = \mathrm{Cr}_{\mathcal{F}} \circ \Delta_{|-|}^{\mathrm{op}}$*, we have a commutative diagram*

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, "\mathrm{Cut}"] \arrow[d, "\Delta_{|-|}^{\mathrm{op}}"'] & \mathrm{Ass}^{\otimes} \arrow[d, "p"] \\ \Delta_{\mathcal{F}}^{\mathrm{op}} \arrow[r, "\mathrm{Cr}_{\mathcal{F}}"'] & \mathcal{F}_*. \end{tikzcd}$$

The functor Cut *plays an essential role in the theory of planar* ∞*-operads because it is a weak equivalence in the model category of* ∞*-operads by [Lur, Proposition 4.1.2.10]. In particular, this functor allows us to think of* Δ^{op} *as a simplicial model for the associative* ∞*-operad* $\mathrm{Ass}^{\otimes}$*.*

Lemma 2.23. *Let* Φ *be an operator category. The canonical map* $\Phi \to *$ *induces a Cartesian fibration* $\Delta_\Phi \to \Delta$*.*

Proof. The map $\Delta_\Phi \to \Delta$ induced by the canonical map $\Phi \to *$ is the composite of the inclusion $\Delta_\Phi \hookrightarrow \Delta'_\Phi$ and the Grothendieck fibration $\Delta'_\Phi \to \Delta$. Since every morphism in Δ has a Cartesian lift in Δ_Φ, the composite functor is also a Grothendieck fibration and its nerve is a Cartesian fibration. □

Remark 2.24. *The surjective and injective maps, as well as the active and inert maps, form factorization systems on* Δ *which can be lifted along the Cartesian fibration* $\Delta_\Phi \to \Delta$*, to a surjective-injective factorization system, as well as an active-inert factorization system in* Δ_Φ*.*

Definition 2.25. Let $X \in \mathcal{S}$ and let $i_X\colon \{\mathfrak{c}\} \to \mathcal{S}$ denote the functor which carries the object $\mathfrak{c} \in \Delta_\Phi$ to X. We write $\tilde{i}_X\colon \Delta_\Phi^{\mathrm{op}} \to \mathcal{S}$ for the right Kan extension of i_X along the inclusion $\{\mathfrak{c}\} \hookrightarrow \Delta_\Phi^{\mathrm{op}}$ and we denote the corresponding left fibration by $\pi_X\colon \Delta_{\Phi,X}^{\mathrm{op}} \to \Delta_\Phi^{\mathrm{op}}$. Given an object $([m], I) \in \Delta_\Phi$ and $M = \coprod_{0 \leq k \leq m} |I(k)|$, we write $([m], I, x_1, \ldots, x_M)$ for an object in $\Delta_{\Phi,X}$ lying over $([m], I) \in \Delta_\Phi$. If $([m], I)$ is a corolla $\mathfrak{c}_I$, then we write $\mathfrak{c}_I(x_1, \ldots, x_n; x)$ instead of $([m], I, x_1, \ldots, x_M)$. We write $\mathfrak{c}(x_1, \ldots, x_n; x)$, if there is no need to emphasize I.

Remark 2.26. *If we briefly return to the ideas of 2.8 and interpret the objects in Δ_Φ as a forest, then we can think of an object in $\Delta^{\mathrm{op}}_{\Phi,X}$ as a forest whose edges are labelled by objects in the ∞-groupoid X. It is a consequence of this interpretation, together with the definition above, that the fibre $\Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta_\Phi} \{([m], I)\}$ is equivalent a product of size $\coprod_{0\leq k\leq m} |I(k)|$ of the ∞-groupoid X.*

2.3 Categorical Patterns on $\Delta^{\mathrm{op}}_\Phi$

At the end of the last section we introduced a left fibration $\pi_X \colon \Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_\Phi$ for every ∞-groupoid X, which can be regarded as an object in the slice ∞-category $(\mathrm{Cat}_\infty)_{/\Delta^{\mathrm{op}}_\Phi}$. In the next chapter we will define an enriched Φ-∞-operad with respect to a fixed space X of objects to be a morphism lying in a subcategory of the slice ∞-category $(\mathrm{Cat}_\infty)_{/\Delta^{\mathrm{op}}_\Phi}$ whose source object coincoides with $\Delta^{\mathrm{op}}_{\Phi,X}$. Before we can define Φ-∞-operads in the next chapter, we first have to prove the existence of this subcategory of $(\mathrm{Cat}_\infty)_{/\Delta^{\mathrm{op}}_\Phi}$. For this reason, we will use this section to provide an introduction to the theory of categorical patterns as developed in [Lur, Appendix B]. One of the main uses of categorial patterns is to generate nice model structures describing ∞-categories of the above form. More precisely, they describe model structures on slice categories of the form $\mathrm{sSet}^+_{/K}$, where sSet^+ denotes the category of marked simplicial sets and $K \in \mathrm{sSet}^+$. Here, a marked simplicial set is a simplicial set with a distinguished collection of 1-simplices (see [Lur09, Definition 3.1.0.1] for more details). This additional property provides a tool to study subcategories of slice categories whose morphisms preserve a certain property of 1-simplices. As an example, the ∞-category of ∞-operads is a subcategory of $(\mathrm{Cat}_\infty)_{/\mathcal{F}_*}$ whose morphisms preserve inert maps.

We will start this section by defining categorical patterns on $\Delta^{\mathrm{op}}_\Phi$ and compare the ∞-categories induced by these patterns with the ∞-category of ∞-operads.

Definition 2.27. A categorical pattern is given by a tuple $(\mathcal{C}, I, \mathrm{Dia})$, where

- $\mathcal{C}$ is an ∞-category.
- I is a collection of 1-simplices in $\mathcal{C}$ containing all degenerate morphisms. We call the elements of I *marked morphisms* and we write $\mathcal{C}^\natural$ for the marked simplicial set $(\mathcal{C}, I)$.
- Dia is a collection of functors of the form $q \colon K^\triangleleft \to \mathcal{C}$ between ∞-categories, which map every morphism in $K^\triangleleft$ to a marked one.

Remark 2.28. *In [Lur, Definition B.0.19] the data of a categorical pattern on $\mathcal{C}$ also contains a collection T of 2-simplices of $\mathcal{C}$ containing all degenerate ones. The definition presented above is a special case of the original definition: Given a categorical pattern as defined above, we can regarded it as a categorical pattern in Lurie's sense by choosing T to be the set all 2-simplices of $\mathcal{C}$. Of course, Lurie's definition permits other choices of T, but we will not require this additional degree of generality.*

Our simplified version of categorical patterns allows us to provide an easier description of the fibrant objects in the model category induced by a pattern. The following theorem is due to Lurie ([Lur, Theorem B.0.20] and [Lur, Definition B.0.19]).

Theorem 2.29. *Let $(\mathcal{C}, I, \mathrm{Dia})$ be a categorical pattern on an ∞-category $\mathcal{C}$. There is a left proper combinatorial simplicial model structure on the category $(\mathrm{sSet}^+)_{/\mathcal{C}^\natural}$ such that:*

- *A morphism is a cofibration if and only if it is a monomorphism between the underlying simplicial sets. In particular, every object is cofibrant.*
- *The object $p\colon (\mathcal{D}, J) \to \mathcal{C}^\natural$ in $(\mathrm{sSet}^+)_{/\mathcal{C}^\natural}$ is fibrant if an only if the following conditions are satisfied:*
  1. *The underlying map of simplicial sets $p\colon \mathcal{D} \to \mathcal{C}$ is an inner fibration.*
  2. *If $\alpha\colon C \to C'$ is a map in $\mathcal{C}$ and $D \in \mathcal{D}$ lies over C then there exists a p-coCartesian morphism $\tilde{\alpha}\colon C \to C'$ lifting α.*
  3. *If $q\colon K^\triangleleft \to \mathcal{C}$ is an element in Dia and $q'\colon \mathcal{D} \times_\mathcal{C} K^\triangleleft \to K^\triangleleft$ denotes the coCartesian fibration induced by pulling back p along q, then q' corresponds to a limit diagram $K^\triangleleft \to \mathrm{Cat}_\infty$.*
  4. *Given an element $q\colon K^\triangleleft \to \mathcal{C}$ in Dia and an arbitrary coCartesian section $K^\triangleleft \to \mathcal{D} \times_\mathcal{C} K^\triangleleft$ of q', the composite of s and the projection $\mathcal{D} \times_\mathcal{C} K^\triangleleft \to \mathcal{D}$ is a p-limit diagram.*

Remark 2.30. *From now on we will abuse notation and write $p\colon \mathcal{D} \to \mathcal{C}^\natural$ for an object $p\colon (\mathcal{D}, J) \to \mathcal{C}^\natural$ in $(\mathrm{sSet}^+)_{/\mathcal{C}^\natural}$, i.e. we will not explicitly mention the marking of $\mathcal{D}$.*

Example 2.31. *For $\langle m\rangle \in \mathcal{F}_*$, let $\langle m\rangle^{\mathrm{Cr}}$ be the nerve of the full subcategory of $(\mathcal{F}_*)_{\langle m\rangle/}$ spanned by inert morphisms in $\mathcal{F}_*$ (see Definition 2.17) of the form $\langle m\rangle \to \langle 1\rangle$ and suppose $\mathrm{Dia}^0_{\mathcal{F}_*}$ denotes the set $\coprod_{\langle m\rangle\in\mathcal{F}_*} q_{\langle m\rangle}\colon \langle m\rangle^{\mathrm{Cr},\triangleleft} \to \mathcal{F}_*$ of functors, where each functor $q_{\langle m\rangle}$ is given by the canonical inclusion map which carries the cone point to $\langle m\rangle$. Then the usual categorical pattern on $\mathcal{F}_*$ which induces the ∞-operadic model structure on $\mathrm{sSet}^+_{/\mathcal{F}_*}$ is given by the tuple*

$$\mathcal{F}_*^{\natural,0} \colon = (\mathcal{F}_*, I_{\mathcal{F}_*}, \mathrm{Dia}^0_{\mathcal{F}_*}),$$

where $I_{\mathcal{F}_*}$ denotes the set of inert morphisms. In thise case, the model category $\mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}}$ conicides with the ∞-operadic model structure on $\mathrm{sSet}^+_{/\mathcal{F}_*}$ as defined in [Lur, Definition 2.1.4.2]. The fibrant objects in $\mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}}$ are exactly the ∞-operads in sense of Lurie (see [Lur, Definition 2.1.1.10]). Therefore, we write Op_∞ for the ∞-category associated to $\mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}}$.

For $\langle m\rangle \in \mathcal{F}_*$, let $\langle m\rangle^{\mathrm{el}}$ be the nerve of the full subcategory of $(\mathcal{F}_*)_{\langle m\rangle/}$ spanned by inert morphisms in $\mathcal{F}_*$ (see Definition 2.17) of the form $\langle m\rangle \to \langle i\rangle$, $i \in \{0,1\}$, and suppose $\mathrm{Dia}_{\mathcal{F}_*}$ denotes the set $\coprod_{\langle m\rangle\in\mathcal{F}_*} q_{\langle m\rangle}\colon \langle m\rangle^{\mathrm{el},\triangleleft} \to \mathcal{F}_*$ of functors, where each functor $q_{\langle m\rangle}$ is given by the canonical inclusion map which carries the cone point to $\langle m\rangle$. By definition, the generalized ∞-operadic model structure on $\mathrm{sSet}^+_{/\mathcal{F}_*}$ as defined in [Lur, Remark 2.3.2.4] coincides with the one induced by the categorical pattern $\mathcal{F}_*^\natural$ given by the tuple

$$\mathcal{F}_*^\natural = (\mathcal{F}_*, I_{\mathcal{F}_*}, \mathrm{Dia}_{\mathcal{F}_*}).$$

The fibrant objects in $\mathrm{sSet}^+_{/\mathcal{F}_*^\natural}$ are exactly the generalized ∞-operads as defined in [Lur, Definition 2.3.2.1]. Therefore, we write $\mathrm{Op}_\infty^{\mathrm{gen}}$ for the ∞-category associated to $\mathrm{sSet}^+_{/\mathcal{F}_*^\natural}$. It follows from [Lur, Corollary 2.3.2.6] that the identity functor on $\mathrm{sSet}^+_{/\mathcal{F}_*}$ induces an adjunction

$$\mathrm{Op}_\infty^{\mathrm{gen}} \rightleftarrows \mathrm{Op}_\infty,$$

which exhibits ∞-category of ∞-operads as a localization of that of generalized ∞-operads.

We will need to define two categorical patterns on the category Δ_Φ which differ in the set of diagrams Dia. First, however, we're going to introduce some simplifications to our notation to avoid at least some of the notational overhead involved in the process.

Notation 2.32. *Let $([m], I)$ be an object in Δ_Φ.*

- *We let $(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$ denote the category $\Delta_\Phi^{\mathrm{el,op}} \times_{\Delta_\Phi^{\mathrm{in,op}}} (\Delta_\Phi^{\mathrm{in,op}})_{([m],I)/}$ and write $(\Delta_\Phi^{\mathrm{el,op}})^{\triangleleft}_{([m],I)/}$ for the category $\Delta^0 * (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$.*
- *We will denote the full subcategory of $\Delta_\Phi^{\mathrm{el,op}}$ spanned by the corollas by $\Delta_\Phi^{\mathrm{Cr,op}}$ and let $\Delta_\Phi^{\mathrm{Cr,op}}{}_{/([m],I)}$ denote the pullback category $\Delta_\Phi^{\mathrm{Cr,op}} \times_{\Delta_\Phi^{\mathrm{in,op}}} (\Delta_\Phi^{\mathrm{in,op}})_{([m],I)/}$. Furthermore, we shall write $(\Delta_\Phi^{\mathrm{Cr,op}})^{\triangleleft}_{([m],I)/}$ for the category $\Delta^0 * (\Delta_\Phi^{\mathrm{Cr,op}})_{([m],I)/}$.*

Remark 2.33. *It follows that an object in $(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$ is given by an inert morphism in Δ_Φ which has either the form $\mathfrak{e} \to ([m], I)$ or $\mathfrak{c}_J \to ([m], I)$. Definition 2.15 clearly implies that there is an equivalence of categories*

$$(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/} \simeq (\Delta_{\mathcal{F}}^{\mathrm{el,op}})_{([m],|I|)/}$$

and that the category $\Delta_\Phi^{\mathrm{Cr,op}}{}_{/([m],I)}$ can be identified with the set $\coprod_{k\in\mathbf{m}} |I(k)|$.

Definition 2.34. Let I_Φ denote the set of inert morphisms in the category $\Delta_\Phi^{\mathrm{op}}$.

- Let $p_{([m],I)} \colon (\Delta_\Phi^{\mathrm{el,op}})^{\triangleleft}_{([m],I)/} \to \Delta_\Phi^{\mathrm{op}}$ denote the canonical inclusion map which carries the cone point to $([m], I)$. We write Dia_Φ for the collection of functors

$$\coprod_{([m],I)\in\Delta_\Phi^{\mathrm{op}}} \{p_{([m],I)} \colon (\Delta_\Phi^{\mathrm{el,op}})^{\triangleleft}_{([m],I)/} \to \Delta_\Phi^{\mathrm{op}}\}.$$

  Let $\Delta_\Phi^{\mathrm{op},\natural}$ be the categorical pattern (see Definition 2.27) on the on the category $\Delta_\Phi^{\mathrm{op}}$ given by the tuple

$$\Delta_\Phi^{\mathrm{op},\natural} = (\Delta_\Phi^{\mathrm{op}}, I_\Phi, \mathrm{Dia}_\Phi).$$

- Let $p^0_{([m],I)} \colon (\Delta_\Phi^{\mathrm{Cr,op}})^{\triangleleft}_{([m],I)/} \to \Delta_\Phi^{\mathrm{op}}$ denote the canonical inclusion map which carries the cone point to $([m], I)$. We write Dia^0_Φ for the collection of functors

$$\coprod_{([m],I)\in\Delta_\Phi^{\mathrm{op}}} \{p^0_{([m],I)} \colon (\Delta_\Phi^{\mathrm{Cr,op}})^{\triangleleft}_{([m],I)/} \to \Delta_\Phi^{\mathrm{op}}\}$$

  and write $\Delta_\Phi^{\mathrm{op},\natural,0}$ for the categorical pattern given by the tuple

$$\Delta_\Phi^{\mathrm{op},\natural,0} = (\Delta_\Phi^{\mathrm{op}}, I_\Phi, \mathrm{Dia}^0_\Phi).$$

Definition 2.35. By Theorem 2.29, the categorical patterns $\Delta_\Phi^{\mathrm{op},\natural,0}$ and $\Delta_\Phi^{\mathrm{op},\natural}$ induce model categories $\mathrm{sSet}^+_{/\Delta_\Phi^{\mathrm{op},\natural,0}}$ and $\mathrm{sSet}^+_{/\Delta_\Phi^{\mathrm{op},\natural}}$, respectively.

- We write $\mathrm{coCart}^\Phi_{\mathrm{Seg}}$ for the ∞-category associated to $\mathrm{sSet}^+_{/\Delta_\Phi^{\mathrm{op},\natural,0}}$ and we call its objects *coCartesian Segal fibrations on $\Delta_\Phi^{\mathrm{op}}$*.

- Similarly, we write $\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}}$ for the ∞-category associated to $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural}_{\Phi}}$ and we call its objects *generalized coCartesian Segal fibrations on* $\Delta^{\mathrm{op}}_{\Phi}$.

By Theorem 2.29 the model structures on $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural,0}_{\Phi}}$ and $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural}_{\Phi}}$ are simplicial and every object in these model categories is cofibrant. Therefore, the ∞-categories $\mathrm{coCart}^{\Phi}_{\mathrm{Seg}}$ and $\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}}$ can be identified with the respective full simplicial subcategories of $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural,0}_{\Phi}}$ and $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural}_{\Phi}}$ spanned by fibrant objects. In the remark below we want to apply the description of fibrant object provided by Theorem 2.29 to these model categories.

Remark 2.36. *By unwinding the description of fibrant objects in Theorem 2.29 and using the obvious equivalence* $(\Delta^{\mathrm{el,op}}_{\Phi})_{([m],I)/} \simeq (\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],|I|)/}$, *we see that an object* $p\colon \mathcal{O} \to \Delta^{\mathrm{op}}_{\Phi}$ *in* $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural}_{\Phi}}$ *is fibrant if and only if it satisfies the following conditions:*

1. *The morphism* $p\colon \mathcal{O} \to \Delta^{\mathrm{op}}_{\Phi}$ *is an inner fibration.*
2. *For each inert map* $\alpha\colon ([m],I) \to ([n],J)$ *in* $\Delta^{\mathrm{op}}_{\Phi}$ *and every object* $x \in \mathcal{O}$ *lying over* $([m],I)$, *there exists a* p*-coCartesian morphism* $\tilde{\alpha}\colon x \to \alpha_! x$ *lifting* α.
3. *For every object* $([m],I) \in \Delta^{\mathrm{op}}_{\Phi}$, *the canonical map*
$$\mathcal{O}_{([m],I)} \to \lim_{\alpha\in(\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],|I|)/}} \mathcal{O}_{\alpha([m],I)}$$
is an equivalence of ∞*-categories.*
4. *For every object* $X \in \mathcal{O}_{([m],I)}$ *is a* p*-limit of the diagram* $q\colon (\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],|I|)/} \to \mathcal{O}$ *which takes an object* $\alpha \in (\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],|I|)/}$ *to the object* $\alpha_! X$ *and each morphism to an inert one.*

Using the equivalence ${\Delta^{\mathrm{Cr,op}}_{\mathcal{F}}}_{/([m],I)} \simeq \coprod_{k\in\mathbf{m}} |I(k)|$, *Definition 2.27 implies that an object* $p\colon \mathcal{O} \to \Delta^{\mathrm{op}}_{\Phi}$ *in the model category* $\mathrm{sSet}^{+}_{/\Delta^{\mathrm{op},\natural,0}_{\Phi}}$ *is fibrant if and only if it satisfies the following conditions:*

1'. *The morphism* $p\colon \mathcal{O} \to \Delta^{\mathrm{op}}_{\Phi}$ *is an inner fibration.*

2'. *For each inert map* $\alpha\colon ([m],I) \to ([n],J)$ *in* $\Delta^{\mathrm{op}}_{\Phi}$ *and every object* $x \in \mathcal{O}$ *lying over* $([m],I)$, *there exists a* p*-coCartesian morphism* $\tilde{\alpha}\colon x \to \alpha_! x$ *lifting* α.

3'. *For every object* $([m],I) \in \Delta^{\mathrm{op}}_{\Phi}$, *the canonical map*
$$\mathcal{O}_{([m],I)} \to \prod_{\mathfrak{c}_J \in \coprod_k |I_k|} \mathcal{O}_{\mathfrak{c}_J}$$
is an equivalence.

4'. *For every object* $X \in \mathcal{O}_{([m],I)}$ *is a* p*-product of the objects* $\mathfrak{c}_{J,!}X$, *for* $\mathfrak{c}_J \in \coprod_k |I_k|$.

In the following we would like to provide a more concrete reformulation of the conditions 4 and 4' from above. First, let us recall the definition of relative limits in [Lur09, Definition 4.3.1.1]: Let $p\colon \mathcal{O} \to \mathcal{P}$ be an inner fibration, let $\overline{f}\colon K^{\triangleleft} \to \mathcal{O}$ be a diagram and let f denote its restriction $\overline{f}_{|K}$. Then $\overline{f}$ is a p-limit diagram if and only if the natural map
$$\varphi\colon \mathcal{O}_{\overline{f}} \to \mathcal{O}_{f} \times_{\mathrm{P}_{pf}} \mathrm{P}_{p\overline{f}}$$

is a trivial fibration. Since the map φ is a right fibration by [Lur09, Proposition 2.1.2.1], it is a trivial fibration if and only if it is a equivalence. Moreover, since the maps $\mathcal{O}_{\overline{f}} \to\to \mathcal{O}$ and $\mathcal{O}_f \times_{\mathrm{P}_{pf}} \mathrm{P}_{p\overline{f}} \to \mathcal{O}$ given by projects are also right fibrations, the map φ is an equivalence if and only if its fibre over any object $Z \in \mathcal{O}$ is an equivalence. Using the fact that $K^{\triangleleft}$ has an initial object given by the cone point $-\infty$, there are equivalences $\mathcal{O}_{\overline{f}} \simeq \mathcal{O}_X$ and $\mathrm{P}_{p\overline{f}} \simeq \mathrm{P}_{pX}$, where $X = \overline{f}(-\infty)$. Therefore, the fibre of φ over X is given by

$$\mathrm{Map}_{\mathcal{O}}(Z, X) \to \lim_{k \in K} \mathrm{Map}_{\mathcal{O}}(Z, f(k)) \times_{\lim_{k\in K} \mathrm{Map}_{\mathcal{O}}(pZ, pf(k))} \mathrm{Map}_{\mathcal{P}}(p(Z), p(X)).$$

We see that the condition for φ to be an equivalence is equivalent to requiring the map above to be an equivalence, in other words, the following commutative diagram is a pullback:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{O}}(Z, X) \arrow[r] \arrow[d] & \lim_{k\in K} \mathrm{Map}_{\mathcal{O}}(Z, f(k)) \arrow[d] \\
\mathrm{Map}_{\mathcal{P}}(p(Z), p(X)) \arrow[r] & \lim_{k\in K} \mathrm{Map}_{\mathcal{O}}(pZ, pf(k)).
\end{tikzcd}$$

This square is Cartesian if and only if the fibres of the vertical maps are equivalent, i.e., for every $\beta \in \mathrm{Map}_{\mathcal{P}}(p(Z), p(X))$, the induced of map of fibres

$$\mathrm{Map}^{\beta}_{\mathcal{O}}(Z, X) \to \lim_{k\in K} \mathrm{Map}^{p\overline{f}(-\infty \to k)\circ\beta}_{\mathcal{O}}(Z, f(k))$$

is an equivalence. If $\overline{q} \colon (\Delta^{\mathrm{el,op}}_{\Phi})^{\triangleleft}_{([m],I)/} \to \mathcal{O}$ denotes the extension of q which carries $-\infty$ to X, then by applying this observation to our situation, we obtain that condition 4 above is equivalent to requiring the map

$$\mathrm{Map}^{\beta}_{\mathcal{O}}(Z, X) \to \lim_{\alpha \in (\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],|I|)/}} \mathrm{Map}^{\alpha\circ\beta}_{\mathcal{O}}(Z, \alpha_! X)$$

to be an equivalence for every map $\beta \colon p(Z) \to ([m], I)$ in $\Delta^{\mathrm{op}}_{\Phi}$. Similarly, condition 4' is equivalent to requiring the map

$$\mathrm{Map}^{\beta}_{\mathcal{O}}(Z, X) \to \prod_{\mathfrak{c}_J \in \coprod_k |I_k|} \mathrm{Map}^{\mathfrak{c}_J\circ\beta}_{\mathcal{O}}(Z, \mathfrak{c}_{J,!} X)$$

to be an equivalence for every map $\beta \colon p(Z) \to ([m], I)$ in $\Delta^{\mathrm{op}}_{\Phi}$.

Remark 2.37. *The observation above and the proof of [Lur, Proposition 2.1.2.12] then imply that, for every $X \in \mathcal{S}$, the left fibration $\Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_{\Phi}$ is an object in* $\mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural}_{\Phi}}$.

Lemma 2.38. *The inclusion functor* $\mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural,0}_{\mathcal{F}}} \hookrightarrow \mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural}_{\mathcal{F}}}$ *induces a Quillen adjunction*

$$\mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural}_{\mathcal{F}}} \rightleftarrows \mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural,0}_{\mathcal{F}}}.$$

In particular, the ∞-category $\mathrm{coCart}^{\mathcal{F}}_{\mathrm{Seg}}$ *is a localization of* $\mathrm{coCart}^{\mathcal{F},\mathrm{gen}}_{\mathrm{Seg}}$.

Proof. Since cofibrations in both model categories $\mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural}_{\mathcal{F}}}$ and $\mathrm{sSet}^+_{/\mathcal{F}^{\natural,0}_*}$ are monomorphisms, we only need to verify that weak equivalences in $\mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural}_{\mathcal{F}}}$ are also weak equivalences in $\mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural,0}_{\mathcal{F}}}$. By [Lur, Remark B.2.6] and [Lur, Definition B.2.1], a morphism $f \colon q_1 \to q_2$ in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}}$ is a weak equivalence in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural}}$, if and only if the induced map $f^*\colon \mathrm{Map}^\sharp_{\Delta_{\mathcal{F}}^{\mathrm{op}}}(q_2,p) \to \mathrm{Map}^\sharp_{\Delta_{\mathcal{F}}^{\mathrm{op}}}(q_1,p)$ is a homotopy equivalence of Kan complexes for every fibrant object p in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural}}$, and similarly, f is a weak equivalence in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural,0}}$ if and only if the induced map f^* is a homotopy equivalence for every fibrant object $p \in \mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural,0}}$. Therefore, the claim follows, if the inclusion functor $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural,0}} \hookrightarrow \mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural}}$ preserves fibrant objects.

Let $p\colon \mathcal{O} \to \Delta_{\mathcal{F}}^{\mathrm{op},\natural,0}$ be a fibrant object in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural,0}}$. We want to show that it is fibrant as an object in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural}}$. By the remark above, we have to verify that p satisfies four conditions. The first two conditions are obviously satisfied. For the third condition, we want to show that the canonical map

$$\mathcal{O}_{([m],I)} \to \lim_{\alpha\in(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}} \mathcal{O}_{\alpha([m],I)}$$

is an equivalence of ∞-categories. Since p is fibrant in $\mathrm{sSet}^+_{/\Delta_{\mathcal{F}}^{\mathrm{op},\natural,0}}$, we have an equivalence $\mathcal{O}_{([m],I)} \simeq \prod_{\mathfrak{c}_J\in\coprod_k |I_k|} \mathcal{O}_{\mathfrak{c}_J}$ which implies that the fibre $\mathcal{O}_{\mathfrak{e}}$ of p over $\mathfrak{e}$ is weakly contractible. This means that we can identify the ∞-category $\lim_{\alpha\in(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}} \mathcal{O}_{\alpha([m],I)}$ with the product $\prod_{\mathfrak{c}_J\in\coprod_k |I_k|} \mathcal{O}_{\mathfrak{c}_J}$ of ∞-categories. For the last condition we have to show that, for objects $([m],I) \in \Delta_{\mathcal{F}}^{\mathrm{op}}$ and $X \in \mathcal{O}_{([m],I)}$, a p-product of $\mathfrak{c}_{J,!}X$ can be extended to a p-limit of the diagram $q\colon (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/} \to \mathcal{O}$ by carrying a morphism $\mathfrak{c}_J \to \mathfrak{e} \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$ to the morphism $\mathfrak{c}_{J,!}X \to *$, where $*$ denotes the unique object in $\mathcal{O}_{\mathfrak{e}} \simeq \{*\}$. But this is always possible because $*$ is p-terminal. □

Remark 2.39. *The full subcategory of the ∞-category $\mathrm{coCart}_{\mathrm{Seg}}^{\mathcal{F},\mathrm{gen}}$ spanned by left fibrations is equivalent to the ∞-category of symmetric Segal operads in the sense of [Bar13, Definition 2.6]. Similarly, one can identify the ∞-category of Segal Φ-operads with the full subcategory of the ∞-category $\mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$ spanned by left fibrations.*

Lemma 2.40. *The map $\Delta_f^{\mathrm{op}}\colon \Delta_\Phi^{\mathrm{op}} \to \Delta_\Psi^{\mathrm{op}}$ induced by an operator morphism $f\colon \Phi \to \Psi$ in* OpCat *is compatible with the categorical patterns $\Delta_\Phi^{\mathrm{op},\natural}$ and $\Delta_\Psi^{\mathrm{op},\natural}$. The same is true for categorical patterns $\Delta_\Phi^{\mathrm{op},\natural,0}$ and $\Delta_\Psi^{\mathrm{op},\natural,0}$.*

Proof. We only prove the first statement, the second can be shown similarly. Since $\mathcal{F}$ is terminal in OpCat, there is commutative diagram

$$\begin{tikzcd}
\Delta_\Phi^{\mathrm{op}} \arrow[rr, "\Delta_f^{\mathrm{op}}"] \arrow[dr] & & \Delta_\Psi^{\mathrm{op}} \arrow[dl] \\
& \Delta_{\mathcal{F}}^{\mathrm{op}}, &
\end{tikzcd} \tag{2.2}$$

where the diagonal maps are induced by the unique maps $|-|\colon \Phi \to \mathcal{F}$ and $|-|\colon \Psi \to \mathcal{F}$. Thus, the map Δ_f^{op} preserves inert morphisms because the functor $\Delta_{|-|}^{\mathrm{op}}$ preserves and detects inert morphisms. The equivalences $(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/} \simeq (\Delta_{\mathcal{F}}^{\mathrm{el,op}})_{([m],|I|)/} \simeq (\Delta_\Psi^{\mathrm{el,op}})_{([m],I)/}$ imply that the functor Δ_f^{op} also carries elements in Dia_Φ to elements in Dia_Ψ. Hence, the functor Δ_f^{op} is compatible with the categorical patterns. □

Lemma 2.41. *The functor* $\mathrm{Cr}_{\mathcal{F}}\colon \Delta^{\mathrm{op}}_{\mathcal{F}} \to \mathcal{F}_*$ *given by Lemma 2.20 is compatible with the categorical patterns* $\Delta^{\mathrm{op},\natural,0}_{\mathcal{F}}$ *and* $\mathcal{F}^{\natural,0}_*$.

Proof. The categorical patterns $\Delta^{\mathrm{op},\natural,0}_{\mathcal{F}}$ and $\mathcal{F}^{\natural,0}_*$ are given by the respective tuples

$$(\Delta^{\mathrm{op}}_{\mathcal{F}}, I_{\mathcal{F}}, \mathrm{Dia}^0_{\mathcal{F}}) \text{ and } (\mathcal{F}_*, I_{\mathcal{F}_*}, \mathrm{Dia}^0_{\mathcal{F}_*}).$$

We have to verify that the functor $\mathrm{Cr}_{\mathcal{F}}$ preserves inert morphisms and the composition with $\mathrm{Cr}_{\mathcal{F}}$ carries objects in $\mathrm{Dia}^0_{\mathcal{F}}$ to objects in $\mathrm{Dia}^0_{\mathcal{F}_*}$.

By definition, a morphism f in $\Delta^{\mathrm{op}}_{\mathcal{F}}$ is inert if and only if it corresponds to a morphism $(\alpha,\phi)\colon ([m],I) \to ([n],J) \in \Delta_{\mathcal{F}}$ such that α is inert. By the pullback condition in Definition 2.6 and by the description of $\mathrm{Cr}_{\mathcal{F}}$, we have that the functor $\mathrm{Cr}_{\mathcal{F}}(f)\colon \langle \sum_{j=1}^{n} |J(j)|\rangle \to \langle \sum_{j=1}^{m} |I(j)|\rangle$ is given by $J(\alpha(k)) \ni x \mapsto y \in I(k)$, if $x = \phi_k(y)$, and $x \mapsto *$ otherwise. Hence, the functor $\mathrm{Cr}_{\mathcal{F}}$ preserves inert morphisms. If $\langle M\rangle = \mathrm{Cr}_{\mathcal{F}}([m],I)$, then both categories $(\Delta^{\mathrm{Cr},\mathrm{op}}_{\mathcal{F}})_{/([m],I)}$ and $\langle M\rangle^{\mathrm{Cr}}$ (see Example 2.31) can be identified with the set $\coprod_{k\in\mathbf{m}} |I(k)|$ and the composition with $\mathrm{Cr}_{\mathcal{F}}$ carries objects in $\mathrm{Dia}^0_{\mathcal{F}}$ to objects in $\mathrm{Dia}^0_{\mathcal{F}_*}$. □

Corollary 2.42. *For every operator category* Φ*, the functor* $\mathrm{Cr}_{\Phi}\colon \Delta^{\mathrm{op}}_{\Phi} \to \mathcal{F}_*$ *as introduced in Definition 2.15 is compatible with the categorical patterns* $\Delta^{\mathrm{op},\natural,0}_{\Phi}$ *and* $\mathcal{F}^{\natural,0}_*$.

Proof. Since the functor Cr_{Φ} is a composition of $\Delta^{\mathrm{op}}_{|-|}$ and $\mathrm{Cr}_{\mathcal{F}}$ and a composition of functors which are compatible with patterns is again compatible with patterns, the claim follows from Lemma 2.41 and Lemma 2.40. □

Remark 2.43. *The previous lemma together with [Lur, Proposition B.2.9] proves the existence of a left Quillen functor* $\mathrm{Cr}_{\Phi,!}\colon \mathrm{sSet}^+_{/\Delta^{\mathrm{op},\natural,0}_{\Phi}} \to \mathrm{sSet}^+_{/\mathcal{F}^{\natural,0}_*}$. *In particular, we obtain an adjunction of* ∞*-categories*

$$\mathrm{Cr}_{\Phi,!}\colon \mathrm{coCart}^{\Phi}_{\mathrm{Seg}} \rightleftarrows \mathrm{Op}_\infty : \mathrm{Cr}^*_{\mathcal{F}}$$

which can be composed with the localization adjunction

$$\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}} \rightleftarrows \mathrm{coCart}^{\Phi}_{\mathrm{Seg}}$$

of Lemma 2.38. By abusing the notation, we will write $(\mathrm{Cr}_{\mathcal{F},!}, \mathrm{Cr}^*_{\mathcal{F}})$ *for the composed adjunction.*

Remark 2.44. *If we write* $*^\natural$ *for the categorical pattern given by the tuple* $(\Delta^0, I, T, \mathrm{Dia})$*, where* I *is the set of all edges,* T *is the set of all 2-simplices of* Δ^0 *and* Dia *is given by a collection of diagrams* $\{p_\alpha\colon K^{\triangleleft}_\alpha \to \Delta^0, \alpha \in A\}$*, where each simplicial set* K_α *is weakly contractible and each map* p_α *is constant. By [Lur, Remark B.0.28], the model structure on* $(\mathrm{sSet}^+_{/*^\natural})$ *coincides with the coCartesian model structure. Furthermore, [Lur09, Proposition 3.3.1.8] implies that a map of simplicial sets which has a Kan complex as target is a Cartesian fibration if and only if it is a coCartesian fibration. Hence, the Cartesian and the coCartesian model structure on* sSet *coincide, because they are simplicial model structures with the same cofibrations and fibrant objects. By applying [Lur09, Proposition 3.1.5.3], we see that* $\mathrm{sSet}^+_{/*^\natural}$ *is Quillen equivalent to the Joyal model structure on* sSet.

Definition 2.45. The categorical pattern $*^\natural$ introduced above together with [Lur, Definition B.1.9] and [Lur, Remark B.2.5] provide a left Quillen bifunctor

$$\mathrm{sSet}^+_{/*^\natural} \times \mathrm{sSet}^+_{/\Delta_\Phi^{\mathrm{op},\natural}} \to \mathrm{sSet}^+_{/\Delta_\Phi^{\mathrm{op},\natural}}$$

which induces a bifunctor

$$\mathrm{Cat}_\infty \times \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}} \to \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$$

of presentable ∞-categories which preserves colimits in each variable. Let

$$\mathrm{Alg}^{\mathrm{gen}}_{(-)/\Delta_\Phi^{\mathrm{op}}}(-)\colon \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen},\mathrm{op}} \times \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}} \to \mathrm{Cat}_\infty$$

be the functor given by adjunction. Similarly, we obtain

$$\mathrm{Cat}_\infty \times \mathrm{coCart}_{\mathrm{Seg}}^{\Phi} \to \mathrm{coCart}_{\mathrm{Seg}}^{\Phi} \quad \text{and} \quad \mathrm{Alg}_{(-)/\Delta_\Phi^{\mathrm{op}}}(-)\colon \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{op}} \times \mathrm{coCart}_{\mathrm{Seg}}^{\Phi} \to \mathrm{Cat}_\infty\,.$$

The objects in the ∞-categories $\mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}/\Delta_\Phi^{\mathrm{op}}}(\mathcal{P})$ and $\mathrm{Alg}_{\mathcal{O}/\Delta_\Phi^{\mathrm{op}}}(\mathcal{P})$ are called *$\mathcal{O}$-algebras in $\mathcal{P}$*.

Remark 2.46. *If $p\colon \mathcal{O} \to \Delta_\Phi^{\mathrm{op}}$ and $q\colon \mathcal{P} \to \Delta_\Phi^{\mathrm{op}}$ are (generalized) coCartesian Segal fibrations, then an $\mathcal{O}$-algebra in $\mathcal{P}$ is given by a commutative diagram*

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{P} \arrow[dl, "q"] \\ & \Delta_\Phi^{\mathrm{op}}, & \end{tikzcd}$$

where f preserves inert morphisms.

Lemma 2.47. *If $p\colon \mathcal{O} \to \Delta_\Phi^{\mathrm{op}}$ and $q\colon \mathcal{P} \to \Delta_\Phi^{\mathrm{op}}$ are generalized coCartesian Segal fibrations, then a morphism f between them is an $\mathcal{O}$-algebra in $\mathcal{P}$ if and only if f preserves those inert morphisms in $\mathcal{O}$ whose image in Δ^{op} under $\Delta_\pi^{\mathrm{op}} \circ p$ is of the form $[m] \to [i]$ for $i \in \{0,1\}$.*

Proof. By Remark 2.46, we only need to show that the functor $f\colon \mathcal{O} \to \mathcal{P}$ preserves all inert morphisms. Suppose $X \to Y$ is an inert morphism in $\mathcal{O}$ lying over the inert map $\alpha\colon ([m],I) \to ([n],J)$ in $\Delta_\Phi^{\mathrm{op}}$, then we have to show that the q-coCartesian morphism $f(X) \to \alpha_! f(X)$ induces an equivalence $\alpha_! f(X) \to f(Y)$. By Definition 2.34, the objects $\alpha_! f(X), f(Y) \in \mathcal{D}$ are equivalent to $\lim_{\beta \in \Delta_\Phi^{\mathrm{el,op}}{}_{([n],J)/}} \beta_!(\alpha_! f(X))$ and $\lim_{\beta \in \Delta_\Phi^{\mathrm{el,op}}{}_{([n],J)/}} \beta_! f(Y)$, respectively. Therefore, we only need to verify that, for every $\beta \in \Delta_\Phi^{\mathrm{el,op}}{}_{([n],J)/}$ the induced map $\varphi\colon \beta_!\alpha_! f(X) \simeq (\beta\alpha)_! f(X) \to \beta_! f(Y)$ is an equivalence in $\mathcal{D}$. Since the projections of $\beta\circ\alpha$ and α in Δ^{op} are of the form $[m] \to [i]$ and $[n] \to [i]$ for $i \in \{0,1\}$, respectively, the assumption on f implies that $(\beta\alpha)_! f(X) \simeq f(\beta\alpha)_! X)$ and $\beta_! f(Y) \simeq f(\beta_! Y)$. It follows that the map φ is an equivalence, because Y is equivalent to $\alpha_! X$. $\square$

2.4 The Dendroidal Category Ω

The dendroidal category Ω was introduced by Moerdijk and Weiss in [MW07] in order to define dendroidal sets, which provide another approach to ∞-operads. The category Ω was originally defined as a category of trees whose morphisms are given by morphisms between the free operads generated by these trees. In this section we want to recall another definition provided by Kock [Koc11]. The combinatorial nature of his definition allows us to compare an important subcategory of $\Delta_{\mathcal{F}}$ with Ω in Section 2.6.

Definition 2.48. A *polynomial endofunctor* is a diagram of sets

$$T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0.$$

We call a polynomial endofunctor as above a *tree* if the following conditions are satisfied:

1. The involved sets T_i are all finite.
2. The function t is injective.
3. The function s is injective and there is a unique element R called the *root* lying in the complement of its image.
4. Define a successor function $\sigma\colon T_0 \to T_0$ by $\sigma(R) = R$ and $\sigma(e) = t(p(e))$ for $e \in s(T_2)$. Then for every e there exists some $k \geq 0$ such that $\sigma^k(e) = R$.

Notation 2.49. *Let T be a tree and let $e, e' \in T_0$. We say e and e' are* comparable*, if there is some $k \geq 0$ such that either $\sigma^k(e) = e'$ or $\sigma^k(e') = e$, and* incomparable *otherwise.*

Remark 2.50. *The intuition behind this notion of a "tree" is as follows: We interpret T_0 as the set of edges of the tree, T_1 as the set of vertices, and T_2 as the set of pairs (v, e) where e is an incoming edge of v. The function s is the projection $s(v, e) = e$, the function p is the projection $p(v, e) = v$, and the function t assigns to each vertex its unique outgoing edge.*

Definition 2.51. For a tree $T \in \Omega$ given by $T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$, we call an edge $e \in T_0$ a *leaf*, if it does not lie in the image of t, and an *inner edge*, if it lies in the image of t and $e \neq R$.

Remark 2.52. *The name "polynomial endofunctor" comes from the fact that such a diagram induces an endofunctor of* $\mathrm{Set}_{/X_0}$ *given by $t_! p_* s^*$. We refer the reader to [Koc11] for a more thorough discussion of this.*

Definition 2.53. A morphism of polynomial endofunctors $f\colon X \to Y$ is a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[d, "f_0"'] & X_2 \arrow[l] \arrow[r] \arrow[d, "f_2"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & X_0 \arrow[d, "f_0"'] \\
Y_0 & Y_2 \arrow[l] \arrow[r] & Y_1 \arrow[r] & Y_0
\end{tikzcd}$$

such that the middle square is Cartesian. We write Ω_{in} for the category of trees and morphisms of polynomial endofunctors between them; we will refer to these morphisms as the *inert* morphisms between trees, or as *embeddings* of subtrees.

Remark 2.54. *By [Koc11, Proposition 1.1.3] every morphism of polynomial endofunctors between trees is injective, which justifies calling these morphisms embeddings.*

Definition 2.55. We write C_n for the *n-corolla*, namely the tree

$$\{0,\dots,n\} \hookleftarrow \{1,\dots,n\} \to \{0\} \hookrightarrow \{0,\dots,n\}.$$

We write η for the *edge*, namely the trivial tree

$$* \hookleftarrow \emptyset \to \emptyset \hookrightarrow *.$$

Definition 2.56. We define Ω_{el} to be the full subcategory of Ω_{in} spanned by the objects η and C_n, $n \geq 0$. Furthermore, we will write Ω_{Cr} for the full subcategory of Ω_{in} spanned by objects C_n, $n \geq 0$.

Remark 2.57. *It follows directly from the definition of morphisms of polynomial endofunctors that Ω_{Cr} is a groupoid.*

Definition 2.58. If T is a tree, let $\mathrm{sub}(T)$ be the set of subtrees of T, i.e. the set of morphisms $T' \to T$ in Ω_{in}, and let $\mathrm{sub}'(T)$ be the set of subtrees of T with a marked leaf, i.e. the set of pairs of morphisms $(\mathfrak{e} \to T', T' \to T)$. We then write $\overline{T}$ for the polynomial endofunctor

$$T_0 \leftarrow \mathrm{sub}'(T) \to \mathrm{sub}(T) \to T_0,$$

where the first map sends a marked subtree to its marked edge, the second is the obvious projection, and the third sends a subtree to its root.

Definition 2.59. The dendroidal category Ω has trees as objects and the morphisms of polynomial endofunctors $\overline{T} \to \overline{T}'$ as morphisms between the object T and T'.

Remark 2.60. *By [Koc11, Corollary 1.2.10], the polynomial endofunctor $\overline{\overline{T}}$ is in fact the free polynomial monad generated by T, and the category Ω is a full subcategory of the Kleisli category of the free polynomial monad. This means that a morphism $\overline{\overline{T}} \to \overline{\overline{T}}'$ is uniquely determined by the composite $T \to \overline{\overline{T}} \to \overline{\overline{T}}'$. It follows that Ω_{in} is a subcategory of Ω; we say that a morphism in Ω is* inert *if it lies in the image of Ω_{in}.*

Lemma 2.61. *[Koc11, Lemma 1.3.5] Any morphism $\overline{\overline{T}} \to \overline{\overline{T}}'$ in Ω is uniquely determined by the underlying map $T_0 \to T'_0$ of the corresponding sets of edges.*

Corollary 2.62. *[Koc11, 1.3.6 Corollary] If S is an object in Ω and $S \neq \eta$, then every map $S \to T$ in Ω is determined by its value on the corollas in S_1.*

Definition 2.63. We say that a morphism $\phi\colon T \to T'$ in Ω is *active* if it takes the leaves of T to the leaves of T' (bijectively) and the root of T to the root of T'.

Remark 2.64. *In [Koc11] the inert morphisms are called* free, *and the active ones* boundary-preserving. *Our terminology follows that of Barwick [Bar13] and Lurie [Lur].*

Proposition 2.65. *[Koc11, Proposition 1.3.13] The active and inert morphisms form a factorization system on Ω.* □

In Definition 2.25 we introduced, for every $X \in \mathcal{S}$, a category $\Delta^{\mathrm{op}}_{\Phi,X}$ together with a left fibration $\Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_{\Phi}$ such that objects in $\Delta^{\mathrm{op}}_{\Phi,X}$ can be thought of an object in $\Delta^{\mathrm{op}}_{\Phi}$ where the edges are labelled by objects in X. In Definition 3.1 we used $\Delta^{\mathrm{op}}_{\Phi,X}$ to define enriched ∞-operads. Now we want to define a category Ω_X whose objects are trees in Ω with a labelling of the edges. The construction presented below is the same as in Definition 2.25. This category allows us to introduce enriched Ω-∞-operads in Definition **??**.

Definition 2.66. Let X be an ∞-groupoid and let $i_X \colon \{\eta\} \to \mathcal{S}$ denote the functor which carries the trivial tree η to X. We write $p_X \colon \Omega_X^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ for the left fibration associated to the right Kan extension $\tilde{i}_X \colon \Omega^{\mathrm{op}} \to \mathcal{S}$ of i_X along the inclusion $\{\eta\} \hookrightarrow \Omega^{\mathrm{op}}$. An object in Ω_X will be denoted by $(S, \{x_i\}_{i \in S_0})$, where $\{x_i\}_{i\in S_0}, x_i \in X$ is the labelling of the edges in the tree S. If $S = C_n$ for some n, then we will often write $C_n(x_1, \ldots, x_n; x_0)$ for $(C_n, \{x_i\}_{0 \le i \le n})$.

Definition 2.67.

1. We say S *is a corolla in* T, if $S = C_n$ for some n and if there exists an inert morphism $S \to T$ in Ω.
2. An object $(S, \{x_i\}_{i\in S_0})$ in Ω_X is called a *corolla*, if $S = C_n$ for some n. For objects $(S, \{x_i\}_{i\in S_0})$ and $(S, \{x_i\}_{i\in S_0})$ in Ω_X, we say $(S, \{x_i\}_{i\in S_0})$ *is a corolla in* $(S, \{x_i\}_{i\in S_0})$, if there exists a map $f \colon (S, \{x_i\}_{i\in S_0}) \to (S, \{x_i\}_{i\in S_0})$ such that $p_X(f)$ exhibits S as a corolla in T.

Let $S \in \Omega$ be given by the polynomial endofunctor $S_0 \xleftarrow{s} S_2 \xrightarrow{p} S_1 \xrightarrow{t} S_0$. Every element $s \in S_1$ induces an inert map $C_n \to S$ in Ω:

$$\begin{tikzcd}
\{1,\ldots,n,s\} \arrow[d] & \{1,\ldots,n\} \arrow[l, hook'] \arrow[r] \arrow[d] & \{s\} \arrow[r, hook] \arrow[d] & \{1,\ldots,n,s\} \arrow[d] \\
S_0 & S_2 \arrow[l] \arrow[r] & S_1 \arrow[r] & S_0,
\end{tikzcd}$$

where the set $\{1,\ldots,n\}$ is uniquely determined by requiring the middle square to be a pullback. Hence, we can think of S_1 as the set of corollas in S as well as the subset of $\mathrm{sub}(S)$ consisting of corollas.

Let $f \colon T \to S$ be a morphism in Ω given by the morphism of polynomial endofunctors:

$$\begin{tikzcd}
T_0 \arrow[d, "f_0"'] & \mathrm{sub}'(T) \arrow[l] \arrow[r] \arrow[d, "f_2"'] & \mathrm{sub}(T) \arrow[r] \arrow[d, "f_1"'] & T_0 \arrow[d, "f_0"'] \\
S_0 & \mathrm{sub}'(S) \arrow[l] \arrow[r] & \mathrm{sub}(S) \arrow[r] & S_0.
\end{tikzcd} \tag{2.3}$$

By identifying T_1 with the a subset of $\mathrm{sub}(T)$ consisting of corollas as above, we can restrict the map $f_1 \colon \mathrm{sub}(T) \to \mathrm{sub}(S)$ to a map $T_1 \to f_1(T_1) \subseteq \mathrm{sub}(S)$.

Given $t, t' \in T_1$, we claim that the subtrees $f_1(t)$ and $f_1(t')$ in S have disjoint sets of corollas. If the roots of the corollas t and t' are incomparable, then, by [Koc11, Proposition 1.3.7], their images in S under f remain incomparable. In particular, their images cannot have a corolla (or even an edge) in common. Now, let us assume that the roots of t and t' are comparable. Without loss of generality, we may assume that there is a unique path in T from the root of t to the root of t' passing through a unique leaf e of t'. Then [Koc11, Proposition 1.3.4] and [Koc11, Proposition 1.3.7] imply that there has to be a path from the root of $f_1(t)$ to a leaf of $f_1(t')$. This implies that even in this case, the subtrees $f_1(t), f_1(t')$ in S have disjoint sets of corollas. This observation also guarantees that the following definition is well-defined:

Definition 2.68. We write $\mathrm{Cr}\colon \Omega^{\mathrm{op}} \to \mathcal{F}_*$ for the assignment defined as follows:

1. If $S \in \Omega$ is given by the polynomial endofunctor $S_0 \xleftarrow{s} S_2 \xrightarrow{p} S_1 \xrightarrow{t} S_0$, then we define $\mathrm{Cr}(S)$ to be pointed set $\langle S_1 \rangle \in \mathcal{F}_*$.
2. If $f\colon T \to S$ is a morphism in Ω, then we define $\mathrm{Cr}(f)\colon \langle S_1 \rangle \to \langle T_1 \rangle$ by

$$\mathrm{Cr}(f)(s) = \begin{cases} t & \text{if } s \in S_1 \text{ and } \exists t \in T_1 \text{ such that } s \text{ is a corolla in } f(t), \\ * & \text{otherwise.} \end{cases}$$

In order to ease notation, we will denote the composite $\Omega_X^{\mathrm{op}} \xrightarrow{p_X} \Omega^{\mathrm{op}} \xrightarrow{\mathrm{Cr}} \mathcal{F}_*$ by Cr again. Since the assignment Cr is obvious compatible with composition, we have the the following result:

Lemma 2.69. *The assignment* Cr *defined above is a functor.*

Remark 2.70. *Informally, for a morphism $f\colon T \to S$ as above, the functor $\mathrm{Cr}(f)$ carries a corolla $s \in S_1$ to the corolla $t \in T_1$, if s is a corolla in the subtree $f_1(t_1) \in sub(S)$, and to the base point $*$ otherwise. In particular, if $f\colon C_n \to S$ is an active morphism in Ω, then it is boundary preserving by Remark 2.64 and $\mathrm{Cr}(f)$ carries every corolla in S to $\{1\} \in \langle 1 \rangle = \mathrm{Cr}(C_n)$.*

2.5 Categorical Patterns on Ω^{op}

In this section, we first define categorical patterns on the dendroidal category Ω^{op}. As their counterparts in Section 2.3, these categorical patterns induce ∞-categories $\mathrm{coCart}^{\Omega}_{\mathrm{Seg}}$ and $\mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$. At the end of this section we will introduce the corolla functor $\mathrm{Cr}\colon \Omega^{\mathrm{op}} \to \mathcal{F}_*$.

Notation 2.71. *Let T be a tree in Ω.*

- *We denote the category $\Omega^{\mathrm{op}}_{\mathrm{el}} \times_{\Omega^{\mathrm{op}}_{\mathrm{in}}} (\Omega^{\mathrm{op}}_{\mathrm{in}})_{T/}$ by $(\Omega^{\mathrm{op}}_{\mathrm{el}})_{T/}$ and write $(\Omega^{\mathrm{op}}_{\mathrm{el}})^{\triangleleft}_{T/}$ for the category $\Delta^0 * \Omega^{\mathrm{op}}_{\mathrm{el}})_{T/}$.*
- *We denote the category $\Omega^{\mathrm{op}}_{\mathrm{Cr}} \times_{\Omega^{\mathrm{op}}_{\mathrm{in}}} (\Omega^{\mathrm{op}}_{\mathrm{in}})_{T/}$ by $(\Omega_{\mathrm{Cr}})_{T/}$ and write $(\Omega^{\mathrm{op}}_{\mathrm{Cr}})^{\triangleleft}_{T/}$ for the category $\Delta^0 * (\Omega^{\mathrm{op}}_{\mathrm{Cr}})_{T/}$.*

It follows that an object in $(\Omega^{\mathrm{op}}_{\mathrm{el}})_{T/}$ is given by an inert morphism in Ω of the form $\eta \to T$ or $C_n \to T$, for an $n \geq 0$.

Definition 2.72. We define

$$\mathrm{Dia}_{\Omega} = \coprod_{T \in \Omega^{\mathrm{op}}} \{p_T\colon (\Omega^{\mathrm{op}}_{\mathrm{el}})^{\triangleleft}_{T/} \to \Omega^{\mathrm{op}}\} \quad \text{and} \quad \mathrm{Dia}^0_{\Omega} = \coprod_{T \in \Omega^{\mathrm{op}}} \{p_T\colon (\Omega^{\mathrm{op}}_{\mathrm{Cr}})^{\triangleleft}_{T/} \to \Omega^{\mathrm{op}}\},$$

where p_T denotes the canonical inclusion map which carries the cone point to T. Let $\Omega^{\mathrm{op},\natural}$ and $\Omega^{\mathrm{op},\natural,0}$ denote the categorical patterns on Ω^{op} be given by the respective tuples

$$(\Omega^{\mathrm{op}}, I_{\Omega}, \mathrm{Dia}_{\Omega}) \text{ and } (\Omega^{\mathrm{op}}, I_{\Omega}, \mathrm{Dia}^0_{\Omega}),$$

where I_{Ω} denotes the set of inert morphisms.

Definition 2.73. By Theorem 2.29, the categorical patterns $\Omega^{\mathrm{op},\natural}$ and $\Omega^{\mathrm{op},\natural,0}$ induce left proper combinatorial simplicial model categories $\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}}$ and $\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural,0}}$, respectively.

- We write $\mathrm{coCart}^{\Omega}_{\mathrm{Seg}}$ for the ∞-category associated to $\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}}$ and we call its objects *coCartesian Segal fibrations on* Ω^{op} *on* Ω^{op}.
- We write $\mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$ for the ∞-category associated to $\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural,0}}$ and we call its objects *generalized coCartesian Segal fibrations on* Ω^{op}

Definition 2.74. If $*^\natural$ denotes the categorical pattern on Δ^0 defined in Remark 2.44, then [Lur, Remark B.2.5] provides a left Quillen bifunctor

$$\mathrm{sSet}^+_{/*^\natural} \times \mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}} \to \mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}}$$

which induces a bifunctor

$$\mathrm{Cat}_\infty \times \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}} \to \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$$

of presentable ∞-categories which preserves colimits in each variable. We define

$$\mathrm{Alg}^{\mathrm{gen}}_{(-)/\Omega^{\mathrm{op}}}(-)\colon\ \mathrm{coCart}^{\Omega,\mathrm{gen},\mathrm{op}}_{\mathrm{Seg}} \times \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}} \to \mathrm{Cat}_\infty$$

to be the functor given by adjunction. Similarly, we obtain

$$\mathrm{Cat}_\infty \times \mathrm{coCart}^{\Omega}_{\mathrm{Seg}} \to \mathrm{coCart}^{\Omega}_{\mathrm{Seg}} \quad \text{and} \quad \mathrm{Alg}_{(-)/\Omega^{\mathrm{op}}}(-)\colon\ \mathrm{coCart}^{\Omega,\mathrm{op}}_{\mathrm{Seg}} \times \mathrm{coCart}^{\Omega}_{\mathrm{Seg}} \to \mathrm{Cat}_\infty\,.$$

The following two lemmata can be shown by using similar arguments as in the proof of Lemma 2.41 and Lemma 2.38.

Lemma 2.75. *The functor* $\mathrm{Cr}\colon \Omega^{\mathrm{op}} \to \mathcal{F}_*$ *from above is compatible with the categorical patterns* $\Omega^{\mathrm{op},\natural,0}_{\mathcal{F}}$ *and* $\mathcal{F}^{\natural,0}_*$. *Therefore, we have a Quillen adjunction by [Lur, Proposition B.2.9]*

$$\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}} \rightleftarrows \mathrm{sSet}^+_{/\mathcal{F}^{\natural,0}_*}\,,$$

which induces an adjunction of the corresponding ∞*-categories*

$$\mathrm{Cr}_!\colon\ \mathrm{coCart}^{\Omega}_{\mathrm{Seg}} \rightleftarrows \mathrm{Op}_\infty : \mathrm{Cr}^*\,.$$

Lemma 2.76. *The inclusion functor* $\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural,0}} \hookrightarrow \mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}}$ *induces a Quillen adjunction*

$$\mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural}} \rightleftarrows \mathrm{sSet}^+_{/\Omega^{\mathrm{op},\natural,0}}\,.$$

In particular, the ∞*-category* $\mathrm{coCart}^{\Omega}_{\mathrm{Seg}}$ *is a localization of* $\mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$.

Notation 2.77. *If* $\mathcal{V}$ *is an* ∞*-operad, then we write* $\mathrm{Cr}^*(\mathcal{V})$ *for the object in* $\mathrm{coCart}^{\Omega}_{\mathrm{Seg}}$ *as well as its image in* $\mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$.

2.6 From $\Delta^1_{\mathcal{F}}$ to Ω

In this section we introduce a full subcategory $\Delta^1_{\mathcal{F}}$ of $\Delta_{\mathcal{F}}$ and we define a functor $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ in Lemma 2.80.

Definition 2.78. Let $\Delta^1_{\mathcal{F}}$ be the full subcategory of $\Delta_{\mathcal{F}}$ spanned by the objects $([n], I)$ such that $I(n) = *$. The active-inert and surjective-injective factorization systems on $\Delta_{\mathcal{F}}$ clearly restrict to factorization systems on $\Delta^1_{\mathcal{F}}$. We write $\Delta^{1,\mathrm{in}}_{\mathcal{F}}$ for the subcategory of $\Delta^1_{\mathcal{F}}$ containing only the inert maps. We write $\Delta^1_{\mathcal{F},\mathrm{el}}$ for $\Delta_{\mathcal{F},\mathrm{el}}$ which we regard as a full subcategory of $\Delta^1_{\mathcal{F}}$. The inclusion functor $i^1\colon \Delta^1_{\mathcal{F}} \hookrightarrow \Delta_{\mathcal{F}}$ induces a map of presheaf categories $i^{1,*}\colon \mathrm{P}(\Delta_{\mathcal{F}}) \to \mathrm{P}(\Delta^1_{\mathcal{F}})$. Moreover, the categorical pattern $\Delta^{\mathrm{op},\natural}_{\mathcal{F}}$ on $\Delta^{\mathrm{op}}_{\mathcal{F}}$ restricts canonically to a categorical pattern $\Delta^{1,\mathrm{op},\natural}_{\mathcal{F}}$ on $\Delta^{1,\mathrm{op}}_{\mathcal{F}}$ and the inlcusion functor i^1 induces a Quillen adjunction

$$i^1_!\colon \mathrm{sSet}_{/\Delta^{1,\mathrm{op},\natural}_{\mathcal{F}}} \rightleftarrows \mathrm{sSet}_{/\Delta^{\mathrm{op},\natural}_{\mathcal{F}}} : i^{1,*}.$$

In this section we will define a functor $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$. On objects, the functor τ takes an object $([n], f)$ in $\Delta^1_{\mathcal{F}}$ to the diagram

$$\coprod_{i=0}^{n} f(i) \xleftarrow{s} \coprod_{i=0}^{n-1} f(i) \xrightarrow{p} \coprod_{i=1}^{n} f(i) \xrightarrow{t} \coprod_{i=0}^{n} f(i),$$

where s and t are the obvious inclusions and p takes $x \in f(i)$ to $f^{i+1}(x) \in f(i+1)$.

If $(\phi, \eta)\colon ([n], f) \to ([m], g)$ is an inert map in $\Delta^1_{\mathcal{F}}$, then we define $\tau(\phi, \eta)$ to be the obvious morphism

$$\begin{tikzcd}
\coprod_{i=0}^{n} f(i) \arrow[d] & \coprod_{i=0}^{n-1} f(i) \arrow[l] \arrow[r] \arrow[d] & \coprod_{i=1}^{n} f(i) \arrow[r] \arrow[d] & \coprod_{i=0}^{n} f(i) \arrow[d] \\
\coprod_{j=0}^{m} g(j) & \coprod_{j=0}^{m-1} g(j) \arrow[l] \arrow[r] & \coprod_{j=1}^{m} g(j) \arrow[r] & \coprod_{j=0}^{m} g(j).
\end{tikzcd}$$

Here the middle square is Cartesian, as required, since by definition η is a Cartesian natural transformation.

To define τ for a general map in $\Delta^1_{\mathcal{F}}$, it is convenient to first introduce an intermediate object between $\tau([n], f)$ and its free monad $\overline{\tau}([n], f)$:

Definition 2.79. For $([n], f) \in \Delta^1_{\mathcal{F}}$, let $\mathrm{sub}_{\Delta^1_{\mathcal{F}}}([n], f)$ denote the set of subtrees of $([n], f)$ given by maps in $\Delta^1_{\mathcal{F}}$, i.e. the set of inert maps $([m], g) \hookrightarrow ([n], f)$ in $\Delta^1_{\mathcal{F}}$, or equivalently the set of pairs $(x \in f(i), 0 \le j \le i)$, corresponding to the subtree

$$f(j)_x \to f(j+1)_x \to \ldots \to f(i-1)_x \to \{x\},$$

where $f(k)_x$ is the fibre of $f^{ki}\colon f(k) \to f(i)$ at x. Similarly, let $\mathrm{sub}'_{\Delta^1_{\mathcal{F}}}([n], f)$ be the set of subtrees in $\mathrm{sub}_{\Delta^1_{\mathcal{F}}}([n], f)$ with a marked leaf, or equivalently the set of triples $(x \in f(i), 0 \le j \le i, y \in f(j)_x)$. We then let $\widetilde{\tau}([n], f)$ denote the polynomial endofunctor

$$\coprod_{i=0}^{n} f(i) \leftarrow \mathrm{sub}'_{\Delta^1_{\mathcal{F}}}([n], f) \to \mathrm{sub}_{\Delta^1_{\mathcal{F}}}([n], f) \to \coprod_{i=0}^{n} f(i),$$

where the first map takes $(x \in f(i), j, y \in f(j)_x)$ to the marked leaf y and the second projects it to (x, j), and the third takes the subtree $(x \in f(i), j)$ to its root x. The definition of τ on inert maps clearly gives an injective map $\widetilde{\tau}([n], f) \hookrightarrow \overline{\tau}([n], f)$ of polynomial endofunctors, and the canonical map $\tau([n], f) \to \overline{\tau}([n], f)$ factors through this.

For a general map $(\phi, \eta)\colon ([n], f) \to ([m], g)$, we then define a map of polynomial endofunctors $\tau([n], f) \to \widetilde{\tau}([m], g)$, i.e.

$$\begin{tikzcd}
\coprod_{i=0}^{n} f(i) \arrow[d] & \coprod_{i=0}^{n-1} f(i) \arrow[l] \arrow[r] \arrow[d] & \coprod_{i=1}^{n} f(i) \arrow[r] \arrow[d] & \coprod_{i=0}^{n} f(i) \arrow[d] \\
\coprod_{j=0}^{m} g(j) & \mathrm{sub}'_{\Delta^1_{\mathcal{F}}}([m], g) \arrow[l] \arrow[r] & \mathrm{sub}_{\Delta^1_{\mathcal{F}}}([m], g) \arrow[r] & \coprod_{j=0}^{m} g(j),
\end{tikzcd}$$

as follows:

- The component $\coprod_{i=0}^{n} f(i) \to \coprod_{j=0}^{m} g(j)$ is the obvious map, given on $f(i)$ by $\eta_i\colon f(i) \to g(\phi(i))$.
- The component $\coprod_{i=1}^{n} f(i) \to \mathrm{sub}_{\Delta^1_{\mathcal{F}}}([m], g)$ is given by

$$(x \in f(i)) \mapsto (\eta_i(x) \in g(\phi(i)), \phi(i-1)).$$

- The component $\coprod_{i=0}^{n-1} f(i) \to \mathrm{sub}'_{\Delta^1_{\mathcal{F}}}([m], g)$ is defined by

$$(x \in f(i)) \mapsto (\eta_{i+1}(f^{i+1}(x)) \in g(\phi(i+1)), \phi(i), \eta_i(x) \in g(\phi(i))_{\eta_{i+1}(f^{i+1}(x))})$$

Since η is a Cartesian natural transformation, the middle square in the diagram above has to be Cartesian as well, so this does indeed define a map of polynomial endofunctors. We then define $\tau(\phi, \eta)$ to be the map $\overline{\tau}([n], f) \to \overline{\tau}([m], g)$ induced by the composite $\tau([n], f) \to \widetilde{\tau}([m], g) \hookrightarrow \overline{\tau}([m], g)$.

Lemma 2.80. *The assignment τ defines a functor $\Delta^1_{\mathcal{F}} \to \Omega$.*

Proof. Since τ clearly preserves identities, it remains to be checked whether it respects composition, i.e. that given a composition of morphisms

$$([n], f) \overset{(\phi,\eta)}{\to} ([m], g) \overset{(\psi,\lambda)}{\to} ([k], h)$$

in $\Delta^1_{\mathcal{F}}$ the maps $\tau((\psi, \lambda) \circ (\phi, \eta))$ and $\tau(\psi, \lambda) \circ \tau(\phi, \eta)$ agree. But by Lemma 2.61 it suffices to show that they are given by the same map on the set of edges. By definition, for $\tau(\phi, \eta)$ this is the map $\coprod_{i=0}^{n} f(i) \to \coprod_{j=0}^{m} g(j)$ given on $f(i)$ by $\eta_i\colon f(i) \to g(\phi(i))$, so it is evident that the two maps agree on the sets of edges. $\square$

Chapter 3

Enriched Φ-∞-Operads

We start the first section of this chapter by defining $\mathcal{V}$-enriched Φ-∞-operads with a fixed space of objects. Using the theory of operator categories developed in the previous chapter, we can associate to each enriched Φ-∞-operads its underlying enriched ∞-category (Definition 3.3). Ater this, the ∞-category of all $\mathcal{V}$-enriched Φ-∞-operads and the ∞-category of monoid objects are presented in Definition 3.9. The monoid objects will be used in the proof of Theorem 3.24 in the next section which is the main result of this chapter. We will also recall completeness of enriched ∞-categories and define an enriched Φ-∞-operad to be complete if its underlying ∞-category is (Definition 3.14).

The next section is devoted to the proof of Theorem 3.24 which states that complete Φ-∞-operads enriched over spaces are equivalent to Barwick's Φ-∞-operads. Using this we show in Corollary 3.32 that complete Φ-∞-operads which are enriched over spaces are essentially the same as Lurie's ∞-operads. This result implies that our theory of enriched ∞-operads generalized that of ∞-operads in the sense of Barwick. Since this model of ∞-operads is equivalent to all known models for ∞-operads by [CHH16], it is justified to regard enriched Φ-∞-operads as a model for enriched ∞-operads.

3.1 $\mathcal{V}$-enriched Φ-∞-operads and $\mathcal{O}$-monoid objects

Definition 3.1. Let $X \in \mathcal{S}$, let $\mathcal{V}$ be a symmetric monoidal ∞-category and let $\mathrm{Cr}^*_\Phi(\mathcal{V})$ be as defined in Remark 2.43. We define a *$\mathcal{V}$-enriched Φ-∞-operad with colour space X* to be a morphism $\mathcal{O}\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathrm{Cr}^*_\Phi(\mathcal{V})$ in $\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}}$. The corresponding ∞-category is given by $\mathrm{Alg}_{\Delta^{\mathrm{op}}_{\Phi,X}/\Delta^{\mathrm{op}}_{\Phi}}(\mathrm{Cr}^*_\Phi(\mathcal{V}))$.

Notation 3.2. *In order to ease notation, we will write $\mathcal{O}\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$ instead of $\mathcal{O}\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathrm{Cr}^*_\Phi(\mathcal{V})$ for a $\mathcal{V}$-enriched Φ-∞-operad.*

The inclusion $\{*\} \hookrightarrow \Phi$ of the terminal object induces a functor $u\colon \Delta \simeq \Delta_* \to \Delta_\Phi$. If $\tilde{i}_X\colon \Delta^{\mathrm{op}}_\Phi \to \mathcal{S}$ denotes the functor given by right Kan extension (see Definition 2.25), then the composite $\tilde{i}_X u\colon \Delta \to \mathcal{S}$ corresponds to a left fibration $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$ which is given by the following pullback square

$$\begin{tikzcd} \Delta^{\mathrm{op}}_X \arrow[r] \arrow[d] & \Delta^{\mathrm{op}}_{\Phi,X} \arrow[d, "\pi_X"] \\ \Delta^{\mathrm{op}} \arrow[r, "u"] & \Delta^{\mathrm{op}}_\Phi, \end{tikzcd}$$

where the right vertical map $\pi_X \colon \Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_\Phi$ is the left fibration associated to $\tilde{i}_X$. Let $u^* \colon \mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}} \to \mathrm{coCart}^{*,\mathrm{gen}}_{\mathrm{Seg}}$ denote the right adjoint functor induced by u. Then by the above considerations we have $u^*\Delta^{\mathrm{op}}_{\Phi,X} \simeq \Delta^{\mathrm{op}}_X$.

Definition 3.3. Let $\mathcal{O} \colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$ be a $\mathcal{V}$-enriched Φ-∞-operad. Its *underlying $\mathcal{V}$-enriched Φ-∞-category* is defined to be the morphism $u^*\mathcal{O} \colon \Delta^{\mathrm{op}}_X \to u^* \mathrm{Cr}^*_\Phi(\mathcal{V})$, which will be denoted by $u^*\mathcal{O} \colon \Delta^{\mathrm{op}}_X \to \mathcal{V}$.

Remark 3.4. *Let $\mathcal{O} \colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$ be a $\mathcal{V}$-enriched Φ-∞-operad and let A be an object in $\Delta^{\mathrm{op}}_{\Phi,X}$ such that $\pi_X(A) = ([m], I) \in \Delta^{\mathrm{op}}_\Phi$. Since π_X is a left fibration, it is in particular a coCartesian fibration and hence there exists a map $p_{([m],I)} \colon (\Delta^{\mathrm{el,op}}_\Phi)^{\triangleleft}_{([m],I)/} \to \Delta^{\mathrm{op}}_{\Phi,X}$ lifting the canonical inclusion $(\Delta^{\mathrm{el,op}}_\Phi)^{\triangleleft}_{([m],I)/} \hookrightarrow \Delta^{\mathrm{op}}_\Phi$ and satisfying the following properties:*

- $p_{([m],I)}([m], I) = A$.
- $p_{([m],I)}$ *carries each morphism to an intert π_X-coCartesian morphism.*

As a morphism in $\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}}$, the $\mathcal{V}$-enriched Φ-∞-operad $\mathcal{O}$ preserves inert morphisms and we obtain $\alpha_!\mathcal{O}(A) \simeq \mathcal{O}(\alpha_! A)$. Therefore, there is an equivalence

$$\mathcal{O}(A) \simeq \lim_{\alpha \in (\Delta^{\mathrm{el,op}}_\Phi)^{\triangleleft}_{([m],I)/}} \mathcal{O}(\alpha_! A),$$

where the $A \to \alpha_! A$ is a π_X-coCartesian lift of α.

Definition 3.5. Given a fibrant object $p \colon \mathcal{O} \to \Delta^{\mathrm{op}}_\Phi$ in $\mathrm{sSet}^{+,\mathrm{gen}}_{/\Delta^{\mathrm{op},\natural}_\Phi}$ we define $\mathcal{O}_{\mathrm{triv}}$ via the following strict pullback square in sSet:

$$\begin{tikzcd} \mathcal{O}_{\mathrm{triv}} \arrow[r] \arrow[d] & \mathcal{O} \arrow[d, "p"] \\ \Delta^{\mathrm{in,op}}_\Phi \arrow[r] & \Delta^{\mathrm{op}}_\Phi. \end{tikzcd} \tag{3.1}$$

Similarly, we define $\mathcal{O}_{\mathrm{Cr}}$ to be the ∞-category given by the strict pullback square in sSet:

$$\begin{tikzcd} \mathcal{O}_{\mathrm{Cr}} \arrow[r] \arrow[d] & \mathcal{O} \arrow[d, "p"] \\ \Delta^{\mathrm{Cr,op}}_\Phi \arrow[r] & \Delta^{\mathrm{op}}_\Phi. \end{tikzcd} \tag{3.2}$$

Remark 3.6. *Note that each morphism between corollas in $\Delta^{\mathrm{op}}_\Phi$ is necessarily an isomorphism and thus inert. Therefore, $\Delta^{\mathrm{Cr,op}}_\Phi$ and $\mathcal{O}_{\mathrm{Cr}}$ are full subcategories of $\Delta^{\mathrm{in,op}}_\Phi$ and $\mathcal{O}_{\mathrm{triv}}$, respectively. As a consequence of the previous definition, the ∞-categories $\Delta^{\mathrm{Cr,op}}_\Phi$ and $\mathcal{O}_{\mathrm{Cr}}$ are given by the* *pullbacks* $\Delta_\Phi^{\mathrm{op}} \times_{\mathcal{F}_*} \{\langle 1\rangle\}$ *and* $\mathcal{O} \times_{\mathcal{F}_*} \{\langle 1\rangle\}$ *which are induced by the functors* Cr_Φ *and* $(\mathrm{Cr}_\Phi \circ p)$, *respectively.*

Definition 3.7. Let $\mathcal{O} \to \Delta_\Phi^{\mathrm{op}}$ be an object of $\mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$ and let $\mathcal{C}$ be an ∞-category admitting finite limits. An *$\mathcal{O}$-monoid object* in $\mathcal{C}$ is a functor $F\colon \mathcal{O} \to \mathcal{C}$ such that the restriction $F|_{\mathcal{O}_{\mathrm{triv}}}$ is given by the right Kan extension of $F|_{\mathcal{O}_{\mathrm{Cr}}}$ along the inclusion $\mathcal{O}_{\mathrm{Cr}} \hookrightarrow \mathcal{O}_{\mathrm{triv}}$. We will write $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ spanned by the $\mathcal{O}$-monoid objects and write $\mathrm{Mon}_{(-)}(\mathcal{C})\colon \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen},\mathrm{op}} \to \mathrm{Cat}_\infty$ for the functor which carries each object $\mathcal{O} \in \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen},\mathrm{op}}$ to the ∞-category $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-monoid objects.

Remark 3.8. *Let* $F\colon \mathcal{O} \to \mathcal{C}$ *be an* $\mathcal{O}$*-monoid object in* $\mathcal{C}$ *as in Definition 3.7. For an object* T *in* $\mathcal{O}$ *lying over* $([m], I) \in \Delta_\Phi^{\mathrm{op}}$, *the description of the right Kan extensions (see [Lur09, Lemma 4.3.2.13]) implies that the object* $F(T)$ *is given by* $\prod_{\mathfrak{c}_I \in \coprod_k |I(k)|} F(\mathfrak{c}_{I,!}T)$, *where* $T \to \mathfrak{c}_{I,!}T$ *denotes the coCartesian lift of the inert map* $([m], I) \to \mathfrak{c}_I$ *in* $\Delta_\Phi^{\mathrm{op}}$.

Definition 3.9. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{V}$ be a symmetric monoidal ∞-category and let $\Delta^{\mathrm{op}}_{\Phi,(-)}\colon \mathcal{S} \to \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$ denote the functor which assigns to each ∞-groupoid $X \in \mathcal{S}$ the object $\Delta^{\mathrm{op}}_{\Phi,X} \in \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$.

1. We write $\mathrm{Alg}_\Phi(\mathcal{V})$ for the ∞-category given by the pullback square

$$\begin{tikzcd} \mathrm{Alg}_\Phi(\mathcal{V}) \arrow[r] \arrow[d] & \Phi\text{-}\mathrm{Alg}(\mathcal{V}) \arrow[d] \\ \mathcal{S} \arrow[r, "\Delta^{\mathrm{op}}_{\Phi,(-)}"'] & \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}, \end{tikzcd} \tag{3.3}$$

   where the right vertical map denotes the Cartesian fibration associated to the functor $\mathrm{Alg}^{\mathrm{gen}}_{(-)/\Delta_\Phi^{\mathrm{op}}}(\mathcal{V})\colon \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen},\mathrm{op}} \to \mathrm{Cat}_\infty$ as defined in Definition 2.45.

2. We write $\mathrm{Mon}_\Phi(\mathcal{C})$ for the ∞-category given by the pullback square

$$\begin{tikzcd} \mathrm{Mon}_\Phi(\mathcal{C}) \arrow[r] \arrow[d] & \Phi\text{-}\mathrm{Mon}(\mathcal{C}) \arrow[d] \\ \mathcal{S} \arrow[r, "\Delta^{\mathrm{op}}_{\Phi,(-)}"'] & \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}, \end{tikzcd} \tag{3.4}$$

   where the right vertical map Φ-$\mathrm{Mon}(\mathcal{C}) \to \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$ denotes the Cartesian fibration associated to the functor $\mathrm{Mon}_{(-)}(\mathcal{C})\colon \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen},\mathrm{op}} \to \mathrm{Cat}_\infty$.

Definition 3.10. Let $\mathcal{O}\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$ be a $\mathcal{V}$-enriched Φ-∞-operad. Let $\mathfrak{c}_I \in \Delta_\Phi^{\mathrm{op}}$ be a corolla and let $\mathfrak{c}_I(x_1, \ldots, x_n; x) \in (\Delta^{\mathrm{op}}_{\Phi,X})_{\mathfrak{c}_I} \simeq X^{|I(0)|+1}$ (see Notation 2.4). We write $\mathcal{O}(x_1, \ldots, x_n; x)$ for the object $\mathcal{O}(\mathfrak{c}_I(x_1, \ldots, x_n; x)) \in \mathcal{V}$. If $f\colon \mathcal{O} \to \mathcal{P}$ is a morphism in $\mathrm{Alg}_\Phi(\mathcal{V})$ lying over $f_0\colon X \to Y \in \mathcal{S}$, then we write

$$\mathcal{O}(x_1, \ldots, x_n; x) \to \mathcal{P}(f(x_1), \ldots, f(x_n); f(x))$$

for the morphism $\mathcal{O}(\mathfrak{c}_I(x_1, \ldots, x_n; x)) \to \mathrm{P}(\mathfrak{c}_I(f_0x_1, \ldots, f_0x_n; f_0x))$.

Recall that a coarse category is a category which has a unique morphism between any pair of objects (X, Y). In particular, a coarse category with two objects has only a unique isomorphism between the them and a functor from this coarse category into another category is nothing but an isomorphism in the target category. The trivial categories defined below is an enriched version of coarse category in the setting of ∞-categories. They are introduced in [GH15, 5.1] in order to define equivalences and completeness in the language of enriched ∞-categories. We will then define enriched Φ-∞-operads to be complete, if their underlying enriched ∞-category is complete.

Let $\mathcal{V}$ be a symmetric monoidal ∞-category. Corollary [Lur, 3.2.1.9] implies that the ∞-category $\mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathcal{V})$ of associative algebras in $\mathcal{V}$ has an initial object $\mathbb{1}_{\mathcal{V}} \colon \Delta^{\mathrm{op}} \to \mathcal{V}$ which carries $[0] \in \Delta$ to the unit object of $\mathcal{V}$.

Definition 3.11. ([GH15, 5.1.6]) For an ∞-groupoid $X \in \mathcal{S}$, we call a $\mathcal{V}$-enriched ∞-category *trivial*, if it is of the form $E_X^{\mathcal{V}} \colon \Delta^{\mathrm{op}}_{*,X} \to \Delta^{\mathrm{op}}_{*} \xrightarrow{\mathbb{1}_{\mathcal{V}}} (\mathrm{Cr}_*)^*\mathcal{V}$ (the $*$ in the subindex denotes the trival operator category). In order to ease notation we write E_X for $E_X^{\mathcal{V}}$ and E^n for $E^{\mathcal{V}}_{\{0,\dots,n\}}$, if the enrichment is clear from the context.

The $\mathcal{V}$-enriched ∞-categories E_X are functorial in X. Therefore, we have the following definition:

Definition 3.12. We define $E^\bullet$ to be the cosimplicial object in the ∞-category of $\mathcal{V}$-enriched ∞-category induced by the order-preserving morphisms between sets $\{0, \dots, n\}$ for $n \in \mathbb{N}$.

For each $n \geq 0$, $E^n \to \mathcal{C}$ should be thought of as an enriched ∞-category with $n+1$ equivalent objects. Providing a functor $E^n \to \mathcal{C}$ of enriched ∞-categories is the same as a choice of $n+1$ equivalent objects in $\mathcal{C}$.

Definition 3.13. ([GH15, 5.1.13], [GH15, Definition 5.2.1]) Let $\mathcal{C}$ be an object in $\mathrm{Alg}_*(\mathcal{V})$. We write $\iota_\bullet\mathcal{C}$ for the simplicial space $\mathrm{Map}_{\mathrm{Alg}_*(\mathcal{V})}(E^\bullet, \mathcal{C})$ and let $\iota_n\mathcal{C}$ denote the Kan complex $\mathrm{Map}(E^n, \mathcal{C})$ for every $n \in \mathbb{N}$. Given a $\mathcal{V}$-enriched ∞-operad $\mathcal{O}$, we write $\iota_n\mathcal{O}$ for $\iota_n u^*\mathcal{O}$, where $u^*\mathcal{O}$ denotes the underlying $\mathcal{V}$-enriched ∞-category of $\mathcal{O}$.

Definition 3.14. Let $\mathcal{V}$ be a symmetric monoidal ∞-category and let $\mathcal{C} \colon \Delta^{\mathrm{op}}_X \to \mathcal{V}$ be a $\mathcal{V}$-enriched ∞-category. We say that $\mathcal{C}$ is *complete* if the canonical map $\iota_0\mathcal{C} \to \iota_1\mathcal{C}$ is an equivalence. If $\mathcal{O} \colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$ is a $\mathcal{V}$-enriched Φ-∞-operad then $\mathcal{O}$ is said to be *complete* if its underlying $\mathcal{V}$-enriched ∞-category as defined in Definition 3.3 $u^*\mathcal{O}$ is complete. We write $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ for the full subcategory of $\mathcal{V}$-enriched Φ-∞-operads spanned by complete objects.

Using the notation of the previous definition, an ∞-operad $\mathcal{O}$ is complete if and only if $\iota_0\mathcal{O} \to \iota_1\mathcal{O}$ is an equivalence.

Remark 3.15. *If Φ is the trivial operator category $*$, then, as shown in [GH15], the ∞-category* $\mathrm{Alg}_{*,\mathrm{cp}}(\mathcal{V})$ *is the correct ∞-category of $\mathcal{V}$-enriched ∞-categoy. Therefore, we will write* $\mathrm{Cat}^{\mathcal{V}}_\infty$ *for* $\mathrm{Alg}_{*,\mathrm{cp}}(\mathcal{V})$.

Since every object in $\mathcal{S}$ is given by a colimit of a constant diagram at the terminal object $* \in \mathcal{S}$, there exists a unique colimit preserving functor $f \colon \mathcal{S} \to \mathcal{V}_{\langle 1 \rangle}$ sending $*$ to the unit $\mathbb{1}$ of $\mathcal{V}$. The adjoint functor theorem (see [Lur09, Definition 5.5.2.9]) and the requirement that the symmetric monoidal ∞-category $\mathcal{V}$ is presentable imply that f has a right adjoint $g \colon \mathcal{V}_{\langle 1 \rangle} \to \mathcal{S}$ given by

$\mathrm{Map}_{\mathcal{V}_{\langle 1\rangle}}(\mathbb{1},-)$. The adjoint pair f,g can be extended to an adjunction between symmetric monoidal ∞-categories

$$F\colon \mathcal{S}^\times \rightleftarrows \mathcal{V} : G$$

such that F is symmetric monoidal and G is lax symmetric monoidal. Then [GH15, Proposition 3.6.18] implies that the adjunction (F,G) induces an adjunction

$$F_*\colon\ \mathrm{Alg}_*(\mathcal{S}) \rightleftarrows \mathrm{Alg}_*(\mathcal{V}) : G_*,$$

which allows us to indentify $E^{\mathcal{V}}_{\{1,\ldots,n\}} \in \mathrm{Alg}_*(\mathcal{V})$ with the image of $E^{\mathcal{S}}_{\{1,\ldots,n\}} \in \mathrm{Alg}_*(\mathcal{S})$ under F_*. The equivalence

$$\mathrm{Map}_{\mathrm{Alg}_*(\mathcal{V})}(E^{\mathcal{V}}_{\{0,\ldots,n\}},\mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Alg}_*(\mathcal{S})}(E^{\mathcal{S}}_{\{0,\ldots,n\}},G_*\mathcal{C})$$

implies that the $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ is complete, if and only if its underlying ∞-category $G_*\mathcal{C}$ is complete in the sense of Rezk [Rez01].

3.2 $\Delta_\Phi^{\mathrm{op}}$-Monoid Objects

We will prove in Proposition 3.21 that the definition of the Cartesian symmetric monoidal ∞-category of spaces allows us to identify an $\mathcal{S}$-enriched Φ-∞-operad with a $\Delta_\Phi^{\mathrm{op}}$-monoid object. The proof of this proposition can be thought of as a $\Delta_\Phi^{\mathrm{op}}$-version of [Lur, Proposition 2.4.1.7] which shows among others things that for any ∞-operad $\mathcal{O}$ an $\mathcal{O}$-algebra in the Cartesian symmetric monoidal ∞-category in spaces is equivalent to an $\mathcal{O}$-monoid in spaces. This result will be essential for the proof of Theorem 3.24 in the following section.

Before we get to the proof of Proposition 3.21, we would like to recall the definition of Cartesian symmetric monoidal ∞-categories in 3.16 and discuss their properties in Proposition 3.20.

Definition 3.16. [Lur, Definition 2.4.1.1] If $p\colon \mathcal{C} \to \mathcal{F}_*$ is an ∞-operad and $\mathcal{D}$ is an ∞-category, we call a functor $\pi\colon \mathcal{C} \to \mathcal{D}$ of ∞-categories a *Cartesian structure* if the following conditions are satisfied:

1. If $x \in \mathcal{C}_{\langle n\rangle}$, $x \simeq x_1 \oplus \ldots \oplus x_n$ with each $x_i \in \mathcal{C}_{\langle 1\rangle}$, then the canonical maps $\pi(x) \to \pi(x_i)$ exhibit $\pi(x)$ as a product $\prod_{1\le i\le n}\pi(x_i)$ in the ∞-category $\mathcal{D}$.
2. If $f\colon x \to y$ is a p-coCartesian morphism lying over an active morphism α in $\mathcal{F}_*$ then $\pi(f)$ is an equivalence in the ∞-category $\mathcal{D}$.
3. The functor π restricts to an equivalence $\mathcal{C}_{\langle 1\rangle} \xrightarrow{\simeq} \mathcal{D}$.

A *Cartesian symmetric monoidal ∞-category* is defined to be a symmetric monoidal ∞-category $p\colon \mathcal{C} \to \mathcal{F}_*$ together with a Cartesian structure π.

The Cartesian product functor $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ of the ∞-category $\mathcal{S}$ of ∞-groupoid induces not only a symmetric monoidal structure on $\mathcal{S}$ but also a Cartesian structure. Hence, the symmetric monoidal ∞-category $\mathcal{S}$ is an example of a Cartesian symmetric monoidal ∞-category. In the following, we define the category $\Gamma^\times$, which allows us to construct Cartesian symmetric monoidal ∞-categories out of ∞-categories which admit finite products. This will be done in Proposition 3.20.

Definition 3.17. [Lur, Notation 2.4.1.2] Let $\Gamma^{\times}$ denote the category given by the following data:

1. The objects in $\Gamma^{\times}$ are of the form $(\langle m\rangle, S)$ with $\langle m\rangle \in \mathcal{F}_*$ and S a subset of $\langle m\rangle^{\circ}$.
2. A morphism $f\colon (\langle m\rangle, S) \to (\langle n\rangle, T)$ in $\Gamma^{\times}$ is given by a map $\alpha\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}_*$ such that the preimage $\alpha^{-1}(T)$ is a subset of S.

Remark 3.18. *It is easy to check that the forgetful functor $\Gamma^{\times} \to \mathcal{F}_*$ is a Grothendieck fibration and admits a canonical section s given by $\langle m\rangle \mapsto (\langle m\rangle, \langle m\rangle^{\circ})$. In particular, its nerve is a Cartesian fibration.*

Definition 3.19. For an ∞-category $\mathcal{C}$, let $\widetilde{\mathcal{C}}^{\times}$ be the simplicial set together with the map $\widetilde{\mathcal{C}}^{\times} \to \mathcal{F}_*$ determined by the following universal property: For every map $K \to \mathcal{F}$ in sSet there is a bijection

$$\mathrm{Hom}_{\mathrm{sSet}_{/\mathcal{F}_*}}(K, \widetilde{\mathcal{C}}^{\times}) \simeq \mathrm{Hom}_{\mathrm{sSet}}(K \times_{\mathcal{F}_*} \Gamma^{\times}, \mathcal{C}).$$

For an object $\langle m\rangle \in \mathcal{F}_*$, the fibre $\widetilde{\mathcal{C}}^{\times}_{\langle m\rangle}$ is given by the ∞-category $\mathrm{Fun}(\{\langle m\rangle\} \times_{\mathcal{F}_*} \Gamma^{\times}, \mathcal{C})$, where $\{\langle m\rangle\} \times_{\mathcal{F}_*} \Gamma^{\times}$ can be identified with the opposite category of the partially ordered sets consisting of subsets of $\langle m\rangle^{\circ}$.

If an object $k \in K$ lies over $\langle m\rangle$, then we write (k, S) for the object $(k, (\langle m\rangle, S))$ in $K \times_{\mathcal{F}_*} \Gamma^{\times}$. Let $\mathcal{C}^{\times}$ denote the full subcategory of $\widetilde{\mathcal{C}}^{\times}$ spanned by objects corresponding to functors $f \in \mathrm{Fun}(k \times_{\mathcal{F}_*} \Gamma^{\times}, \mathcal{C})$ where k lies over $\langle m\rangle \in \mathcal{F}_*$ and such that for every subset $S \subseteq \langle m\rangle$ and $s \in S$, the maps $f(k, S) \to f(k, \{s\})$ exhibit $f(k, S)$ as a product of the objects $\{f(k, \{s\})\}$, $s \in S$, in the ∞-category $\mathcal{C}$.

Proposition 3.20. *[Lur, 2.4.1.5] Let $\mathcal{C}$ be an ∞-category.*

1. *The map $\widetilde{p}\colon \widetilde{\mathcal{C}}^{\times} \to \mathcal{F}_*$ is a coCartesian fibration. If $\overline{\alpha}\colon f \to g$ is a morphism in $\widetilde{\mathcal{C}}^{\times}$ lying over a morphism $\langle m\rangle \to \langle n\rangle$ in $\mathcal{F}_*$, then $\overline{\alpha}$ is p-coCartesian if and only if the induced map $f(\langle m\rangle, \alpha^{-1}T) \to g(\langle n\rangle, T)$ is an equivalence in $\mathcal{C}$ for every subset $T \subseteq \langle n\rangle^{\circ}$.*
2. *The map $\widetilde{p}$ restricts to a coCartesian fibration $p\colon \mathcal{C}^{\times} \to \mathcal{F}_*$ which has the same class of coCartesian morphisms.*
3. *The map $p\colon \mathcal{C}^{\times} \to \mathcal{F}_*$ is a Cartesian symmetric monoidal ∞-category if and only if the ∞-category $\mathcal{C}$ admits finite products. The Cartesian structure $\pi\colon \mathcal{C}^{\times} \to \mathcal{C}$ is given by composition with the section $s\colon \mathcal{F}_* \to \Gamma^{\times}$ defined in Remark 3.18.*

The proposition above provides many examples for Cartesian symmetric monoidal ∞-categories: For each ∞-category $\mathcal{C}$, there exists an associated Cartesian symmetric monoidal ∞-category $\mathcal{C}^{\times}$ as constructed in Definition 3.19.

Proposition 3.21. *Let $\mathcal{C}^{\times}$ be a Cartesian symmetric monoidal ∞-category. For $p\colon \mathcal{O} \to \Delta^{\mathrm{op}}_{\Phi}$ in $\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}}$, the ∞-category $\mathrm{Alg}_{\mathcal{O}/\Delta^{\mathrm{op}}_{\Phi}}(\mathcal{C}^{\times})$ is equivalent to the ∞-category $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-monoid objects in $\mathcal{C}$.*

Proof. The proof of this proposition is divided into three parts. In the first part we basically unwind Definition 3.19 to show that the ∞-category $\mathrm{Alg}_{\mathcal{O}/\Delta^{\mathrm{op}}_{\Phi}}(\mathcal{C}^{\times})$ can be regarded as the full subcategory of $\mathrm{Fun}(\mathrm{Cr}_{\Phi,!}(\mathcal{O}) \times_{\mathrm{N}\,\mathcal{F}_*} \mathrm{N}\,\Gamma^\times, \mathcal{C})$ spanned by the functors F satisfying the two conditions (a) and (b) below. Afterwards, we show that every object in $\mathrm{Mon}_\mathcal{O}(\mathcal{C})$ can be extended via Kan extension to an object in an ∞-category $\mathcal{B}$ and that every object in $\mathcal{B}$ restricts to an $\mathcal{O}$-monoid object in $\mathcal{C}$. Then Corollary [Lur09, 4.3.2.15] provides an equivalence $\mathcal{B} \xrightarrow{\simeq} \mathrm{Mon}_\mathcal{O}(\mathcal{C})$. In the last part of the proof we verify that the defining conditions (a') and (b') of $\mathcal{B}$ are equivalent to conditions (a) and (b), which implies that $\mathcal{B} \simeq \mathrm{Mon}_\mathcal{O}(\mathcal{C})$ is also equivalent to $\mathrm{Alg}_{\mathcal{O}/\Delta_\Phi^{\mathrm{op}}}(\mathcal{C}^\times)$.

1. For $\mathcal{O}$ and $\mathcal{C}^\times$ as above, the adjunction $\mathrm{Cr}_{\Phi,!}\colon \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}} \rightleftarrows \mathrm{Op}_\infty : \mathrm{Cr}_\Phi^*$ induces an equivalence
$$\mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}/\Delta_\Phi^{\mathrm{op}}}(\mathrm{Cr}_\Phi^*\,\mathcal{C}^\times) \simeq \mathrm{Alg}_{\mathrm{Cr}_{\Phi,!}(\mathcal{O})/\mathcal{F}_*}(\mathcal{C}^\times).$$
Since $\mathcal{C}^\times$ is a Cartesian symmetric monoidal ∞-category, Definition 3.19 implies that $\mathrm{Alg}_{\mathrm{Cr}_{\Phi,!}(\mathcal{O})/\mathcal{F}_*}(\mathcal{C}^\times)$ lies in the full subcategory of $\mathrm{Fun}(\mathrm{Cr}_{\Phi,!}(\mathcal{O}) \times_{\mathrm{N}\,\mathcal{F}_*} \mathrm{N}\,\Gamma^\times, \mathcal{C})$ spanned by functors F satisfying the following property:
   (a) For every object (x, S) in $\mathrm{Cr}_{\Phi,!}(\mathcal{O}) \times_{\mathrm{N}\,\mathcal{F}_*} \mathrm{N}\,\Gamma^\times$, the canonical map
   $$F(x,S) \to \prod_{s\in S} F(x,\{s\})$$
   is an equivalence.

   Since the ∞-category $\mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}/\Delta_\Phi^{\mathrm{op}}}(\mathrm{Cr}_\Phi^*\,\mathcal{C}^\times)$ consists of functors which preserve inert morphisms, Proposition 3.20 implies that we can regard $\mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}/\Delta_\Phi^{\mathrm{op}}}(\mathrm{Cr}_\Phi^*\,\mathcal{C}^\times) \simeq \mathrm{Alg}_{\mathrm{Cr}_{\Phi,!}(\mathcal{O})/\mathcal{F}_*}(\mathcal{C}^\times)$ as the full subcategory of $\mathrm{Fun}(\mathrm{Cr}_{\Phi,!}(\mathcal{O}) \times_{\mathrm{N}\,\mathcal{F}_*} \mathrm{N}\,\Gamma^\times, \mathcal{C})$ satisfying the following condition in addtion to (a):
   (b) If $\tilde{\alpha}\colon x \to y$ is an inert morphism in $\mathcal{O}$ such that $(\mathrm{Cr}_\Phi \circ p)(\tilde{\alpha})$ is given by $\alpha\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}_*$, then the induced map $F(x, \alpha^{-1}(S)) \to F(y, S)$ is an equivalence in $\mathcal{C}$ for every subset $S \subseteq \langle n\rangle^\circ$.

2. Let $\mathcal{A}$ denote the full subcategory of $\mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times$ spanned by all objects of the form $(x, \langle m\rangle^\circ)$ such that $(\mathrm{Cr}_\Phi \circ p)(x) = \langle m\rangle$. This ∞-category $\mathcal{A}$ is obviously equivalent to $\mathcal{O}$.

   Let $\mathcal{B}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times, \mathcal{C})$ spanned by those functors F which satisfy the following two conditions:
   (a') The restriction $F_{|\mathcal{A}}$ is an $\mathcal{O}$-monoid object in $\mathcal{C}$.
   (b') The functor F is a right Kan extension of the functor $F_{|\mathcal{A}}$.

   The inclusion $\mathcal{A} \hookrightarrow \mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times$ induces a restriction functor $\mathcal{B} \to \mathrm{Mon}_\mathcal{O}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{O}, \mathcal{C})$. Now we want to show that every functor $F\colon \mathcal{A} \simeq \mathcal{O} \to \mathcal{C}$ admits a right Kan extension along the inclusion $\mathcal{A} \hookrightarrow \mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times$. By [Lur09, Lemma 4.3.2.13], we only need to verify that for every object $(x, S) \in \mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times$, the diagram $\mathcal{A}_{(x,S)/} \to \mathcal{A} \xrightarrow{F} \mathcal{C}$ admits a limit. In the following we show that $\mathcal{A}_{(x,S)/}$ admits a subcategory $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ such that the inclusion functor $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \hookrightarrow \mathcal{A}_{(x,S)/}$ is coinitial. Here, we call a functor $g\colon \mathcal{E} \to \mathcal{E}'$ coinitial if and only if $g^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to (\mathcal{E}')^{\mathrm{op}}$ is cofinal. This implies that the induced functor $\mathcal{A}_{(x,S)/} \to \mathcal{A}$ admits a limit if and only if its restriction $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{A}$ admits a limit and we will show below that the restriction always admits a limit.

Let $\phi\colon (x,S) \to (y,T)$ be an object of $\mathcal{A}_{(x,S)/}$. Then ϕ is induced by a morphism $\tilde{\alpha}\colon x \to y$ in $\mathcal{O}$ lying over $\alpha\colon \langle m\rangle \to \langle n\rangle$. The inert-active factorization system on $\Delta_\Phi^{\mathrm{op}}$ introduced in Remark 2.24 and the fact that $\mathcal{O} \in \mathrm{coCart}_{\mathrm{Seg}}^{\Phi,\mathrm{gen}}$ yield that $\tilde{\alpha}$ can be factorized as $x \overset{\tilde{\alpha}_1}{\to} y' \overset{\tilde{\alpha}_2}{\to} y$ in $\mathcal{O}$ where $\tilde{\alpha}_1$ is inert in $\mathcal{O}$ and $p(\tilde{\alpha}_2)$ is active in $\Delta_\Phi^{\mathrm{op}}$. Let $\alpha_1\colon \langle m\rangle \to \langle k\rangle$ and $\alpha_2\colon \langle k\rangle \to \langle n\rangle$ be the images of $p(\tilde{\alpha}_1)$ and $p(\tilde{\alpha}_2)$ in $\mathcal{F}_*$ under the functor Cr_Φ.

Since $p(\tilde{\alpha}_2)$ is active and $T = \langle n\rangle^\circ$, we have $\langle k\rangle^\circ = \tilde{\alpha}_2^{-1}(T)$, which implies that $\tilde{\alpha}_1^{-1}(\langle k\rangle^\circ) = \alpha^{-1}(T) \subseteq S$. Hence, the morphisms $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ induce maps ϕ_1 and ϕ_2 in $\mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times$ such that the following diagram commutes.

$$\begin{tikzcd} & (x,S) \arrow[dl, "\phi_1"'] \arrow[dr, "\phi"] & \\ (y', \langle k\rangle^\circ) \arrow[rr, "\phi_2"'] & & (y,T) \end{tikzcd} \tag{3.5}$$

Let $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ denote the full subcategory of $\mathcal{A}_{(x,S)/}$ spanned by those objects whose image in $\mathcal{O}$ is an inert morphism. The construction above provides an object ϕ_1 in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ for each object $\phi \in \mathcal{A}_{(x,S)/}$. Hence, the category ${(\mathcal{A}_{(x,S)/})^{\mathrm{in}}}_{/\phi}$ is non-empty. Let (ϕ_1,ϕ_2) be an object in ${(\mathcal{A}_{(x,S)/})^{\mathrm{in}}}_{/\phi}$ given by a commutative diagram of the form 3.5 such that the image of ϕ_1 in $\mathcal{O}$ is inert. The uniqueness of the inert-active factorization of maps in $\Delta_\Phi^{\mathrm{op}}$ implies that the image of the morphism ϕ_2 in $\mathcal{O}$ can be factorized uniquely into an inert map followed by an active map. This shows that the ∞-category ${(\mathcal{A}_{(x,S)/})^{\mathrm{in}}}_{/\phi}$ has a final object which is induced by the inert-active factorization of the image of the map ϕ in $\mathcal{O}$. Furthermore, the dual of [Lur09, Theorem 4.1.3.1] implies that the inclusion $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \hookrightarrow \mathcal{A}_{(x,S)/}$ is coinitial. Therefore, as mentioned above, it suffices to verify that the restriction $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{C}$ admits limits in order to show that the diagram $\mathcal{A}_{(x,S)/} \to \mathcal{C}$ does.

A morphism in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ is given by a commutative diagram of the form 3.5. Then [Lur09, Lemma 2.4.2.7] implies that the images of the morphisms of this diagram in $\mathcal{O}$ have to be inert. Since the space of inert morphisms between each tow objects in $\mathcal{O}$ is contractible, we obtain that every mapping space in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ is trivial. Hence, the ∞-category $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ is a poset.

It follows from the construction that each object in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ is given by an inert map of the form $\phi\colon x \to y$ in $\mathcal{O}$. Let $\alpha\colon ([m'],I) \to ([n'],J)$ denote the projection of ϕ in $\Delta_\Phi^{\mathrm{op}}$ and let $\mathrm{Cr}_\Phi(\alpha)$ be the map in $\mathcal{F}_*$ given by $\langle m\rangle \to \langle n\rangle$. Since α is inert, the map $\mathrm{Cr}_\Phi(\alpha)$ has also to be inert as well, i.e. it induces an inclusion of sets $\langle n\rangle^\circ \hookrightarrow \langle m\rangle^\circ$ (see Definition 2.17). By identifying $\langle n\rangle^\circ$ with the set of corollas in y, the definition of $\mathcal{A}_{(x,S)/}$ then implies that this inclusion is of the form $\langle n\rangle^\circ \hookrightarrow S \subseteq \langle m\rangle^\circ$. As shown above, the poset $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ only contains morphisms whose images in $\mathcal{O}$ are inert. This means that a morphism in the poset $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ is given by an inclusion of subsets of S. Therefore, if $\mathrm{Po}(S)$ denotes the poset of subsets of S, then the poset $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ can be identified with a subcategory $\mathrm{P}(S)$ in $\mathrm{Po}(S)^{\mathrm{op}}$. We claim that $\mathrm{P}(S)$ is a full subcategory. Let $\phi_1, \phi_2 \in (\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ be given by $\phi_1\colon (x,S) \to (y_1, \langle n_1\rangle^\circ)$ and $\phi_2\colon (x,S) \to (y_2, \langle n_2\rangle^\circ)$, respectively. Given a map $\langle n_1\rangle^\circ \to \langle n_2\rangle^\circ$ in $\mathrm{Po}(S)^{\mathrm{op}}$, we have to prove the existence of a map $\phi\colon \phi_1 \to \phi_2$ in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$. Let the projection of ϕ_1 and ϕ_2 in Δ_Φ be given by the respective maps $(\alpha_1, \psi_1)\colon ([n_1'], J_1) \to ([m], I)$ and $(\alpha_2, \psi_2)\colon ([n_2'], J_2) \to ([m], I)$ such that $\mathrm{Cr}_\Phi([n_1'], J_1) = \langle n_1 \rangle$ and $\mathrm{Cr}_\Phi([n_2'], J_2) = \langle n_2 \rangle$. We first claim the existence of a dotted arrow rendering the following diagram in Δ_Φ commutative.

$$\begin{tikzcd}
([n_2'], J_2) \arrow[rr, dotted] \arrow[dr, "(\alpha_2,\psi_2)"'] & & ([n_1'], J_1) \arrow[dl, "(\alpha_1,\psi_1)"] \\
& ([m], I) &
\end{tikzcd}$$

The map $\langle n_1 \rangle^\circ \to \langle n_2 \rangle^\circ$ in $\mathrm{Po}(S)^{\mathrm{op}}$ implies that $\langle n_2 \rangle^\circ \subseteq \langle n_1 \rangle^\circ$. Hence, there exists an inert map $\alpha_0\colon [n_2'] \to [n_1']$ in Δ such that $\alpha_2 = \alpha_1 \circ \alpha_0$. The inclusion of corollas $\langle n_2 \rangle^\circ \subseteq \langle n_1 \rangle^\circ$ and Definition 2.6 imply that the map $\psi_2(n_2')\colon J_2(n_2') \hookrightarrow I(\alpha_2(n_2'))$ factors through $\psi_1(\alpha_0(n_2'))\colon J_1(\alpha_0(n_2')) \to I(\alpha_2(n_2'))$. We then obtain an inert map $(\alpha_0, \psi_0)\colon ([n_2'], J_2) \to ([n_1'], J_1)$ in Δ_Φ by pullback. The coCartesian lift of this map in $\mathcal{O}$ is the unique inert map $y_1 \to y_2$ which induces a map $\phi\colon \phi_1 \to \phi_2$ in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$. Thus, the poset $\mathrm{P}(S)$ is a full subcategory of $\mathrm{Po}(S)^{\mathrm{op}}$ spanned by objects of the form $(\mathrm{Cr}_\Phi \circ p)(y)$, where $x \to y \in (\mathcal{A}_{(x,S)/})^{\mathrm{in}}$.

Since the map $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{C}$ is given by the composite $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{A} \xrightarrow{F} \mathcal{C}$ and $F_{|\mathcal{A}}$ is an $\mathcal{O}$-monoid object in $\mathcal{C}$, Remark 3.8 implies that F carries the object $\phi\colon x \to y$ in $(\mathcal{A}_{(x,S)/})^{\mathrm{in}}$ to the product

$$\prod_{\mathfrak{c} \in (\mathrm{Cr}_\Phi \circ p)(y)} F(\mathfrak{c}).$$

This finite product exists in $\mathcal{C}$ by the assumption that the ∞-operad $\mathcal{C}^\times$ is Cartesian. The observation above implies that the functor $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{C}$ can be identified with the functor $\mathrm{P}(S) \to \mathcal{C}$ which maps a set $(\mathrm{Cr}_\Phi \circ p)(y)$ to the product $\prod_{\mathfrak{c} \in (\mathrm{Cr}_\Phi \circ p)(y)} F(\mathfrak{c})$ and each morphism in $\mathrm{P}(S)$ to the map induced by the inclusion of the corresponding sets of corollas. It follows from the construction that the set S which corresponds to a collection of corollas in x is given by the union

$$\bigcup_{(\mathrm{Cr}_\Phi \circ p)(y) \in \mathrm{P}(S)} (\mathrm{Cr}_\Phi \circ p)(y)$$

of corollas. For every $s \in S$, it is clear that there is a corresponding object $(z, \{s\}) \in (\mathcal{A}_{(x,S)/})^{\mathrm{in}}$, where $z \in \mathcal{O}$ lies over the corolla $\mathfrak{c}$ associated to s, lies in $\mathrm{P}(s)$. By [Arn16, Corollary 2.3.3], the limit of $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{C}$ is given by the product $\prod_{s \in S} F(s) = \prod_{s \in S} F(s, \langle 1 \rangle^\circ)$.

By [Lur09, Lemma 4.3.2.13], the right Kan extension carries each object $F \in \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ to an object in $\mathrm{Fun}(\mathcal{O} \times_{\mathcal{F}_*} \Gamma^\times, \mathcal{C})$ and by Corollary [Lur09, 4.3.2.15], the restriction functor $\mathcal{B} \to \mathrm{Mon}_{\mathcal{O}_{\mathcal{F}}}(\mathcal{C})$ is a trivial fibration.

3. Therefore, the proof is complete, provided the assertions (a) and (b) are equivalent to the assertions (a') and (b').

   For an object $(x, \langle m \rangle^\circ) \in \mathcal{A}$ and an integer i with $1 \leq i \leq m$ we let $\mathfrak{c}^i \in \Delta_\Phi^{\mathrm{op}}$ denote the corolla corresponding to $i \in \langle m \rangle^\circ$. Then the restriction $F_{|\mathcal{A}}$ is an $\mathcal{O}$-monoid object in $\mathcal{C}$ if and only if the p-coCartesian lifts $x \to x_{\mathfrak{c}^i}$ of the inert maps $p(x) \to \mathfrak{c}^i \in \Delta^{\mathrm{op}}_{\Phi}$ induce an equivalence

$$F(x, \langle m \rangle^\circ) \xrightarrow{\simeq} \prod_i F(x_{\mathfrak{c}^i}, \langle 1 \rangle^\circ).$$

Since (b) holds and $x \to x_{\mathfrak{c}^i}$ is inert, the morphism $F(x, \{\mathfrak{c}^i\}) \to F(x_{\mathfrak{c}^i}, \langle 1 \rangle^\circ)$ is an equivalence in $\mathcal{C}$. This equivalence and condition (a) now yield that the map $F(x, \langle m \rangle^\circ) \to \prod_i F(x, \{\mathfrak{c}^i\}) \simeq \prod_i F(x_{\mathfrak{c}^i}, \langle 1 \rangle^\circ)$ is an equivalence as well.

Given an object (x, S), the limit of $(\mathcal{A}_{(x,S)/})^{\mathrm{in}} \to \mathcal{C}$ is given by the product $\prod_{s \in S} F(s, \langle 1 \rangle^\circ)$. Therefore, the description of right Kan extensions implies that condition (b') holds if and only if the induced map

$$F(x, S) \to \prod_{s \in S} F(s, \langle 1 \rangle^\circ)$$

is an equivalence in $\mathcal{C}$. But this map is the composite of the equivalence $F(x, S) \simeq \prod_{s \in S} F(S, \langle 1 \rangle^\circ)$ provided by (a) and the equivalence $\prod_{s \in S} F(S, \langle 1 \rangle^\circ) \simeq \prod_{s \in S} F(s, \langle 1 \rangle^\circ)$ given by condition (b).

Now suppose (a') and (b') are satisfied. We need to show that they imply conditions (a) and (b). It follows from above and (b') that the right Kan extension F induces an equivalence $F(x, S) \simeq \prod_{s \in S} F(s, \langle 1 \rangle^\circ)$ for every object (x, S). To verify condition (a) it is therefore sufficient to show that $F(s, \langle 1 \rangle^\circ)$ is equivalent to $F(x, \{s\})$. But this already follows from the fact that F is a right Kan extension of its restriction $F_{|\mathcal{A}}$.

For condition (b) we have to show that every inert map $x \to y$ in $\mathcal{O}$ induces an equivalence $F(x, \alpha^{-1}(S)) \to F(y, S)$ in $\mathcal{C}$. Since the map $x \to y$ is inert, we have an isomorphism of sets $\alpha^{-1}(S) \cong S$. In this case the conditions (a') and (b') imply that both objects $F(x, \alpha^{-1}(S))$ as well as $F(y, S)$ are equivalent to $\prod_{s \in S} F(s, \langle 1 \rangle^\circ)$ in $\mathcal{C}$.

□

Since we want to show that Φ-∞-operads enriched over ∞-groupoids are essentially Segal Φ-operads, we only need to study the case where the Cartesian ∞-operad $\mathcal{C}^\times$ of Proposition 3.21 equals $\mathcal{S}^\times$.

3.3 Enrichment over Spaces

The aim of this section is to verify Theorem 3.24, which states that complete $\mathcal{S}$-enriched Φ-∞-operads are equivalent to Barwick's Segal Φ-operads as defined in Definition 3.22.

The strategy for the proof of the theorem is as follows: We first verify in Proposition 3.29 that the $\Delta^{\mathrm{op}}_{\Phi}$-monoid objects can be identified with objects in the ∞-category $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi}, \mathcal{S})$. Following the idea of the proof, the next corollary then reveals that the ∞-category of $\Delta^{\mathrm{op}}_{\Phi}$-monoid objects is actually equivalent to $\mathrm{Seg}_{\Phi,X} \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi}, \mathcal{S})$, which can be thought of as Segal Φ-operads with the fixed space of objects X. Using this, it is not hard to prove Theorem 3.24 at the end of this section. As an immediate consequence of this theorem we obtain that complete $\mathcal{S}$-enriched Φ-∞-operads are equivalent to all known models of ∞-operads by using some results of [CHH16].

Let us first recall the definition of complete Segal Φ-operads as introduced in [Bar13].

Definition 3.22. For an object $([m], I) \in \Delta_\Phi^{\mathrm{op}}$, let $p_{([m],I)}\colon (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/} \to \Delta_\Phi^{\mathrm{op}}$ denote the canonical inclusion. Define Seg_Φ to be the full subcategory of $\mathrm{Fun}(\Delta_\Phi^{\mathrm{op}}, \mathcal{S})$ spanned by those functors F such that the canonical map

$$F([m], I) \to \lim_{x \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}} F p_{([m],I)}(x)$$

is an equivalence for every object $([m], I) \in \Delta_\Phi^{\mathrm{op}}$. The objects in Seg_Φ are called *Segal Φ-operads*. For $X \in \mathcal{S}$, let $\mathrm{Seg}_{\Phi,X}$ denote the subcategory of Seg_Φ spanned by those objects which carry $\mathfrak{e} \in \Delta_\Phi^{\mathrm{op}}$ to X together with those natural transformations between these functors which reduce to identities at $\mathfrak{e}$.

Definition 3.23. Let $u^*\colon \mathrm{P}(\Delta_\Phi) \to \mathrm{P}(\Delta)$ be the restriction functor given by the precomposition with the functor $u\colon \Delta \simeq \Delta_* \to \Delta_\Phi$ induced by the inclusion of the terminal object $\{*\} \hookrightarrow \Phi$. We call a Segal Φ-operad $\mathcal{O}$ *complete* if $u^*\mathcal{O}$ is a complete Segal space in the sense of [Rez01]. We write $\mathrm{Seg}_{\Phi,\mathrm{cp}}$ and $\mathrm{Seg}_{\Phi,\mathrm{cp},X}$ for the full subcategory of Seg_Φ and $\mathrm{Seg}_{\Phi,X}$ spanned by the complete Segal Φ-operads.

Theorem 3.24. *If $\mathcal{S}^\times$ denotes the Cartesian symmetric monoidal ∞-category (see Definition 3.19) of spaces, then there exists an equivalence of ∞-categories*

$$\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{S}^\times) \simeq \mathrm{Seg}_{\Phi,\mathrm{cp}}.$$

The proof of the theorem will be presented at the end of this section.

Definition 3.25. [Lur09, Definition 6.1.3.1] Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $p, q\colon K \to \mathcal{C}$ be two functors. We call a natural transformation $\tau\colon p \to q$ a *Cartesian natural transformation* if every edge $f\colon x \to y$ in K induces a pullback square in $\mathcal{C}$:

$$\begin{tikzcd}
p(x) \arrow[r, "p(f)"] \arrow[d, "\tau(x)"'] & p(y) \arrow[d, "\tau(y)"] \\
q(x) \arrow[r, "q(f)"] & q(y).
\end{tikzcd}$$

Theorem 3.26. *[Lur09, Theorem 6.1.3.9] Let $\mathcal{C}$ be an ∞-topos, let K be a small simplicial set and $\tau\colon p \to q$ be a natural transformation between two functors $p, q\colon K^\triangleright \to \mathcal{C}$. If q is a colimit diagram and the restriction $\tau_{|K}$ is a Cartesian transformation, then the functor p is a colimit diagram if and only if τ is a Cartesian transformation.*

In the proof of the next proposition we will use the following two inconspicuous lemmata.

Lemma 3.27. *If $F\colon \Delta_{\Phi,X}^{\mathrm{op}} \to \mathcal{S}$ is a $\Delta_{\Phi,X}^{\mathrm{op}}$-monoid, then there exists a canonical equivalence $\pi_{X,!}F(\mathfrak{e}) \simeq X$ in $\mathcal{S}$.*

Proof. Suppose $\widehat{F}\colon \mathcal{C} \to \Delta_{\Phi,X}^{\mathrm{op}}$ denotes the left fibration associated to F. Then the left Kan extension $\pi_! F(\mathfrak{e})\colon \Delta_\Phi^{\mathrm{op}} \to \mathcal{S}$ corresponds to the left fibration given by the composite $\pi_X \circ \widehat{F}\colon \mathcal{C} \to$

$\Delta^{\mathrm{op}}_{\Phi}$. We then obtain a commutative diagram

$$\begin{tikzcd}
\pi_! F(\mathfrak{e}) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \{\mathfrak{e}\} \arrow[d] \\
\mathcal{C} \arrow[r, "\widehat{F}"'] & \Delta^{\mathrm{op}}_{\Phi,X} \arrow[r, "\pi_X"'] & \Delta^{\mathrm{op}}_{\Phi},
\end{tikzcd}$$

where the two small squares are Cartesian. As a pullback of a left fibration, the left upper horizontal map is also a left fibration and its fibre over $x \in X$ can be identified with $F(x) \in \mathcal{S}$. Since the definition of monoid objects implies that $F(x) \simeq \{*\}$, we see that $\pi_! F(\mathfrak{e}) \to X$ is a left fibration with contractible fibres, which then has to be a trivial fibration by [Lur09, Lemma 2.1.3.4]. $\square$

Lemma 3.28. *For every* $([m], I) \in \Delta^{\mathrm{op}}_{\Phi}$*, the canonical inclusion*

$$\Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta_\Phi} \{([m], I)\} \simeq \Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta_\Phi} \{\mathrm{id}_{([m],I)}\} \hookrightarrow \Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta^{\mathrm{op}}_{\Phi}} (\Delta^{\mathrm{op}}_{\Phi})_{/([m],I)} = (\Delta^{\mathrm{op}}_{\Phi,X})_{/([m],I)}$$

is cofinal.

Proof. If $\mathcal{C} := \Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta_\Phi} \{\mathrm{id}_{([m],I)}\}$ and $\mathcal{D} := \Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta^{\mathrm{op}}_{\Phi}} (\Delta^{\mathrm{op}}_{\Phi})_{/([m],I)}$, then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[r] \arrow[d] & \Delta^{\mathrm{op}}_{\Phi,X} \arrow[d] \\
\{([m], I)\} \arrow[r] & (\Delta^{\mathrm{op}}_{\Phi})_{/([m],I)} \arrow[r] & \Delta^{\mathrm{op}}_{\Phi},
\end{tikzcd}$$

where the big square and the right hand square are Cartesian. According to [Lur09, Theorem 4.1.3.1], to verify that the map $\mathcal{C} \to \mathcal{D}$ is cofinal, it suffices to show that the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{f/}$ is weakly contractible for every object $f \in \mathcal{D}$. The definition of $\mathcal{D}$ implies that f is given by a pair $(([n], J, \{x_i\}), f_0)$, where $([n], J, \{x_i\})$ is an object in $\Delta^{\mathrm{op}}_{\Phi,X}$ and $f_0\colon ([n], J) \to ([m], I)$ is an object in $(\Delta^{\mathrm{op}}_{\Phi})_{/([m],I)}$. By pullback we can extend the diagram above to the following commutative diagram

$$\begin{tikzcd}
& \mathcal{E} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{D}_{f/} \arrow[rr] \arrow[dl] \arrow[dd] & & (\Delta^{\mathrm{op}}_{\Phi,X})_{([n],J,\{x_i\})/} \arrow[dl] \arrow[dd] \\
\{f_0\} \arrow[rr] \arrow[dd] & & ((\Delta^{\mathrm{op}}_{\Phi})_{/([m],I)})_{f_0/} \arrow[rr] \arrow[dd] & & (\Delta^{\mathrm{op}}_{\Phi})_{([n],J)/} \arrow[dd] & \\
& \mathcal{C} \arrow[rr] \arrow[dl] & & \mathcal{D} \arrow[rr] \arrow[dl] & & \Delta^{\mathrm{op}}_{\Phi,X} \arrow[dl] \\
\{([m], I)\} \arrow[rr] & & (\Delta^{\mathrm{op}}_{\Phi})_{/([m],I)} \arrow[rr] & & \Delta^{\mathrm{op}}_{\Phi}, &
\end{tikzcd}$$

where every square is Cartesian by Lemma [Lur09, Lemma 4.4.2.1]. This means that $\mathcal{E}$ can be identified with $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{f/}$ and we only need to see that the upper left diagonal map $\mathcal{E} \to \{f_0\}$

is an equivalence. By [Lur09, Proposition 2.1.2.5], the left fibration $\Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_{\Phi}$ induces a trivial fibration $(\Delta^{\mathrm{op}}_{\Phi,X})_{([n],J,\{x_i\})/} \to (\Delta^{\mathrm{op}}_{\Phi})_{([n],J)/}$. Then the upper big horizontal square of the above diagram implies that this trivial fibration pulls back to $\mathcal{E} \to \{f_0\}$. Hence, the ∞-category $\mathcal{E} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{f/}$ is weakly contractible. □

The following proposition is a generalization of [GH15, Proposition 4.4.3].

Proposition 3.29. *Let X be an object in $\mathcal{S}$ and let $\pi_{X,!}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathcal{S}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi}, \mathcal{S})$ be the functor induced by the left Kan extension along the canonical projection map $\pi_X\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_{\Phi}$. Then a functor $F\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{S}$ is a $\Delta^{\mathrm{op}}_{\Phi,X}$-monoid if and only if $\pi_{X,!}F\colon \Delta^{\mathrm{op}}_{\Phi} \to \mathcal{S}$ is a Segal Φ-operad.*

Proof. To verify the claim of the proposition, we need to successively reformulate the condition for the functor $\pi_{X,!}F$ to be a Segal Φ-operad. Let $p_{([m],I)}\colon (\Delta^{\mathrm{el,op}}_{\Phi})_{([m],I)/} \to \Delta^{\mathrm{op}}_{\Phi}$ denote the canonical projection. By definition, the functor $\pi_{X,!}F$ is a Segal Φ-operad if and only if the canonical map

$$\pi_{X,!}F([m],I) \to \lim_{x\in(\Delta^{\mathrm{el,op}}_{\Phi})_{([m],I)/}} \pi_{X,!}F \circ p_{([m],I)}(x) =: \pi_{X,!}F^{\mathrm{Seg}}([m],I)$$

is an equivalence for every object $([m],I) \in \Delta^{\mathrm{op}}_{\Phi}$. It follows from the description of left Kan extensions along coCartesian fibrations that $\pi_{X,!}F([m],I)$ is given by a colimit of the diagram $(\Delta^{\mathrm{op}}_{\Phi,X})_{/([m],I)} \to \mathcal{S}$ induced by F.

If we write M for the set $\coprod_{k\in\mathbf{m}} |I(k)|$, then, by Remark 2.26, there is an equivalence $\Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta_\Phi} \{([m],I)\} \simeq X^M$. Since the canonical inclusion map $\Delta^{\mathrm{op}}_{\Phi,X} \times_{\Delta_\Phi} \{([m],I)\} \hookrightarrow (\Delta^{\mathrm{op}}_{\Phi,X})_{/([m],I)}$ is cofinal by the previous lemma, the object $\pi_{X,!}F([m],I)$ is given by a colimit of the restriction $F_{|X^M}\colon X^M \to \mathcal{S}$. This means that $\pi_{X,!}F$ is a Segal Φ-operad if and only if $\pi_{X,!}F^{\mathrm{Seg}}([m],I)$ is also a colimit of $F_{|X^M}\colon X^M \to \mathcal{S}$. We now want to show that this condition is equivalent to requiring the natural transformation τ as defined below to be Cartesian. Once this is proven, it is not difficult to see that τ is Cartesian if and only if F is a $\Delta^{\mathrm{op}}_{\Phi,X}$-monoid.

Since the $F_{|X^M}\colon X^M \to \mathcal{S}$ has a colimit $\pi_{X,!}F([m],I)$, the natural map $\pi_{X,!}F([m],I) \to \pi_{X,!}F^{\mathrm{Seg}}([m],I)$ allows us to extend $F_{|X^M}$ to a functor $F^{\triangleright}_{|X^M}\colon (X^M)^{\triangleright} \to \mathcal{S}$ such that $F^{\triangleright}_{|X^M}(\infty) = \pi_{X,!}F^{\mathrm{Seg}}([m],I)$. If we write $\mathrm{cst}\colon X^M \to \mathcal{S}$ for the constant functor mapping X^M to $\{*\} \in \mathcal{S}$, then we get an induced colimit diagram $\mathrm{cst}^{\triangleright}\colon (X^M)^{\triangleright} \to \mathcal{S}$ which carries the cone point ∞ to $X^M \in \mathcal{S}$. Let $\tau\colon F^{\triangleright}_{|X^M} \to \mathrm{cst}^{\triangleright}$ denote the obvious natural transformation. In particular, we have $\tau(\overline{x})\colon F_{|X^M}(\overline{x}) \to \{\overline{x}\} \cong \{*\}$ for every $\overline{x} \in X^M$ and $\tau(\infty)\colon \pi_{X,!}F^{\mathrm{Seg}}([m],I) \to X^M$.

Since every morphism $\overline{x} \to \overline{y}$ in the ∞-groupoid X^M is an equivalence, that commutative diagram

$$\begin{tikzcd} F_{|X^M}(\overline{x}) \arrow[r, "\simeq"] \arrow[d, "\tau(\overline{x})"'] & F_{|X^M}(\overline{y}) \arrow[d, "\tau(\overline{y})"] \\ \mathrm{cst}(\overline{x}) \arrow[r, "="] & \mathrm{cst}(\overline{y}) \end{tikzcd} \tag{3.6}$$

induced by τ is a pullback in $\mathcal{S}$. In other words, the restricted natural transformation $\tau_{|X^M}$ is Cartesian. Since $\mathcal{S}$ is an ∞-topos and $\mathrm{cst}^{\triangleright}$ is a colimit diagram by definition, we can apply Theorem 3.26 to the natural transformation $\tau\colon F^{\triangleright}_{|X^M} \to \mathrm{cst}^{\triangleright}$ and get that τ is Cartesian if and only if $F^{\triangleright}_{|X^M}$ is a colimit diagram, which is equivalent to saying that $\pi_{X,!}F$ is a Segal Φ-operad.

Since the diagram 3.6 is a pullback, the natural transformation τ is Cartesian if and only if, for every $\overline{x} \in X^M$, the unique map $\overline{x} \to \infty$ in X^M induces a pullback square in $\mathcal{S}$:

$$\begin{tikzcd} F_{|X^M}(\overline{x}) \arrow[r] \arrow[d, "\tau(\overline{x})"'] & \pi_{X,!}F^{\mathrm{Seg}}([m],I) \arrow[d, "\tau(\infty)"] \\ \mathrm{cst}(\overline{x}) \arrow[r] & X^M. \end{tikzcd} \tag{3.7}$$

We obtain that the condition for $\pi_{X,!}F$ to be a Segal Φ-operad is equivalent to requiring the natural transformation τ to being Cartesian or, equivalently, to requiring the vertical maps in diagram 3.7 to have equivalent fibres. The fibre of the left vertical map is obviously $F_{|X^M}(\overline{x})$. Since pullbacks commute with limits, the definition of $\pi_{X,!}F^{\mathrm{Seg}}([m],I)$ implies that the fibre of the right vertical map at $\overline{x}$ can be identified with

$$\lim_{(([m],I)\to a)\in(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}}(\pi_{X,!}F(a) \times_{X^M} \{\overline{x}\}).$$

Therefore, the statement of the proposition can be reformulated as follows: A functor $F\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{S}$ is a $\Delta^{\mathrm{op}}_{\Phi,X}$-monoid if and only if $F_{|X^M}(\overline{x}) \to \lim_{(([m],I)\to a)\in(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}}(\pi_{X,!}F(a) \times_{X^M} \{\overline{x}\})$ is an equivalence for every $\overline{x} \in X^M$.

Let $a \in \Delta_\Phi^{\mathrm{el,op}}$ be given by $([n],J)$. Let N be the set $\coprod_{0\leq k\leq n}|J_k|$ and let $N \hookrightarrow M$ denote the inclusion induced by the inert map $([n],J) \to ([m],I)$ in $\Delta_\Phi^{\mathrm{op}}$. Since the canonical map $\pi_{X,!}F(a) \to \pi_{X,!}F^{\mathrm{Seg}}(a)$ is obviously an equivalence, the observation above implies that the induced natural transformation τ is Cartesian and that is an equivalence $F_{|X^N}(\overline{x}') \simeq \pi_{X,!}F(a)\times_{X^N} \{\overline{x}'\}$ for every $\overline{x}' \in X^N$. Furthermore, the inclusion of sets $N \hookrightarrow M$ implies that, for every $\overline{x} = (x_1,\dots,x_M) \in X^M$, there is an element $\overline{x}' \in X^N$ given by $(x_j)_{j\in N}$. Therefore, we have equivalences

$$\pi_{X,!}F(a) \times_{X^M} \{\overline{x}\} \simeq \pi_{X,!}F(a) \times_{X^N} \{(x_j)_{j\in N}\} \simeq F_{|X^N}((x_j)_{j\in N}).$$

Suppose F is a $\Delta^{\mathrm{op}}_{\Phi,X}$-monoid. It follows from the definition that the restriction of F to the fibre $(\Delta^{\mathrm{op}}_{\Phi,X})_{\mathfrak{e}} \simeq X$ is the constant functor at $\{*\} \in \mathcal{S}$. Hence, the description of left Kan extensions implies that $\pi_{X,!}F(\mathfrak{e}) \simeq X$ and we have $\pi_{X,!}F(\mathfrak{e}) \times_{X^M} \{\overline{x}\} \simeq \{*\} \in \mathcal{S}$. This in turn implies that $\lim_{(([m],I)\to a)\in(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}}(F_{|X^N}((x_j)_{j\in N}))$ is given by the product $\prod_{\mathfrak{c}_J} F(\{x_j\}_{j\in|J|+1})$, where $\mathfrak{c}_J$ ranges over the corollas in $([m],I)$. The definition of monoid objects implies that $\prod_{\mathfrak{c}_J} F(\{x_j\}_{j\in|J|+1}) \simeq F(\overline{x})$, therefore, $\pi_{X,!}F$ is a Segal Φ-operad with $\pi_{X,!}F(\mathfrak{e}) \simeq X$.

Conversely, if $\pi_{X,!}F(\mathfrak{e})$ is a Segal Φ-operad, then $\pi_{X,!}F(\mathfrak{e})$ is equivalent to X by Lemma 3.27. It follows that $\pi_{X,!}F(\mathfrak{e})\times_{X^M}\{\overline{x}\}) \simeq \{*\} \in \mathcal{S}$ and $F(\overline{x}) \simeq \lim_{(([m],I)\to a)\in(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}}(F_{|X^N}((x_j)_{j\in N}))$ is equivalent to $\prod_{\mathfrak{c}_J} F(\{x_j\}_{j\in|J|+1})$, which coincides with $F(\overline{x})$, i.e F is a $\Delta^{\mathrm{op}}_{\Phi,X}$-monoid. $\square$

In the following proposition we further characterize the essential image of $\pi_{X,!}|_{\mathrm{Mon}_{\Delta^{\mathrm{op}}_{\Phi,X}}(\mathcal{S})}$.

Corollary 3.30. *Let* $i^*\colon \mathrm{Fun}(\Delta^{\mathrm{op}}_\Phi, \mathcal{S}) \rightleftarrows \mathcal{S} : i_*$ *be the adjunction induced by the obvious inclusion* $i\colon \{\mathfrak{e}\} \hookrightarrow \Delta^{\mathrm{op}}_\Phi$. *There is an equivalence of* ∞*-categories* $\mathrm{Mon}_{\Delta^{\mathrm{op}}_{\Phi,X}}(\mathcal{S}) \simeq \mathrm{Seg}_{\Phi,X}$.

Proof. By Corollary [GHN15, 8.6], the functor $\pi_{X,!}\colon \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathcal{S}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}_\Phi, \mathcal{S})$ given by the left Kan extension induces an equivalence

$$\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathcal{S}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi}, \mathcal{S})_{/i_*X},$$

which carries the full subcategory $\mathrm{Mon}_{\Delta^{\mathrm{op}}_{\Phi,X}}(\mathcal{S})$ of $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X})$ to the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi}, \mathcal{S})_{/i_*X}$ spanned by natural transformations $F \to i_*X$ satisfying the following conditions:

- The canonical map $i^*F = F(\mathfrak{e}) \to X$ defined in Lemma 3.27 is an equivalence.
- $F \to i_*X$ is given by the adjunction unit $F \to i_*i^*F \simeq i_*X$.

By the proposition above, the image of $\pi_{X,!}$ consists of Segal Φ-operads F such that $F(\mathfrak{e}) \simeq X$. Hence, the ∞-category $\mathrm{Mon}_{\Delta^{\mathrm{op}}_{\Phi,X}}(\mathcal{S})$ is equivalent to the full subcategory $\mathrm{Seg}_{\Phi,X}$ of $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi}, \mathcal{S})$ spanned by the Segal Φ-operads F such that $F(\mathfrak{e}) = X$ and natural transformations which restrict to identities at $\mathfrak{e}$. $\square$

Theorem 3.31. *Let $\mathcal{S}^\times$ be the Cartesian symmetric monoidal ∞-category of spaces. There exists an equivalence*

$$\mathrm{Alg}_\Phi(\mathcal{S}^\times) \simeq \mathrm{Seg}_\Phi.$$

Proof. The ∞-category $\mathcal{S}$ admits finite products and hence, by [Lur, Proposition 2.4.1.5], induces a Cartesian symmetric monoidal ∞-category denoted by $\mathcal{S}^\times$. The proof of Proposition 3.21 implies that there exists an equivalence $\mathrm{Alg}_{\mathcal{O}/\Delta^{\mathrm{op}}_{\Phi}}(\mathcal{S}^\times) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{S})$ which is natural in $\mathcal{O}$. We then obtain the commutative diagram

$$\begin{tikzcd}
 & \Phi\text{-}\mathrm{Alg}(\mathcal{S}^\times) \arrow[rr, "\simeq"] \arrow[ddr] & & \Phi\text{-}\mathrm{Mon}(\mathcal{S}) \arrow[ddl] \\
\mathrm{Alg}_\Phi(\mathcal{S}^\times) \arrow[ur] \arrow[rr, "\simeq"] \arrow[dddr] & & \mathrm{Mon}_\Phi(\mathcal{S}) \arrow[ur] \arrow[ddl] & \\
 & & \mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}} & \\
 & \mathcal{S} \arrow[ur, "\Delta^{\mathrm{op}}_{\Phi,(-)}"'] & &
\end{tikzcd} \tag{3.8}$$

by pullback.

The map $\mathrm{ev}_{\mathfrak{e}} \colon \mathrm{Seg}_\Phi \to \mathcal{S}$ given by the evaluation at $\mathfrak{e}$ is a Cartesian fibration by [GH15, Lemma A.1.6]. The proof of Proposition 3.29 implies that the equivalence $\mathrm{Mon}_{\Delta^{\mathrm{op}}_{\Phi,X}}(\mathcal{S}) \xrightarrow{\simeq} \mathrm{Seg}_{\Phi,X}$ of Corollary 3.30 is natural in X. Hence, there exists an equivalence of Cartesian fibrations

$$\begin{tikzcd}
\mathrm{Mon}_\Phi(\mathcal{S}) \arrow[rr, "\simeq"] \arrow[dr] & & \mathrm{Seg}_\Phi \arrow[dl, "\mathrm{ev}_{\mathfrak{e}}"] \\
 & \mathcal{S} &
\end{tikzcd} \tag{3.9}$$

and the equivalence in the claim is given by the composite of equivalences $\mathrm{Alg}_\Phi(\mathcal{S}^\times) \simeq \mathrm{Mon}_\Phi(\mathcal{S}) \simeq \mathrm{Seg}_\Phi$. $\square$

Proof of Theorem 3.24. It easily follows from the constructions above that there is commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\Phi}(\mathcal{S}^{\times}_{\Phi}) \arrow[r, "\simeq"] \arrow[d, "u^*"'] & \mathrm{Seg}_{\Phi} \arrow[d, "u^*"] \\ \mathrm{Alg}_{\mathcal{F}}(\mathcal{S}^{\times}_{*}) \arrow[r, "\simeq"] & \mathrm{Seg}_{*} \end{tikzcd}$$

where the horizontal maps are equivalences by Theorem 3.31. Therefore, the upper horizontal induces an equivalence $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{S}^{\times}_{\Phi}) \simeq \mathrm{Seg}_{\Phi,\mathrm{cp}}$. □

Corollary 3.32. *The ∞-category $\mathrm{Alg}_{\mathcal{F},\mathrm{cp}}(\mathcal{S}^{\times}_{\mathcal{F}})$ is equivalent to the ∞-category Op_{∞} of ∞-operads in the sense of Lurie.*

Proof. Since the ∞-category of complete Segal $\mathcal{F}$-operads is equivalent to the ∞-category Op_{∞} of ∞-operads in the sense of Lurie by [Bar13, Theorem 10.16], we obtain the equivalence $\mathrm{Alg}_{\mathcal{F},\mathrm{cp}}(\mathcal{S}^{\times}_{\mathcal{F}}) \simeq \mathrm{Op}_{\infty}$ we are searching for. □

Remark 3.33. *For every $\mathcal{V}$-enriched Φ-∞-operad $\mathcal{O}\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$, the object $\mathcal{O}(x_1, \dots, x_n; x)$ lies in $\mathcal{V}$. Moreover, if $\mathcal{V} = \mathcal{S}^{\times}$, then Theorem 3.24 implies that the Kan complex $\mathcal{O}(x_1, \dots, x_n; x)$ can be identified with the mapping space $\mathrm{Map}_{\mathcal{O}}(x_1, \dots, x_n; x)$ of $\mathcal{O}$ regarded as a Segal operad.*

Chapter 4

The Φ-Presheaf Model

We begin the first section by recalling the definition of presentable symmetric monoidal ∞-categories. Then we study a generalization of these categories which are called κ-accessible symmetric monoidal ∞-categories in Definition 4.1. These are symmetric monoidal ∞-categories which satisfy certain cardinality conditions. In Proposition 4.7 we prove that these ∞-categories always admit small symmetric monoidal subcategories. This result will be important for the next section, where we define a presheaf model for enriched ∞-operads, as well as compare this model and its dendroidal counterpart introduced in Chapter 7.

In the second section we introduce the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ of Segal Φ-presheaves in Definition 4.13 as a presheaf model for enriched ∞-operads. Proposition 4.20 implies that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ is presentable, if the ∞-category $\mathcal{V}$ is presentable symmetric monoidal. The strategy of the proof is to show that $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ is given by an accessible localization with respect to a set of maps which encodes the defining conditions for being Segal Φ-presheaves.

The last section is devoted to the proof of Theorem 4.22 which states that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ is equivalent to $\mathrm{Alg}_\Phi(\mathcal{V})$ if $\mathcal{V}$ is presentable symmetric monoidal. In other words, if we enrich over a presentable symmetric monoidal ∞-category, then the two different approaches to enriched ∞-operads introduced so far are equivalent.

4.1 Presentability and Accessibility

Let us recall the definition of presentable symmetric monoidal ∞-categories.

Definition 4.1. Let $\mathcal{V} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category and let $\mathcal{V}_{\langle 1\rangle}$ denote the fibre of $\mathcal{V}$ at $\langle 1\rangle$. We say $\mathcal{V}$ is *presentable symmetric monoidal*, if the following conditions are satisfied:

1. The ∞-category $\mathcal{V}_{\langle 1\rangle}$ is presentable.
2. For every $\langle n\rangle \in \mathcal{F}_*$, the functor $\otimes_n \colon \mathcal{V}_{\langle n\rangle} \to \mathcal{V}_{\langle 1\rangle}$ induced by the unique active morphism $\langle n\rangle \to \langle 1\rangle$ preserves small colimits in each variable.

By weakening the conditions above, we obtain the following:

Definition 4.2. Given a regular cardinal κ, we say a symmetric monoidal ∞-category $\mathcal{V}$ is *κ-accessible symmetric monoidal*, if the following conditions are satisfied:

1. The ∞-category $\mathcal{V}_{\langle 1\rangle}$ is κ-accessible, where $\mathcal{V}_{\langle 1\rangle}$ denotes the fibre of $\mathcal{V} \to \mathcal{F}_*$ at $\langle 1\rangle$.
2. For every $\langle n\rangle \in \mathcal{F}_*$, the functor $\otimes_n : \mathcal{V}_{\langle n\rangle} \to \mathcal{V}_{\langle 1\rangle}$ induced by the unique active morphism $\langle n\rangle \to \langle 1\rangle$ preserves small κ-filtered colimits.

We call a symmetric monoidal ∞-category *accessible symmetric monoidal*, if it is κ-accessible symmetric monoidal for some cardinal κ.

The condition 2 of the definition above is equivalent to requiring the functor $\otimes_n$ to preserve small κ-filtered colimits in each variable by [Lur09, Proposition 5.5.8.6].

Definition 4.3. [Lur09, 5.4.2.8] Suppose κ' and κ are two regular cardinals. We write $\kappa \ll \kappa'$ if $(\kappa_0')^{\kappa_0} < \kappa'$, for every $\kappa_0' < \kappa'$ and every $\kappa_0 < \kappa$.

Lemma 4.4. *Let κ and κ' be two regular cardinals such that $\kappa \ll \kappa'$. If $\mathcal{C}$ is a κ-accesible ∞-category, then an object in $\mathcal{C}$ is κ'-compact if and only if it is a κ'-small κ-filtered colimit of κ-compact objects.*

Proof. Let $\mathcal{C}^\kappa$ and $\mathcal{C}^{\kappa'}$ denote the full subcategories of $\mathcal{C}$ spanned by the κ-compact and κ'-compact objects, respectively. If we write $\mathcal{C}'$ for the full subcategory of $\mathcal{C}$ spanned by the colimits of all κ'-small κ-filtered diagrams in $\mathcal{C}^\kappa$, then $\mathcal{C}'$ is essentially small. This is because $\mathcal{C}$ is locally small and the collection of all equivalence classes of κ'-small κ-filtered diagrams is bounded. Since $\mathcal{C}^{\kappa'}$ is closed under κ'-small colimits by [Lur09, Corollary 5.3.4.15], the ∞-category $\mathcal{C}'$ is a full subcategory of $\mathcal{C}^{\kappa'}$.

The proof of [Lur09, Proposition 5.4.2.11] shows that $\mathcal{C}'$ generates $\mathcal{C}$ under small κ'-filtered colimits. According to [Lur09, Lemma 5.4.2.4], the ∞-category $\mathcal{C}^{\kappa'} \subseteq \mathcal{C} = \mathrm{Ind}_{\kappa'}(\mathcal{C}')$ is given by the idempotent completion of $\mathcal{C}'$. Since $\mathcal{C}'$ is already idempotent complete by [Lur09, Proposition 4.4.5.15], the ∞-categories $\mathcal{C}'$ and $\mathcal{C}^{\kappa'}$ coincide. $\square$

Definition 4.5. Let $\mathcal{V} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category and let $\mathcal{V}_{\langle 1\rangle}$ denote the fibre of $\mathcal{V}$ over $\langle 1\rangle \in \mathcal{F}_*$. For a cardinal κ, let $\mathcal{V}^\kappa_{\langle 1\rangle}$ denote the full subcategory of $\mathcal{V}_{\langle 1\rangle}$ spanned by κ-compact objects in $\mathcal{V}_{\langle 1\rangle}$. We also write $\mathcal{V}^\kappa$ for the full subcategory of $\mathcal{V}$ spanned by objects lying in $(\mathcal{V}^\kappa_{\langle 1\rangle})^n \subseteq (\mathcal{V}_{\langle 1\rangle})^n \simeq \mathcal{V}_{\langle n\rangle}$ for some $\langle n\rangle \in \mathcal{F}_*$.

Remark 4.6. *We would like to emphasize that $\mathcal{V}^\kappa$ is in general* not *the full subcategory of the ∞-category $\mathcal{V}$ spanned by κ-compact objects.*

Proposition 4.7. *Suppose $\mathcal{V}$ is a κ-accessible symmetric monoidal ∞-category with unit $\mathbb{1}$, then there exists a regular cardinal κ' such that the full subcategory $\mathcal{V}^{\kappa'}$ is a symmetric monoidal ∞-category.*

Proof. Since $\mathcal{V}$ is κ-accessible symmetric monoidal, the ∞-category $\mathcal{V}^\kappa$ is essentially small and we can choose a cardinal κ' such that $\kappa' \gg \kappa$, $\mathbb{1} \in \mathcal{V}^{\kappa'}_{\langle 1\rangle}$ and $v \otimes w \in \mathcal{V}^{\kappa'}_{\langle 1\rangle}$ for two every objects $v, w \in \mathcal{V}^\kappa_{\langle 1\rangle}$.

If $w \in \mathcal{V}^{\kappa'}_{\langle 1\rangle}$, then, by Lemma 4.4, there exists a κ'-small κ-filtered colimit diagram $f : I^\triangleright \to \mathcal{V}_{\langle 1\rangle}$ such that $f(i) \in \mathcal{V}^\kappa_{\langle 1\rangle}$ and $w = \mathrm{colim}_{i\in I} f(i)$. The tensor product preserves colimits in each variable by the definition of being κ-accessible symmetric monoidal. Therefore, for every $v \in \mathcal{V}^\kappa_{\langle 1\rangle}$, we have that $v \otimes (\mathrm{colim}_{i\in I} f(i)) \simeq \mathrm{colim}_{i\in I}(v \otimes f(i))$ is a κ'-small colimit of κ'-compact objects. Hence, $\mathrm{colim}_{i\in I}(v \otimes f(i))$ lies in $\mathcal{V}^{\kappa'}_{\langle 1\rangle}$ by [Lur09, Corollary 5.3.4.15]. By symmetry, we have that $v \otimes w \in \mathcal{V}^{\kappa'}_{\langle 1 \rangle}$, if $v \in \mathcal{V}^{\kappa'}_{\langle 1 \rangle}$ and $w \in \mathcal{V}^{\kappa}_{\langle 1 \rangle}$. Using the fact that the tensor product preserves colimits in each variable once again, we obtain that that $\mathcal{V}^{\kappa'}$ is closed under tensor products, i.e. it is a symmetric monoidal ∞-category. □

Corollary 4.8. *Every presentable symmetric monoidal ∞-category admits a small symmetric monoidal full subcategory of the form $\mathcal{V}^{\kappa}$.*

Proof. The corollary follows directly from the proposition above, since every presentable symmetric monoidal ∞-category is κ-accessible symmetric monoidal for some κ. □

4.2 Segal Φ-Presheaves

Definition 4.9. Let $\mathcal{V} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category and let $p \colon \mathcal{F}_* \to \mathrm{Cat}_\infty$ be the associated functor. Let $\mathcal{V}_\otimes \to \mathcal{F}_*^{\mathrm{op}}$ denote the Cartesian fibration associated to p. We write $p^\vee \colon \mathcal{V}^\vee \to \Delta_\Phi$ for the Cartesian fibration given by the following pullback diagram

$$\begin{tikzcd} \mathcal{V}^\vee \arrow[r] \arrow[d, "p^\vee"'] & \mathcal{V}_\otimes \arrow[d] \\ \Delta_\Phi \arrow[r, "\mathrm{Cr}_\Phi^{\mathrm{op}}"'] & \mathcal{F}_*^{\mathrm{op}}. \end{tikzcd}$$

Note that the fibre $(\mathcal{V}^\vee)_{\mathfrak{c}}$ is equivalent to $\mathcal{V}_{\langle 1 \rangle}$ for a corolla $\mathfrak{c} \in \Delta_\Phi$, .

Remark 4.10. *We would like to emphasize that the ∞-category $\mathcal{V}^\vee$ above depends on the operator category Φ. We left the index Φ out in order to avoid the notational overhead.*

Remark 4.11. *The definition above allows us to think of an object in $\mathcal{V}^\vee$ as an object $([m], I) \in \Delta_\Phi$ such that each corolla in $([m], I)$ is labelled by an object of $\mathcal{V}_{\langle 1 \rangle}$. Therefore, if $\mathfrak{c}$ is a corolla in $\mathcal{V}^\vee$, then we write $(\mathfrak{c}, v)$ for an object in $\mathcal{V}^\vee$ lying over $\mathfrak{c}$ and labelled by $v \in \mathcal{V}_{\langle 1 \rangle}$. In particular, for $\Phi = \mathcal{F}$, each object in $\mathcal{V}^\vee$ can be regarded as a forest whose vertices are labelled by $\mathcal{V}_{\langle 1 \rangle}$.*

Notation 4.12. *For the rest of the chapter, we will fix a presentable symmetric monoidal ∞-category $\mathcal{V}$ and a regular cardinal κ such that full subcategory $\mathcal{V}^\kappa$ is symmetric monoidal and $\mathcal{V}_{\langle 1 \rangle}$ is κ-presentable. The existence of such a $\mathcal{V}$ is provided by Corollary 4.8.*

Definition 4.13. A presheaf $F \colon \mathcal{V}^{\vee,\mathrm{op}} \to \mathcal{S}$ in $\mathrm{P}(\mathcal{V}^\vee)$ is called a *Segal Φ-presheaf*, if the following conditions are satisfied:

1. If $\bar{\mathfrak{e}}$ denotes the essentially unique object in $(\mathcal{V}^\vee)_{\mathfrak{e}} \simeq \{\bar{\mathfrak{e}}\}$, then, for every corolla $\mathfrak{c}_I \in \Delta_\Phi$ with $|I| = n$, the functor $(\mathcal{V}^{\vee,\mathrm{op}})_{\mathfrak{c}_I} \simeq \mathcal{V}^{\mathrm{op}}_{\langle 1 \rangle} \to \mathcal{S}_{/F(\bar{\mathfrak{e}})^{n+1}}$ induced by the $p^\vee$-Cartesian lifts of the $n+1$-many morphisms $\mathfrak{e} \to \mathfrak{c}_I$ preserves all small limits.
2. Recall that $(\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$ denotes the category $\Delta_\Phi^{\mathrm{el,op}} \times_{\Delta_\Phi^{\mathrm{in,op}}} (\Delta_\Phi^{\mathrm{in,op}})_{([m],I)/}$. The functor F *satisfies the Segal condition*, i.e., for every object $v \in \mathcal{V}^{\vee,\mathrm{op}}$ lying over $([m], I)$, the canonical map
$$F(v) \to \lim_{\alpha \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}} F(\alpha^* v)$$
is an equivalence, where $\alpha^* v \to v$ is the $p^\vee$-Cartesian lift of the inert map α (corresponding to the coCartesian morphism in $\mathcal{V}$).

We write $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ for the full subcategory of presheaves spanned by Segal Φ-presheaves.

The previous definition generalizes Segal presheaves introduced in [GH15, 4.5]. We can recover the theory of Segal presheaves by choosing the trivial operator category $*$ for Φ.

Remark 4.14. *Suppose $F \in \mathrm{P}(\mathcal{V}^\vee)$ is a presheaf which satisfies condition 2 in Definition 4.13, then F satisfies condition 1 if and only if it satisfies the following:*

1'. *Let $([m], I) \in \Delta_\Phi$ contain n edges and let $\bar{\mathfrak{e}} \in (\mathcal{V}^\vee)_{\mathfrak{e}}$ denote the unique object. The functor $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}_{/f(\bar{\mathfrak{e}})^n}$ induced by the n-many $p^\vee$-Cartesian lifts of the morphisms $\mathfrak{e} \to ([m], I)$ preserves small limits.*

Lemma 4.15. *The following assertions are always true:*

1. *Every presheaf $F_0 \colon \mathcal{V}^{\kappa,\vee,\mathrm{op}} \to \mathcal{S}$ admits a right Kan extension $F \colon \mathcal{V}^{\vee,\mathrm{op}} \to \mathcal{S}$.*
2. *A functor $F \colon \mathcal{V}^{\vee,\mathrm{op}} \to \mathcal{S}$ is a right Kan extension of its restriction $F_{|\mathcal{V}^{\kappa,\vee,\mathrm{op}}}$ if and only if F preserves all small κ-cofiltered limits of the form*
$$K^{\triangleleft} \to \mathcal{V}^{\vee,\mathrm{op}},$$
such that its restriction to K factorizes through a fibre $(\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{([m],I)}$.

Proof. Since products of cofiltered diagrams are cofiltered and $\mathcal{V}$ is κ-presentable, we have that the ∞-category $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \simeq (\mathcal{V}_{\langle 1\rangle})^M$ is freely generated by $(\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{([m],I)} \simeq (\mathcal{V}^{\kappa,\mathrm{op}}_{\langle 1\rangle})^M$ under κ-cofiltered limits for $M = \coprod_{k\in\mathbf{m}} |I(k)|$ and every object $([m], I) \in \Delta_\Phi$. The map $p^\vee$ is a Cartesian fibration, therefore $p^{\vee,\mathrm{op}} \colon \mathcal{V}^{\vee,\mathrm{op}} \to \Delta_\Phi^{\mathrm{op}}$ is a coCartesian fibration and by [Lur09, Corollary 4.3.1.16], for every object $([m], I) \in \Delta_\Phi$, a functor $q \colon K^{\triangleleft} \to (\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)}$ is a limit diagram in the fibre $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)}$ if and only if it is a $p^{\vee,\mathrm{op}}$-limit diagram in $\mathcal{V}^{\vee,\mathrm{op}}$. According to [Lur09, Lemma 5.3.1.18], every cofiltered ∞-category K is weakly contractible which allows us to regard the constant functor $p^\vee \circ q$ as a limit diagram in $\Delta_\Phi^{\mathrm{op}}$. Assume now that q is a limit diagram. Then it follows from [Lur09, Proposition 4.3.1.5.] that $K^{\triangleleft} \to (\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \subseteq \mathcal{V}^{\vee,\mathrm{op}}$ is a limit diagram in $\mathcal{V}^{\vee,\mathrm{op}}$. Since the ∞-category $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)}$ is freely generated by $(\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{([m],I)}$ under small κ-cofiltered limits of the form $K^{\triangleleft} \to (\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{([m],I)}$, for every object $([m], I) \in \Delta_\Phi^{\mathrm{op}}$, the observation above shows that every object in the ∞-category $\mathcal{V}^{\vee,\mathrm{op}}$ lies in the full subcategory of $\mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}}$ spanned by those objects which are limits of diagrams of the from
$$K \to (\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{([m],I)} \subseteq \mathcal{V}^{\kappa,\vee,\mathrm{op}} \hookrightarrow \mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}}. \tag{4.1}$$

Now we want to show that $\mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}}$ contains the ∞-category $\mathcal{V}^{\vee,\mathrm{op}}$ as a full subcategory. Let v and w be two objects in $\mathcal{V}^{\vee,\mathrm{op}}$ lying over $([m], I)$ and $([n], J)$, respectively. Then there are cofiltered diagrams $p \colon K \to \mathcal{V}^{\vee,\mathrm{op}}$ and $q \colon L \to \mathcal{V}^{\vee,\mathrm{op}}$ factorizing through the fibres $\mathcal{V}^{\vee,\mathrm{op}}_{([m],I)}$ and $\mathcal{V}^{\vee,\mathrm{op}}_{([n],J)}$, respectively, such that $\lim_{k\in K} p(k) = v$ and $\lim_{l\in L} q(l) = w$. Regarded as objects in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}}$, the mapping space $\mathrm{Map}_{\mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}}}(v, w)$ is given by
$$\lim_{l\in L} \operatorname{colim}_{k\in K^{\mathrm{op}}} \mathrm{Map}_{\mathcal{V}^{\kappa,\vee}}(q(l), p(k)).$$

To verify that $\mathcal{V}^{\vee,\mathrm{op}}$ is a full subcategory of $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$, we need to show that the corresponding mapping space $\mathrm{Map}_{\mathcal{V}^{\vee,\mathrm{op}}}(v, w)$ is of the same form. Since it is clear that $\mathrm{Map}_{\mathcal{V}^{\vee,\mathrm{op}}}(v, w) \simeq$

$\lim_{l \in L} \mathrm{Map}_{\mathcal{V}^\vee}(q(l), \mathrm{colim}_{k \in K^{\mathrm{op}}} p(k))$, it suffices to check that the canonical map

$$\mathrm{colim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{\mathcal{V}^\vee}(q(l), p(k)) \to \mathrm{Map}_{\mathcal{V}^\vee}(q(l), \mathrm{colim}_{k \in K^{\mathrm{op}}} p(k))$$

is an equivalence. Hence, we wish to see that the induced commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{\mathcal{V}^\vee}(q(l), p(k)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{V}^\vee}(q(l), \mathrm{colim}_{k \in K^{\mathrm{op}}} p(k)) \arrow[d] \\
\mathrm{Hom}_{\Delta_\Phi}(([n], J), ([m], I)) \arrow[r, "\mathrm{id}"'] & \mathrm{Hom}_{\Delta_\Phi}(([n], J), ([m], I))
\end{tikzcd}$$

is Cartesian. In other words, for every object $\alpha \in \mathrm{Hom}_{\Delta_\Phi}(([n], J), ([m], I))$, the fibres of the vertical maps over α have to be equivalent. The simplicial set K^{op} is filtered and filtered colimits commute with pullbacks in $\mathcal{S}$. Furthermore $\mathcal{V}^\vee \to \Delta_\Phi$ is a Cartesian fibration, and thereofore by [Lur09, Proposition 2.4.4.2], the fibre of the left vertical map over α can be identified with $\mathrm{colim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{\mathcal{V}^\vee_{([n],J)}}(q(l), \alpha^* p(k))$. Using the fact that $\mathcal{V}^{\kappa,\vee}_{([n],J)}$ is a product of $\mathcal{V}^\kappa$ and taking into accoount filtered colimits commute with finite limits, we obtain

$$\mathrm{colim}_{k \in K^{\mathrm{op}}} \mathrm{Map}_{\mathcal{V}^\vee_{([n],J)}}(q(l), \alpha^* p(k)) \simeq \mathrm{Map}_{\mathcal{V}^\vee_{([n],J)}}(q(l), \mathrm{colim}_{k \in K^{\mathrm{op}}} \alpha^* p(k)).$$

[Lur09, Proposition 2.4.4.2] implies that the fibre of the right vertical map over α is given by $\mathrm{Map}_{\mathcal{V}^\vee}(q(l), \alpha^* \mathrm{colim}_{k \in K^{\mathrm{op}}} p(k))$. Therefore, it suffices to show that the canonical map

$$\mathrm{colim}_{k \in K^{\mathrm{op}}} \alpha^* p(k) \to \alpha^* \mathrm{colim}_{k \in K^{\mathrm{op}}} p(k)$$

is an equivalence in $\mathcal{V}^\vee$. If $f\colon \mathcal{V} \to \mathcal{F}_*$ denotes the structure map of $\mathcal{V}$, then the objects $\mathrm{colim}_{k \in K^{\mathrm{op}}} \alpha^* p(k)$ and $\alpha^* \mathrm{colim}_{k \in K^{\mathrm{op}}} p(k)$ correspond to objects $\mathrm{colim}_{k \in K^{\mathrm{op}}} \mathrm{Cr}_\Phi(\alpha)_!(p(k))$ and $\mathrm{Cr}_\Phi(\alpha)_!(\mathrm{colim}_{k \in K^{\mathrm{op}}} p(k))$ given by the f-coCartesian lifts of $\mathrm{Cr}_\Phi(\alpha)$, respectively. The claim then follows from the observation that $\mathrm{colim}_{k \in K^{\mathrm{op}}} \mathrm{Cr}_\Phi(\alpha)_!(p(k))$ and $\mathrm{Cr}_\Phi(\alpha)_!(\mathrm{colim}_{k \in K^{\mathrm{op}}} p(k))$ are equivalent, because the fibre transport map $\mathrm{Cr}_\Phi(\alpha)_!$ preserves colimits.

Every functor $F_0\colon \mathcal{V}^{\kappa,\vee,\mathrm{op}} \to \mathcal{S}$ admits a right Kan extension $\tilde{F}\colon \mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}} \to \mathcal{S}$ because the ∞-category $\mathcal{S}$ has all limits. Obviously, every object of $\mathcal{V}^{\vee,\mathrm{op}}$ lies in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$. Therefore, if we write F for the restriction $\tilde{F}_{|\mathcal{V}^{\vee,\mathrm{op}}}$ along the inclusion $\mathcal{V}^{\vee,\mathrm{op}} \hookrightarrow \mathrm{P}(\mathcal{V}^{\kappa,\vee})^{\mathrm{op}}$, then $F\colon \mathcal{V}^{\vee,\mathrm{op}} \to \mathcal{S}$ is a right Kan extension of F_0 which proves (1).

Since the right Kan extension F preserves all cofiltered limits of the form 4.1 and right Kan extensions of F_0 are unique up to equivalence, we obtain the "only if" part of (2).

For the "if" part of (2) we need to show that a functor $F'\colon \mathcal{V}^{\vee,\mathrm{op}} \to \mathcal{S}$ is a right Kan extension of F_0 if it preserves all cofiltered limits of the form 4.1 and $F'_{|\mathcal{V}^{\kappa,\vee,\mathrm{op}}} = F_0$. By the universal property of the right Kan extension F, we have a natural transformation $\tau\colon F \to F'$ such that $\tau(v)\colon F(v) \to F'(v)$ is an equivalence in $\mathcal{S}$ for every $v \in \mathcal{V}^{\kappa,\vee,\mathrm{op}}$. If $\mathcal{C}$ denotes the full subcategory of $\mathcal{V}^{\vee,\mathrm{op}}$ spanned by those objects v such that $\tau(v)$ is an equivalence in $\mathcal{S}$, then the definition of F' says that $\mathcal{V}^{\kappa,\vee,\mathrm{op}} \subseteq \mathcal{C}$ and we want to show that $\mathcal{V}^{\vee,\mathrm{op}} = \mathcal{C}$. We observe that the ∞-category $\mathcal{C}$ is closed under κ-cofiltered limits of the form 4.1, because both functors F and F' preserve these limits. Since $\mathcal{V}^{\vee,\mathrm{op}}$ is freely generated by $\mathcal{V}^{\kappa,\vee,\mathrm{op}}$ under these limits, we have $\mathcal{V}^{\vee,\mathrm{op}} = \mathcal{C}$. The functor F', which is equivalent to a right Kan extension F, is then a right Kan extension itself by the essential uniqueness. $\square$

Corollary 4.16. *If we write* $\mathrm{P}'(\mathcal{V}^\vee)$ *for the full subcategory of* $\mathrm{P}(\mathcal{V}^\vee)$ *spanned by presheaves such that their restrictions to every fibre* $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}$ *preserve* κ*-cofiltered limits, then there is an equivalence of* ∞*-categories* $\mathrm{P}'(\mathcal{V}^\vee) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})$*. In particular, the* ∞*-category* $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ *(see Definition 4.13) is a full subcategory of* $\mathrm{P}'(\mathcal{V}^\vee) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})$*.*

Proof. By the previous lemma, we have $F \in \mathrm{P}'(\mathcal{V}^\vee)$ if and only if F is a right Kan extension of its restriction $F_{|\mathcal{V}^{\kappa,\vee,\mathrm{op}}}$. According to [Lur09, Corollary 4.3.2.16], the canonical functor $\mathrm{P}(\mathcal{V}^\vee) \to \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ restricts to a trivial fibration $\mathrm{P}'(\mathcal{V}^\vee) \to \mathrm{P}(\mathcal{V}^{\kappa,\vee})$. $\square$

Lemma 4.17. *Let* K, L *be simplicial sets and* $\mathcal{C}$ *an* ∞*-category. Suppose* $\overline{f}\colon K^{\triangleleft} \to \mathcal{D}$ *is a limit diagram and let* $\{-\infty\}$ *denote the cone point of* $K^{\triangleleft}$*. Then a diagram* $\overline{g}\colon L^{\triangleleft} \to \mathcal{D}_{/\overline{f}(-\infty)}$ *is a limit diagram if and only if* $\overline{g} * f\colon L^{\triangleleft} * K \to \mathcal{D}$ *is a limit diagram in* $\mathcal{D}$

Proof. Let f and g denote the respective restrictions $\overline{f}_{|K}\colon K \to \mathcal{D}$ and $\overline{g}_{|L}\colon L \to \mathcal{D}_{/\overline{f}(-\infty)}$. Since the inclusion of the cone point $\{-\infty\} \hookrightarrow L^{\triangleleft}$ is left anodyne, [Lur09, Proposition 2.1.2.5] and the assumption that $\overline{f}$ is a limit diagram induce the following equivalences

$$\mathcal{D}_{/\overline{f}(-\infty)} \simeq \mathcal{D}_{/\overline{f}} \simeq \mathcal{D}_{/f}.$$

This implies that $\overline{g}$ is a limit diagram if and only if the canonical map

$$(\mathcal{D}_{/f})_{/\overline{g}} \simeq (\mathcal{D}_{/\overline{f}(-\infty)})_{/\overline{g}} \to (\mathcal{D}_{/\overline{f}(-\infty)})_{/g} \simeq (\mathcal{D}_{/f})_{/g}$$

is an equivalence. The definition of slice categories implies that $(\mathcal{D}_{/f})_{/\overline{g}} \simeq \mathcal{D}_{/\overline{g}*f}$ and $(\mathcal{D}_{/f})_{/g} \simeq \mathcal{D}_{/g*f}$. Thus, by the 2-of-3 property, we have that $\overline{g}$ is a limit diagram in $\mathcal{D}_{/\overline{f}(-\infty)}$ if and only if $\overline{g} * f\colon L^{\triangleleft} * K \to \mathcal{D}$ is a limit diagram in $\mathcal{D}$. $\square$

Definition 4.18. Let $\mathcal{C}$ be an ∞-category, let $z \in \mathcal{C}$ and let S be a collection of morphisms in $\mathcal{C}$. We say z is *S-local* if, for every morphism $f\colon x \to y$ in S, the induced map $\mathrm{Map}_{\mathcal{C}}(y,z) \to \mathrm{Map}_{\mathcal{C}}(x,z)$ is an isomorphism in the homotopy category of spaces.

Definition 4.19. If $\mathcal{C}$ is an ∞-category admitting all small colimits, then a collection S of morphisms in $\mathcal{C}$ is called *strongly saturated*, if it satisfies the following conditions:

1. If
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\ x' \arrow[r, "f'"'] & y' \end{tikzcd}$$
is a pushout square in $\mathcal{C}$ and $f \in S$, then $f' \in S$.

2. The full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by S is stable under small colimits.

3. Given a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "g"'] & & y \arrow[dl, "h"] \\ & z, & \end{tikzcd}$$

where two of the three maps f, g and h lie in S, then so thus the third.

Proposition 4.20. *Let $\mathcal{V}$ be as in Notation 4.12. The ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ introduced in Definition 4.13 is presentable.*

Proof. The proof consists of three parts. First we use Corollary 4.16 and Remark 4.14 to show that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is equivalent to the full subcategory of $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ spanned by objects which satisfy the conditions (a) and (b), which are similar to the two conditons mentioned in Definition 4.13. Then we prove that a presheaf $F \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ satisfies condition (a) if and only if it is local with respect to a set $S_{\Phi}^{\lim}$ of morphisms in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$. Similarly, the last part of the proof reveals that F satisfies condition (b) if and only if it is local with respect to another set S_{Φ}^{Seg} of morphisms in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$. This means that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is equivalent to the full subcategory of $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ spanned by objects which are local with respect to elements in $S_{\Phi}^{\lim}$ as well as in a set S_{Φ}^{Seg}. Therefore, as an accessible localization of the presentable ∞-category $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$, the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is also presentable.

Let $F \in \mathrm{P}(\mathcal{V}^{\vee})$ be a presheaf that satisfies condition 1' of Remark 4.14. This means, for every object object $([m], I) \in \Delta_{\Phi}^{\mathrm{op}}$ the functor $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}_{/f(\bar{\mathfrak{e}})^n}$ induced by the $n = \coprod_{0 \le k \le m} |I(k)|$-many $p^{\vee}$-Cartesian lifts of the morphisms $\mathfrak{e} \to ([m], I)$ preserves small limits. By [Lur, Corollary 4.2.3.12], this functor $((\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}_{/f(\bar{\mathfrak{e}})^n}$ preserves all small limits if and only if it preserves all κ-small limits as well as all κ-cofiltered limits. Since cofiltered diagrams are weakly contractible,$(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}_{/f(\bar{\mathfrak{e}})^n}$ preserves κ-cofiltered limits if and only if its composite with the forgetful functor $\mathcal{S}_{/f(\bar{\mathfrak{e}})^n} \to \mathcal{S}$ does. Hence, a presheaf F satisfies condition 1' of Remark 4.14 if and only if the following assertions hold:

- The induced functor $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}_{/f(\bar{\mathfrak{e}})^n}$ preserves κ-small limits.
- The composite unctor $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}$ preserves κ-cofiltered limits.

By Corollary 4.16, a presheaf $F \in \mathrm{P}(\mathcal{V}^{\vee})$ such that the induced functor $(\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}$ preserves κ-cofiltered limits can be identified with an object in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$. Let $p^{\kappa,\vee} \colon \mathcal{V}^{\kappa,\vee} \to \Delta_{\Phi}$ be the Cartesian fibration associated to the symmetric monoidal ∞-category $\mathcal{V}^{\kappa}$ (see Notation 4.12). It follows from Remark 4.14 and the observations above that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is the full subcategory of $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ spanned by presheaves F satisfying the following conditions:

(a) For every corolla $\mathfrak{c}_I \in \Delta_{\Phi}^{\mathrm{op}}$, the functor $(\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{c}_I} \to \mathcal{S}_{/f(\bar{\mathfrak{e}})^{|I|+1}}$ induced by F preserves κ-small limits.

(b) If $v \in \mathcal{V}^{\kappa,\vee,\mathrm{op}}$ lies over an object $([m], I) \in \Delta_{\Phi}^{\mathrm{op}}$, then the canonical map

$$F(v) \to \lim_{\alpha \in (\Delta_{\Phi}^{\mathrm{el},\mathrm{op}})_{([m],I)/}} F(\alpha^* v)$$

is an equivalence, where $\alpha^* v \to v$ is the $p^{\kappa,\vee}$-Cartesian lift of the inert map α (corresponding to the coCartesian morphism in $\mathcal{V}$).

We are now going to show that $F \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ satisfies these two conditions if and only if it is local with respect to a set of morphisms.

Let $p_n\colon \mathbf{n+1} \to \mathcal{S}$ denote the constant diagram at the object $F(\bar{\mathfrak{e}}) \in \mathcal{S}$. The product $F(\bar{\mathfrak{e}})^{n+1}$ is a limit of p_n in $\mathcal{S}$. If $\overline{G}\colon L^{\triangleleft} \to (\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{c}_I}$ is a limit diagram in $(\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{c}_I}$, then by Lemma 4.17, $F' \circ \overline{G}\colon L^{\triangleleft} \to \mathcal{S}_{/F(\bar{\mathfrak{e}})^{n+1}}$ is a limit diagram if and only if the induced diagram $(F' \circ \overline{G}) * p_n\colon L^{\triangleleft} * (\mathbf{n+1}) \to \mathcal{S}$ is a limit diagram in $\mathcal{S}$. If $q_n\colon \mathbf{n+1} \to \mathcal{V}^{\kappa,\vee,\mathrm{op}}$ denotes the constant functor at $\bar{\mathfrak{e}} \in (\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{e}}$, then $(F' \circ \overline{G}) * p_n$ is clearly given by the composition of $\overline{G} * q_n\colon L^{\triangleleft} * (\mathbf{n+1}) \to \mathcal{V}^{\kappa,\vee,\mathrm{op}}$ and $F\colon \mathcal{V}^{\kappa,\vee,\mathrm{op}} \to \mathcal{S}$.

Let Q denote the set of all functors $(\overline{G} * q_n)^{\mathrm{op}}\colon (L^{\triangleleft} * (\mathbf{n+1}))^{\mathrm{op}} \to \mathcal{V}^{\kappa,\vee}$ such that $\overline{G}, q_n$ are as above, $n \geq 0$, and L is κ-small. If an element $q \in Q$ is of the form $q\colon K^{\triangleright} \to \mathcal{V}^{\kappa,\vee}$, then the composition of q with the Yoneda embedding $\mathcal{V}^{\kappa,\vee} \to \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ induces a map of presheaves

$$\operatorname{colim}_{k\in K} q(k) \to q(\infty), \tag{4.2}$$

where we leave the Yoneda embedding implicit by abusing notation. Let $S_\Phi^{\lim}$ denote the strongly saturated set of morphisms in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ of the form 4.2 and let $q \in Q$. The Yoneda lemma and the observation above imply that a presheaf $F \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ induces a functor $\mathcal{V}^{\kappa,\mathrm{op}} \to \mathcal{S}_{/F(\bar{\mathfrak{e}})^{n+1}}$ which preserves κ-small limits if and only if F is local with respect to $S_\Phi^{\lim}$.

For $v \in \mathcal{V}^{\kappa,\vee}$ lying over $([m], I)$, let $p(v)\colon (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/} \to \mathcal{V}^{\kappa,\vee}$ denote the functor which carries $\alpha \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$ to the object $\alpha^*(v) \in \mathcal{V}^{\kappa,\vee}$ given by the Cartesian lift of α. By leaving the Yoneda embedding implicit, we define S_Φ^{Seg} to be the strongly saturated set generated by morphisms in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ of the form

$$\operatorname{colim}_{w\in p(v)} w \to v$$

with $v \in \mathcal{V}^{\kappa,\vee}$. The Yoneda lemma implies that a presheaf $F \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ satisfies the second condition of Definition 4.13 if and only if it is local with respect to the set S_Φ^{Seg}.

Definition 4.13 and Corollary 4.16 imply that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ can be regarded as a full subcategory of $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ spanned by those presheaves F which are local with respect to elements in both $S_\Phi^{\lim}$ as well as S_Φ^{Seg}. If $\overline{S}_\Phi$ denotes the strongly saturated set generated by the set $S_\Phi^{\lim} \coprod S_\Phi^{\mathrm{Seg}}$, then there is an equivalence of ∞-categories

$$\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})[\overline{S}_\Phi^{-1}].$$

Since $\mathcal{V}^{\kappa,\vee}$ is essentially small, the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is given by the accessible localization of the presentable ∞-category $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ and is presentable as well. □

Notation 4.21. *1. Let us recall the definitions of the sets $S_\Phi^{\lim}, S_\Phi^{\mathrm{Seg}}$ and $\overline{S}_\Phi$ from the proof of the previous proposition:*

Let Q denote the set of all functors $(\overline{G} * q_n)^{\mathrm{op}}\colon (L^{\triangleleft} * (\mathbf{n+1}))^{\mathrm{op}} \to \mathcal{V}^{\kappa,\vee}$, where

(a) the functor $\overline{G}\colon L^{\triangleleft} \to (\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{c}_I}$ is a small limit diagram and $\mathfrak{c}_I$ a corolla,

(b) the functor $q_n\colon \mathbf{n+1} \to \mathcal{V}^{\kappa,\vee,\mathrm{op}}$ is the constant functor at $\bar{\mathfrak{e}} \in (\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{e}}$.

By leaving the Yoneda embedding implicit, we define $S_\Phi^{\lim}$ to be the strongly saturated set (see Definition 4.19) generated by the set

$$\{\operatorname{colim}_{k\in K} q(k) \to q(\infty), q \in Q\}$$

of morphisms of presheaves.

For $v \in \mathcal{V}^{\kappa,\vee}$ lying over $([m], I)$, let $p(v)\colon (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/} \to \mathcal{V}^{\kappa,\vee}$ denote the functor which carries $\alpha \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$ to the object $\alpha^*(v) \in \mathcal{V}^{\kappa,\vee}$ given by the Cartesian lift of α. We define S_Φ^{Seg} to be the strongly saturated set generated by the set

$$\{\mathrm{colim}_{w \in p(v)} w \to v\}$$

of morphisms of presheaves. We write $\overline{S}_\Phi$ for the strongly saturated set generated by $S_\Phi^{\lim} \coprod S_\Phi^{\mathrm{Seg}}$.

2. *If Φ is the trivial operator category $*$, then S_*^{Seg} coincides with the set of Segal equivalences. Therefore, we will call elements in S_Φ^{Seg}* Segal equivalences *even if Φ is an arbitrary operator category.*

3. *We will often write v_{Seg} for the object $\mathrm{colim}_{w\in p(v)} w$ where $\mathrm{colim}_{w\in p(v)} w \to v$ is an element of S_Φ^{Seg}.*

4. *For $n \geq 0$, we write Δ^n_{Seg} for the pushout product $\Delta^{(0,1)} \coprod_{\{1\}} \Delta^{(1,2)} \coprod_{\{2\}} \cdots \coprod_{\{n-1\}} \Delta^{(n-1,n)}$.*

4.3 Enrichement via Presheaf Model

The main result of this section is Theorem 4.22, which implies that the concept of Segal Φ-presheaves is equivalent to that of enriched Φ-∞-operads introduced in the first chapter. This allows us to regard Segal Φ-presheaves as another incarnation of enriched Φ-∞-operads.

Theorem 4.22. *Let $\mathcal{V}$ be as in Notation 4.12. Then the ∞-category $\mathrm{Alg}_\Phi(\mathcal{V})$ of $\mathcal{V}$-enriched Φ-operads is equivalent to the ∞-category of Segal Φ-presheaves $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$.*

This theorem can be regarded as a generalization of [GH15, Theorem 4.5.3] which can be recovered from the theorem above by choosing the trivial operator category $*$ for Φ.

In the proof we will use the same notation as [GHN15]. Therefore, we first recall the following definition.

Definition 4.23. [GHN15, 9.1] Let $\mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We write $\mathcal{C}^{\triangleleft}_{\mathcal{D}}$ for the pushout

$$\mathcal{C} \times \Delta^1 \coprod_{\mathcal{C}\times\{0\}} \mathcal{D}$$

and $\mathcal{C}^{\triangleright}_{\mathcal{D}}$ for the pushout

$$\mathcal{C} \times \Delta^1 \coprod_{\mathcal{C}\times\{1\}} \mathcal{D}.$$

Moreover, we have the following equivalences: $(\mathcal{C}^{\triangleleft}_{\mathcal{D}})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{\triangleright}_{\mathcal{D}^{\mathrm{op}}}$ and $(\mathcal{C}^{\triangleright}_{\mathcal{D}})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{\triangleleft}_{\mathcal{D}^{\mathrm{op}}}$.

Notation 4.24. *In the proof of Theorem 4.22 below, if p is a (co)Cartesian fibration or a left/right fibration, then we will write $\widetilde{p}$ for the associated functor.*

Proof of Theorem 4.22. By the definition above, the functor $p^{\vee,\mathrm{op}}\colon \mathcal{V}^{\vee,\mathrm{op}} \to \Delta_\Phi^{\mathrm{op}}$ induces an ∞-category $(\mathcal{V}^{\vee,\mathrm{op}})^{\triangleright}_{\Delta_\Phi^{\mathrm{op}}}$ together with a canonical inclusion functor $j\colon \Delta_\Phi^{\mathrm{op}} \hookrightarrow (\mathcal{V}^{\vee,\mathrm{op}})^{\triangleright}_{\Delta_\Phi^{\mathrm{op}}} \simeq (\mathcal{V}^{\vee,\triangleleft}{}_{\Delta_\Phi})^{\mathrm{op}}$. By [GHN15, Proposition 9.5], the functor j induces a Cartesian fibration

$$j^*\colon \mathrm{P}(\mathcal{V}^{\vee,\triangleleft}_{\Delta_\Phi}) \to \mathrm{P}(\Delta_\Phi)$$

whose associated funtor $\mathrm{P}(\Delta_\Phi)^{\mathrm{op}} \simeq \mathrm{RFib}(\Delta_\Phi)^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sends a right fibration $\mathcal{C} \to \Delta_\Phi$ to the ∞-category of presheaves $\mathrm{P}(\mathcal{C} \times_{\Delta_\Phi} \mathcal{V}^\vee)$.

If $\tilde{\delta}_{\Phi,(-)}\colon \mathcal{S} \to \mathrm{P}(\Delta_\Phi)$ denotes the functor which assigns each object $X \in \mathcal{S}$ to the presheaf $\tilde{i}_X\colon \Delta_\Phi^{\mathrm{op}} \to \mathcal{S}$ as defined in Definition 2.6, then we write $q\colon \mathcal{E} \to \mathcal{S}$ for the Cartesian fibration given by the pullback diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "q"'] & \mathrm{P}(\mathcal{V}^{\vee,\triangleleft}_{\Delta_\Phi}) \arrow[d, "j^*"] \\
\mathcal{S} \arrow[r, "\tilde{\delta}_{\Phi,(-)}"'] & \mathrm{P}(\Delta_\Phi).
\end{tikzcd}$$

It follows from the construction that, for every $X \in \mathcal{S}$, the fibre of q over X can be identified with the ∞-category $\mathrm{P}(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee)$ which is equivalent to $\mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z})$ by [Lur09, Definition 3.2.2.13], where $\mathcal{Z} \to \Delta_\Phi^{\mathrm{op}}$ is a Cartesian fibration which we are going to describe now.

If $\tilde{\mathcal{Z}}$ denotes the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ spanned by the left fibrations, then, since left fibrations are closed under pullbacks, the evaluation at $\{1\}$ induces a Cartesian fibration $\mathrm{ev}_1\colon \tilde{\mathcal{Z}} \to \mathrm{Cat}_\infty$. If $p^{\vee,\mathrm{op}}$ denotes the coCartesian fibration $\mathcal{V}^{\vee,\mathrm{op}} \to \Delta_\Phi^{\mathrm{op}}$, then the Cartesian fibration $\mathcal{Z} \to \Delta_\Phi^{\mathrm{op}}$ from above is given by the following pullback diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] & \tilde{\mathcal{Z}} \arrow[d, "\mathrm{ev}_1"] \\
\Delta_\Phi^{\mathrm{op}} \arrow[r, "\tilde{p}^{\vee,\mathrm{op}}"'] & \mathrm{Cat}_\infty,
\end{tikzcd}$$

where $\tilde{p}^{\vee,\mathrm{op}}$ is defined as in Notation 4.24. Hence, an object in $\mathrm{P}(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee) \simeq \mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z})$ can be interpreted as a functor $F\colon \Delta_{\Phi,X}^{\mathrm{op}} \times \Delta^1 \to \mathrm{Cat}_\infty$ such that $F(-,\{1\})$ is given by the composite of the projection $\Delta_{\Phi,X}^{\mathrm{op}} \to \Delta_\Phi^{\mathrm{op}}$ and the functor $\tilde{p}^{\vee,\mathrm{op}}\colon \Delta_\Phi^{\mathrm{op}} \to \mathrm{Cat}_\infty$. If we write $\mathcal{Z}_0$ for the full subcategory of $\mathcal{Z}$ spanned by objects F such that, for every $z \in \Delta_{\Phi,X}^{\mathrm{op}}$, the left fibration $F(z,-)\colon \Delta^1 \to \mathrm{Cat}_\infty$ is associated to a representable functor, then $\mathcal{Z}_0 \simeq \mathcal{V}$ and we have an equivalence of ∞-categories

$$\mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z}_0) \simeq \mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{V}).$$

Using this equivalence we have that, for every $X \in \mathcal{S}$, the ∞-category $\mathrm{Alg}_{\Phi,X}(\mathcal{V})$ is equivalent to the full subcategory of $\mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z}_0)$ spanned by those functors which preserve inert morphisms. Furthermore, the equivalence $\mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z}) \simeq \mathrm{P}(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee)$ implies that $\mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z}_0)$ can be identified with the full subcategory $\mathrm{P}_0(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee) \subseteq \mathrm{P}(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee)$ spanned by presheaves $(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee)^{\mathrm{op}} \to \mathcal{S}$ such that, for every $z \in \Delta_{\Phi,X}^{\mathrm{op}}$ lying over $([m], I) \in \Delta_\Phi^{\mathrm{op}}$, the induced functor $(\{z\} \times_{\Delta_\Phi} \mathcal{V}^\vee)^{\mathrm{op}} \simeq (\mathcal{V}^{\vee,\mathrm{op}})_{([m],I)} \to \mathcal{S}$ preserves all small limits.

Let $p^{\vee,*}(\delta_{\Phi,X})\colon \Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee \to \mathcal{V}^\vee$ denote the right fibration given by the pullback:

$$\begin{tikzcd}
\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee \arrow[r] \arrow[d, "p^{\vee,*}(\delta_{\Phi,X})"'] & \Delta_{\Phi,X} \arrow[d, "\delta_{\Phi,X}"] \\
\mathcal{V}^\vee \arrow[r, "p^\vee"'] & \Delta_\Phi.
\end{tikzcd}$$

The left Kan extension along $(p^{\vee,*}(\delta_{\Phi,X}))^{\mathrm{op}}$ induces a map

$$p^{\vee,*}(\delta_{\Phi,X})_!\colon\ \mathrm{P}(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee) \to \mathrm{P}(\mathcal{V}^\vee)_{/\tilde{p}^{\vee,*}(\delta_{\Phi,X})},$$

which is an equivalence by [GHN15, Definition 9.9].

The description of the ∞-category $\mathrm{P}_0(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee)$ implies that this equivalence $p^{\vee,*}(\delta_{\Phi,X})_!$ carries $\mathrm{P}_0(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee)$ to the full subcategory $\mathrm{P}_0^X \subseteq \mathrm{P}(\mathcal{V}^\vee)_{/\tilde{p}^{\vee,*}(\delta_{\Phi,X})}$ spanned by functors such that, for every $([m], I) \in \Delta_\Phi$, the induced functor

$$(\mathcal{V}^\vee)_{([m],I)} \to \mathcal{S}_{/\tilde{\delta}_{\Phi,X}([m],I)}$$

preserves all small limits. In particular, if we regard an object of $\mathrm{P}(\mathcal{V}^\vee)_{/\tilde{p}^{\vee,*}(\delta_{\Phi,X})}$ as a functor $F\colon \mathcal{V}^{\vee,\mathrm{op}} \times \Delta^1 \to \mathcal{S}$, we have that

$$F(x, 0) \to F(x, 1) = \tilde{\delta}_{\Phi,X}(\mathfrak{e}) = X$$

is an equivalence for every $x \in (\mathcal{V}^{\vee,\mathrm{op}})_{\mathfrak{e}} \simeq \{*\}$, because the functor $\{*\} \simeq (\mathcal{V}^\vee)_{\mathfrak{e}} \to \mathcal{S}_{/X}$ preserves terminal objects and id_X is terminal in $\mathcal{S}_{/X}$. Thus, if $([m], I)$ has n-many edges, then $\tilde{\delta}_{\Phi,X}([m], I) \simeq X^n \simeq F(x, 0)^n$. Since $\tilde{\delta}_{\Phi,X}([m], I)$ is a terminal object in P_0^X, the natural projection map $\mathrm{P}(\mathcal{V}^\vee)_{/\tilde{p}^{\vee,*}(\delta_{\Phi,X})} \to \mathrm{P}(\mathcal{V}^\vee)$ carries P_0^X to an equivalent ∞-category $\mathrm{P}_0'^X \subseteq \mathrm{P}(\mathcal{V}^\vee)$ spanned by presheaves G such that, for every $x \in (\mathcal{V}^{\vee,\mathrm{op}})_{\mathfrak{e}}$ and every $([m], I) \in \Delta_\Phi$, the induced functor

$$(\mathcal{V}^\vee)_{([m],I)} \to \mathcal{S}_{/G(x)^n} \tag{4.3}$$

preserves all small limits. The chain of equivalences

$$\mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{V}) \simeq \mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{Z}_0) \simeq \mathrm{P}_0(\Delta_{\Phi,X} \times_{\Delta_\Phi} \mathcal{V}^\vee) \simeq \mathrm{P}_0^X \simeq \mathrm{P}_0'^X$$

carries the full subcategory $\mathrm{Alg}_{\Phi,X}(\mathcal{V}) \subseteq \mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{V})$ to a full subcategory $\mathcal{P}'^X_{\mathrm{Alg}} \subseteq \mathrm{P}_0'^X$ and we can identify $\mathrm{Alg}_\Phi(\mathcal{V})$ with the full subcategory of $\mathcal{E}$ spanned by objects lying in $\mathcal{P}'^X_{\mathrm{Alg}}$ for some $X \in \mathcal{S}$. In other words, the Cartesian fibration $q\colon \mathcal{E} \to \mathcal{S}$ restricts the Cartesian fibration $\mathrm{Alg}_\Phi(\mathcal{V}) \to \mathcal{S}$ as defined in Definition 3.9. In order to complete the proof we want to show that $\mathcal{P}'^X_{\mathrm{Alg}}$ coincides with the fibre of $\mathrm{ev}_*\colon\ \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \to \mathcal{S}$ at $X \in \mathcal{S}$, where ev_* denotes the evaluation functor at $* \in (\mathcal{V}^\vee)_{\mathfrak{e}} \simeq \{*\}$.

A functor $F \in \mathrm{Fun}_{\Delta_\Phi^{\mathrm{op}}}(\Delta_{\Phi,X}^{\mathrm{op}}, \mathcal{V})$ lies in $\mathrm{Alg}_{\Phi,X}(\mathcal{V})$ if and only if it preserves inert morphisms and according to Lemma 2.47 this is equivalent to requiring that, for every $([m], I) \in \Delta_\Phi^{\mathrm{op}}$ and every $x \in (\Delta_{\Phi,X}^{\mathrm{op}})_{([m],I)}$ lying over $([m], I)$, the functor F preserves inert morphisms of the form $x \to \alpha_! x$ for every $\alpha \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}$. In other words, F preserves inert morphisms if and only if the canonical maps $\alpha_! F(x) \to F(\alpha_! x)$ are equivalences for every $([m], I) \in \Delta_\Phi^{\mathrm{op}}$ and every $x \in (\Delta_{\Phi,X}^{\mathrm{op}})_{([m],I)}$. This means that a presheaf $G \in \mathrm{P}(\mathcal{V}^\vee)$ lies in the image of the functor $\mathrm{Alg}_{\Phi,X}(\mathcal{V}) \simeq \mathcal{P}'^X_{\mathrm{Alg}} \subseteq \mathrm{P}(\mathcal{V}^\vee)$ if and only if the following two conditions are satisfied:

1. G lies in P'^X_0, which is by equation 4.3 equivalent to requiring the induced functor $(\mathcal{V}^\vee)_{([m],I)} \to \mathcal{S}_{/G(x)^n}$ to preserve small limits for every $([m], I) \in \Delta_\Phi$.
2. $G(v) \simeq \lim_{\alpha \in (\Delta_\Phi^{\mathrm{el,op}})_{([m],I)/}} G(\alpha^* v)$ for every object $v \in \mathcal{V}^\vee$.

Using Remark 4.14, we see that the ∞-categories $\mathrm{Alg}_{\Phi,X}(\mathcal{V}) \simeq \mathcal{P}'^X_{\mathrm{Alg}}$ are equivalent to the fibre of $\mathrm{ev}_*\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \to \mathcal{S}$ at $X \in \mathcal{S}$. □

Remark 4.25. *Let $\mathcal{V}^\vee$ be as above and let $([m], I) \in \Delta_\Phi^{\mathrm{op}}$ be an object with n edges. If $M = \coprod_{k \in \mathbf{m}} |I(k)|$, then that the fibre $(\mathcal{V}^\vee)_{([m],I)}$ is equivalent to $(\mathcal{V}_{\langle 1 \rangle})^M$ and the proof of Theorem 4.22 implies that the equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ carries an object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$ lying over $X \in \mathcal{S}$ to the Segal Φ-presheaf determined by the assignment*

$$(\mathcal{V}^\vee)_{([m],I)} \ni v \mapsto \mathrm{colim}_{(x_i)_i \in X^n} \mathrm{Map}_{(\mathcal{V}^\vee)_{([m],I)}}(\mathcal{O}([m], I, \{x_i\}_i), v) \in \mathcal{S}.$$

Example 4.26. *Theorem 4.22 asserts that a morphism $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$ corresponds to a map in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$, which we denote by $\tilde{f}\colon \tilde{\mathcal{O}} \to \tilde{\mathcal{P}}$. If $\tilde{f}$ lies over $f_0\colon X \to Y$ in $\mathcal{S}$, then for a corolla $\mathfrak{c}_I \in \Delta_\Phi$ and objects $v \in (\mathcal{V}^\vee)_{\mathfrak{c}_I}$, $\bar{\mathfrak{e}} \in (\mathcal{V}^\vee)_{\mathfrak{e}}$, the equivalence $(\mathcal{V}^\vee)_{\mathfrak{c}_I} \simeq \mathcal{V}$ and the remark above imply that*

$$\tilde{\mathcal{O}}(\bar{\mathfrak{e}}) \simeq \mathrm{colim}_{x \in X} \mathrm{Map}_{(\mathcal{V}^\vee)_{\mathfrak{e}}}(\mathcal{O}([0], *, x), \bar{\mathfrak{e}}) \simeq \mathrm{colim}_{x \in X}\{*\} \simeq X.$$

If $|I| = n$, then the left fibration $\tilde{\mathcal{O}}(v) \to \prod_{\mathfrak{e} \in \Delta^{\mathrm{el}}_{\Phi/\mathfrak{c}_I}} \tilde{\mathcal{O}}(\bar{\mathfrak{e}}) = \tilde{\mathcal{O}}(\bar{\mathfrak{e}})^{n+1}$ is of the form

$$\tilde{\mathcal{O}}(v) \simeq \mathrm{colim}_{\{x_i\}_i \in X^{n+1}} \mathrm{Map}_{(\mathcal{V}^\vee)_{\mathfrak{c}_I}}(\mathcal{O}(\mathfrak{c}_I(\{x_i\}_i), v) \to X^{n+1},$$

where $\mathfrak{c}_I(\{x_i\}_i)$ is as defined in Notation 2.4. In particular, for an object $(x_1, \ldots, x_n, x) \in X^{n+1}$, we have

$$\{(x_1, \ldots, x_n, x)\} \times_{X^{n+1}} \tilde{\mathcal{O}}(v) \simeq \mathrm{Map}_{\mathcal{V}}(\mathcal{O}(\mathfrak{c}_I(x_1, \ldots, x_n; x), v).$$

Chapter 5

Tensor Product

In this chapter we are going to define the tensor product of enriched ∞-operads with ∞-categories. It allows us to regard the ∞-category of $\mathcal{V}$-enriched ∞-operads as a module over the symmetric monoidal ∞-category Cat_∞ with the Cartesian product.

This chapter contains three sections. We will define the tensor product at the presheaf level in the first section. Although the notion of the tensor product is easily introduced and it is not hard to show that the resulting functor preserves colimits in each variable, we have to work hard to prove that the tensor product between presheaf categories descends to one of Segal presheaf categories. To show this we have to verify that the tensor product preserves the Segal equivalences, i.e. elements in the sets S_Φ^{Seg} and $S_{\mathrm{Seg}}^{\lim}$. This is done in the following two sections and we prove the main result of this chapte in Theorem 5.5 at the end of the third section.

5.1 Tensor Product of Presheaves

Notation 5.1. *For the whole chapter, we will fix a presentable symmetric monoidal ∞-category $\mathcal{V}$ and a regular cardinal κ such that the full subcategory $\mathcal{V}^\kappa$ is symmetric monoidal and $\mathcal{V}_{\langle 1\rangle}$ is κ-presentable. The existence of such a $\mathcal{V}$ is provided by Corollary 4.8.*

Definition 5.2. Let $\mathcal{V}^\vee \to \Delta_\Phi$ be the Cartesian fibration as defined in Definition 4.9 and let the map $\omega\colon \mathcal{V}^\vee \to \mathcal{V}^\vee \times \Delta$ denote the inclusion $\mathcal{V}^\vee \simeq \mathcal{V}^\vee \times_\Delta \Delta \hookrightarrow \mathcal{V}^\vee \times \Delta$. Define

$$\mu\colon \mathrm{P}(\mathcal{V}^\vee) \times \mathrm{P}(\Delta) \to \mathrm{P}(\mathcal{V}^\vee \times \Delta)$$

to be the functor adjoint to the functor $(\mathrm{P}(\mathcal{V}^\vee) \times \mathcal{V}^{\vee,\mathrm{op}}) \times (\mathrm{P}(\Delta) \times \Delta^{\mathrm{op}}) \to \mathcal{S}$ given by the composition of the evaluations and the product functor $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$. We write

$$\otimes\colon \mathrm{P}(\mathcal{V}^\vee) \times \mathrm{P}(\Delta) \to \mathrm{P}(\mathcal{V}^\vee)$$

for the composite functor $\omega^* \circ \mu$.

It follows directly from the definition of the tensor product that, for $(F, G) \in \mathrm{P}(\mathcal{V}^\vee) \times \mathrm{P}(\Delta)$ and $X \in \mathcal{V}^\vee$ lying over $([m], I) \in \Delta_\Phi$, the ∞-groupoid $(F \otimes G)(X)$ is given by $F(X) \times G([m])$.

Remark 5.3. *If $f\colon K^{\triangleright} \to \mathrm{P}(\mathcal{V}^{\vee})$ is a colimit diagram in $\mathrm{P}(\mathcal{V}^{\vee})$ such that ∞ denotes the cone point, then, for every $G \in \mathrm{P}(\Delta)$ and $X \in \mathcal{V}^{\vee}$ lying over $([m], I)$, we have $(f(\infty) \otimes G)(X) = f(\infty)(X) \times G([m])$. Since colimits of presheaves are computed pointwise and products preserve colimits in $\mathcal{S}$, we further have $f(\infty)(X) \times G([m]) \simeq \mathrm{colim}_{k\in K}(f(k)(X) \times G([m])) \simeq \mathrm{colim}_{k\in K}(f(k) \otimes G)(X)$. Hence, we have $(\mathrm{colim}_{k\in K} f(k)) \otimes G \simeq \mathrm{colim}_{k\in K}(f(k) \otimes G)$, i.e. the tensor product $\otimes\colon \mathrm{P}(\mathcal{V}^{\vee}) \times \mathrm{P}(\Delta) \to \mathrm{P}(\mathcal{V}^{\vee})$ preserves colimits in the first variable. A similar argument shows that $\otimes$ also preserves colimits in the second variable.*

Notation 5.4. *Let $\mathrm{P}_{\mathrm{Seg}}(\Delta)$ denote the ∞-category of Segal spaces as defined in [Rez01], i.e. $\mathrm{P}_{\mathrm{Seg}}(\Delta) \simeq \mathrm{Seg}_*$ as defined in Definition 4.13.*

The main statement of this section is the following theorem.

Theorem 5.5. *If $\mathcal{V}$ is as defined in Notation 5.1 and $\mathcal{V}^{\vee}$ is as defined in Definition 4.9, then the functor $\otimes$ as given in Definition 5.2 descends to a functor of Segal Φ-presheaves*

$$\otimes\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \times \mathrm{P}_{\mathrm{Seg}}(\Delta) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$$

which preserves colimits in each variable.

We will provide the proof of this theorem at the end of this chapter.

Corollary 5.6. *There exists a functor*

$$\mathrm{Alg}^{\Phi}_{\mathrm{Seg}}(-,-)\colon (\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}))^{\mathrm{op}} \times \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \to \mathrm{P}_{\mathrm{Seg}}(\Delta)$$

which preserves limits in each variable.

Proof. The claim follows from the following general result. For two ∞-categories $\mathcal{A}$ and $\mathcal{B}$, let $\mathrm{Fun}^{L}(\mathcal{A},\mathcal{B})$ and $\mathrm{Fun}^{R}(\mathcal{A},\mathcal{B})$ denote the ∞-categories of left and right adjoint functors between $\mathcal{A}$ and $\mathcal{B}$, respectively. For presentable ∞-categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, we have that the ∞-category of functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits in each variable is equivalent to the ∞-category $\mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Fun}^{L}(\mathcal{D},\mathcal{E}))$. There is a chain of equivalences of ∞-categories

$$\mathrm{Fun}^{L}(\mathcal{C}, \mathrm{Fun}^{L}(\mathcal{D},\mathcal{E})) \simeq \mathrm{Fun}^{R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{L}(\mathcal{D},\mathcal{E})^{\mathrm{op}}) \simeq \mathrm{Fun}^{R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{R}(\mathcal{E},\mathcal{C}))$$

where the second equivalence is induced by the equivalence $\mathrm{Fun}^{L}(\mathcal{D},\mathcal{E})^{\mathrm{op}} \simeq \mathrm{Fun}^{R}(\mathcal{E},\mathcal{C})$ given by [Lur09, Proposition 5.2.6.2]. Since the ∞-category $\mathrm{Fun}^{R}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}^{R}(\mathcal{E},\mathcal{C}))$ is equivalent to the ∞-category of functors between $\mathcal{C}^{\mathrm{op}} \times \mathcal{E}$ and $\mathcal{D}$ which preserve limits in each variable, we see that, if $\mathcal{C} = \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$, $\mathcal{D} = \mathrm{P}_{\mathrm{Seg}}(\Delta)$ and $\mathcal{E} = \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$, then the equivalences above carries the functor $\otimes$ to a functor $\mathrm{Alg}^{\Phi}_{\mathrm{Seg}}(-,-)\colon (\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}))^{\mathrm{op}} \times \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \to \mathrm{P}_{\mathrm{Seg}}(\Delta)$ which preserves limits in each variable. □

Notation 5.7. *We will abuse notation and leave the Yoneda embedding implicit in the sequel of this chapter.*

The following lemma will be used fairly often in the material below. Given a Cartesian fibration $\mathcal{E} \to \mathcal{C}$, although we do not know whether it induces a Cartesian fibration $p_!\colon \mathrm{P}(\mathcal{E}) \to \mathrm{P}(\mathcal{C})$, the lemma guarantees nevertheless the existence of $p_!$-Cartesian lifts of morphisms between representable presheaves.

Lemma 5.8. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a Cartesian fibration and let $p^*\colon \mathrm{P}(\mathcal{C}) \to \mathrm{P}(\mathcal{E})$ denote the induced functor. If $\bar{s} \to \bar{t}$ a p-Cartesian morphism lying over $s \to t$ in $\mathcal{C}$, then there is a pullback square in $\mathrm{P}(\mathcal{E})$*

$$\begin{tikzcd} \bar{s} \arrow[r] \arrow[d] & \bar{t} \arrow[d] \\ p^*s \arrow[r] & p^*t, \end{tikzcd}$$

where the vertical maps are the adjunction units $\bar{s} \to p^*p_!\bar{s} \simeq p^*s$ and $\bar{s} \to p^*p_!\bar{s} \simeq p^*t$, respectively.

Proof. We have to show that for every presheaf $F \in \mathrm{P}(\mathcal{E})$, the commutative square

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{P}(\mathcal{E})}(F, \bar{s}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{P}(\mathcal{E})}(F, \bar{t}) \arrow[d] \\ \mathrm{Map}_{\mathrm{P}(\mathcal{E})}(F, p^*s) \arrow[r] & \mathrm{Map}_{\mathrm{P}(\mathcal{E})}(F, p^*t) \end{tikzcd}$$

is a pullback square of simplicial sets. Since every object in the presheaf category $\mathrm{P}(\mathcal{E})$ is given by a colimit of representable objects we can assume without loss of generality that F is represented by an object $\bar{x} \in \mathcal{E}$ lying over $x \in \mathcal{C}$. In this case, the adjunction $(p_!, p^*)$ implies that the above square is of the form

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{E}}(\bar{x}, \bar{s}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{E}}(\bar{x}, \bar{t}) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(x, s) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(x, t), \end{tikzcd}$$

which is a pullback of simplicial sets because the morphism $\bar{s} \to \bar{t}$ was p-Cartesian. □

5.2 Preserving Segal Equivalences in S_Φ^{Seg}

The main result of this section is Proposition 5.16 which implies that the tensor product introduced in the previous section preserves all elements in S_Φ^{Seg}.

Definition 5.9. Let $p^\vee\colon \mathcal{V}^\vee \to \Delta_\Phi$ be the Cartesian fibration as defined in Definition 4.9 and let $p\colon \mathcal{V}^\vee \to \Delta$ denote the composition of $p^\vee$ and the Cartesian fibration $p'\colon \Delta_\Phi \to \Delta$ given by projection. For a morphism $\varphi\colon [m] \to [n]$ in Δ and an object $x \in \mathcal{V}^\vee$ lying over $[n]$, we write $\varphi^*x \to x$ for the p-Cartesian lift of φ at x.

We define $\mathcal{J}^n$ to be the partially ordered set of faces of Δ^n (meaning injective maps $[m] \hookrightarrow [n]$ in Δ). For $i_1 < i_2 < \ldots < i_k$, we write $(i_1, \ldots, i_k)$ for the object in $\mathcal{J}^n$ given by the non-empty subsets of $\{i_1, \ldots, i_k\} \subseteq \{0, \ldots, n\}$. Given a full subcategory (i.e. a partially ordered subset) $\mathcal{J}' \subseteq \mathcal{J}^n$ and $x \in \mathcal{V}^\vee$ of length n, we let $x(\mathcal{J}')$ denote the colimit in $\mathcal{P}(\mathcal{V}^\vee)$ over $\varphi \in \mathcal{J}'$ of φ^*x. If $\partial\mathcal{J}^n$ denotes the subcategory of $\mathcal{J}^n$ containing all objects except $(0, \ldots, n)$, then we write ∂x for $x(\partial\mathcal{J}^n)$. If $\mathcal{J}^n_k$ is the full subcategory of $\partial\mathcal{J}^n$ containing all objects except $(0, \ldots, k-1, k+1, \ldots, n)$, then we write $\Lambda^n_k x$ for $x(\mathcal{J}^n_k)$.

Similarly, for an object $([m], I) \in \Delta_\Phi$, we write $([m], I)(\mathcal{J}')$ for the colimit in $\mathrm{P}(\Delta_\Phi)$ over

$\phi \in \mathcal{J}'$ of $\phi^*([m], I)$, where $\phi^*([m], I) \to ([m], I)$ is the p'-Cartesian lift of ϕ, and we write $\Lambda^n_k([m], I)$ for $([m], I)(\mathcal{J}^n_k)$.

Notation 5.10. *For an object $(i_1, \dots, i_k) \in \mathcal{J}^n$, we write $\overline{(i_1, \dots, i_k)}$ for the full subcategory of $\mathcal{J}^n$ spanned by non-empty subsets $(i_{j1}, \dots, i_{jm}) \subseteq (i_1, \dots, i_k)$. We write $\bigcup_{k \in K} \overline{(i_{k1}, \dots, i_{km_k})}$ for the full subcategory of $\mathcal{J}^n$ spanned by objects which lie in at least one of the subcategories $\overline{(i_{k1}, \dots, i_{km_k})}$, $k \in K$.*

Lemma 5.11. *For an object $x \in \mathcal{V}^\vee$ of length $n \geq 2$ and $0 < k < n$, the map $\Lambda^n_k x \to x$ in $\mathrm{P}(\mathcal{V}^\vee)$ lies in S^{Seg}_Φ (see Notation 4.21), where Λ^n_k is as defined above.*

Proof. Let x_{Seg} be as denoted in Notation 4.21. The Segal equivalence $x_{\mathrm{Seg}} \to x$ is the composite of $f \colon x_{\mathrm{Seg}} \to \Lambda^n_k x$ and $\Lambda^n_k x \to x$. Hence, the 2-of-3 property implies that it suffices to verify that f is a Segal equivalence. We prove the statement by induction on the length of $x \in \mathcal{V}^\vee$.

If x is of length 2, then, by definition $\Lambda^2_i x$ is the pushout $x(0,1) \coprod_{x(1)} x(1,2)$, and x_{Seg} is equivalent to $x(0,1)_{\mathrm{Seg}} \coprod_{x(1)_{\mathrm{Seg}}} x(1,2)_{\mathrm{Seg}}$. Since the set S^{Seg}_Φ is strongly saturated, it is closed under colimits and we have that $x_{\mathrm{Seg}} \to \Lambda^2_i x$ is a Segal equivalence.

Now let us assume that the statement of the lemma is true for $x \in \mathcal{V}^\vee$ of length smaller than $n \geq 2$ and $k < n-1$. Then we see that, for x of length n, the map f factors as $g^n \colon x_{\mathrm{Seg}} \to A$ and $h^n \colon A \to \Lambda^n_k x$, where A denotes $x(0, \dots, n-1) \coprod_{x(n-1)} x(n-1, n)$. The map g^n is given by a pushout diagram

$$\begin{tikzcd}
x(0, \dots, n-1)_{\mathrm{Seg}} \arrow[r] \arrow[d] & x_{\mathrm{Seg}} \arrow[d, "g^n"] \\
x(0, \dots, n-1) \arrow[r] & A,
\end{tikzcd}$$

where the left vertical map is a Segal equivalence by the induction hypothesis. Therefore, we only need to show that the map $h^n \colon A \to \Lambda^n_k x$ is a Segal equivalence. We will show that h^n is given by a composite of of $h^n_0 \colon A \to x(\mathcal{J}^n(0))$ and $h^n_i \colon x(\mathcal{J}^n(i-1)) \to x(\mathcal{J}^n(i))$ in $\mathrm{P}(\mathcal{V}^\vee)$, for $i \in \{1, \dots, n-1\} \setminus \{k\}$ such that each morphism h^n_i is a Segal equivalence.

For $i \in \{0, \dots, n-1\} \setminus \{k\}$, let

$$\mathcal{J}^n(i) = \bigcup_{0 \leq j \leq i, j \neq k} \overline{(0, \dots j-1, j+1, \dots, n)} \cup \overline{(0, \dots, n-1)}.$$

Roughly speaking, the presheaf $x(\mathcal{J}^n(i))$ is given by gluing the one n-face of x which does not contain $\{i\}$ to $x(\mathcal{J}^n(i-1))$. In particluar, the definition implies that the subcategory $\mathcal{J}^n(n-1)$ contains all objects of $\mathcal{J}^n$ except $(0, \dots, n)$ and $(0, \dots, k-1, k+1, \dots, n)$. Hence, the presheaf $x(\mathcal{J}^n(n-1))$ coincides with $\Lambda^n_k x$.

For every object $(i_1, \dots, i_k) \in \mathcal{J}^n$, the category $\overline{(i_1, \dots, i_k)}$ has a terminal object $(i_1, \dots, i_k)$ and [Lur09, Definition 4.2.3.10] provides commutative diagram

$$\begin{tikzcd}
x(1, \dots, n-1) \arrow[r] \arrow[d] & x(1, \dots, n-1) \coprod_{x(n-1)} x(n-1, n) \arrow[r, "h'_0"] \arrow[d] & x(1, \dots, n) \arrow[d] \\
x(0, \dots, n-1) \arrow[r] & A \arrow[r, "h^n_0"'] & x(\mathcal{J}^n(0)),
\end{tikzcd}$$

such that the left square as well as the outer square are pushout squares. Let $h_0^n\colon A \to x(\mathcal{J}^n(0))$ be the lower horizontal map of the right square, which is a pushout accordingt to [Lur09, Lemma 4.4.2.1]. The induction hypothesis implies that the map h_0' in the diagram is a Segal equivalence which implies that h_0^n is also a Segal equivalence.

For $i \in \{1,\dots,n-1\} \setminus \{k\}$, let $\mathcal{J}'^n(i)$ be given by

$$\mathcal{J}'^n(i) = \bigcup_{0\leq j<i, j\neq k} \overline{(0,\dots j-1, j+1,\dots,i-1,i+1,\dots,n)} \cup \overline{(0,\dots,i-1,i+1,\dots,n-1)}$$

and let $l_i\colon \mathcal{J}'^n(i) \to \mathcal{J}^n(i-1), i \in \{1,\dots,n-1\}\setminus\{k,+1\}$ and $l_{k+1}\colon \mathcal{J}'^n(k+1) \to \mathcal{J}^n(k-1)$ denote the canonical inclusion functors. For $i \in \{1,\dots,n-1\}\setminus\{k\}$, the category $\mathcal{J}'^n(i)$ is a full subcategory of $\overline{(0,\dots,i-1,i+1,\dots,n)}$ and there is a canonical map

$$h_i'^n\colon x(\mathcal{J}'^n(i)) \to x\overline{(0,\dots,i-1,i+1,\dots,n)} \simeq x(0,\dots,i-1,i+1,\dots,n),$$

where the equivalence is provided by the fact that the category $\overline{(0,\dots,i-1,i+1,\dots,n)}$ has a terminal object given by $(0,\dots,i-1,i+1,\dots,n)$.

If $i \in \{1,\dots,n-1\}\setminus\{k,k+1\}$, we define the map $h_i^n\colon x(\mathcal{J}^n(i-1)) \to x(\mathcal{J}^n(i))$ of presheaves to be given by the commutative diagram

$$\begin{tikzcd} x(\mathcal{J}'^n(i)) \arrow[r, "l_i"] \arrow[d, "h_i'^n"'] & x(\mathcal{J}^n(i-1)) \arrow[d, "h_i^n"] \\ x(0,\dots,i-1,i+1,\dots,n) \arrow[r] & x(\mathcal{J}^n(i)), \end{tikzcd} \tag{5.1}$$

which is a pushout diagram by [Lur09, Definition 4.2.3.10]. For $i = k+1$, we define the map $h_{k+1}^n\colon x(\mathcal{J}^n(k-1)) \to x(\mathcal{J}^n(k+1))$ of presheaves to be given by the commutative diagram

$$\begin{tikzcd} x(\mathcal{J}'^n(k+1)) \arrow[r, "l_{k+1}"] \arrow[d, "h_{k+1}'^n"'] & x(\mathcal{J}^n(k-1)) \arrow[d, "h_{k+1}^n"] \\ x(0,\dots,k,k+2,\dots,n) \arrow[r] & x(\mathcal{J}^n(k+1)), \end{tikzcd} \tag{5.2}$$

which is again a pushout diagram by [Lur09, Definition 4.2.3.10].

It is easy to see that in the case $n = 2$ the presheaves A and $x(\mathcal{J}^n(0))$ coincide with $x(0,1)\coprod_{x(1)} x(1,2)$, g^2 is a Segal equivalence and h_1^2 is the identity. Hence, we can assume that the maps g^m and h_i^m are Segal equivalences, for all $m < n$ and $i \in \{1,\dots,m-1\}\setminus\{k\}$.

The definition of the maps h_i^n implies that $h^n\colon A \to \Lambda_k^n x$ is given by the composite $h_{n-1}^n \circ \dots h_{k+1}^n \circ h_{k-1}^n \circ \dots \circ h_0^n$. Since every map h_i^n is given by a pushout of the map $h_i'^n\colon x(\mathcal{J}'^n(i)) \to x(0,\dots,i-1,i+1,\dots,n)$ and the Segal equivalence $x(0,\dots,i-1,i+1,\dots,n)_{\mathrm{Seg}} \to x(0,\dots,i-1,i+1,\dots,n)$ is given by the composition of $h_i'^n$ and

$$h_i''^n\colon x(0,\dots,i-1,i+1,\dots,n)_{\mathrm{Seg}} \to x(\mathcal{J}'^n(i)),$$

the 2-of-3 property implies that we only need to verify that the maps $h_i''^n$ are Segal equivalences.

Under the identification $(0,\ldots,i-1,i+1,\ldots,n) \simeq (0,\ldots,n-1)$, we have $x(\mathcal{J}'^n(i)) \simeq x(\mathcal{J}^{n-1}(i-1))$ and $h''^n_i \simeq h^{n-1}_{i-1} \circ \ldots \circ h^{n-1}_0 \circ g^{n-1}$. The induction hypothesis and the 2-of-3 property imply that h''^n_i is a Segal equivalence.

A dual argument implies that $x_{\mathrm{Seg}} \to \Lambda^n_k x$ is a Segal equivalence for $k > 1$. □

Lemma 5.12. *For every object x in Δ_Φ, the map $\tilde{f}\colon x \otimes \Lambda^2_1 \coprod_{\partial x \otimes \Lambda^2_1} \partial x \otimes \Delta^2 \to x \otimes \Delta^2$ of presheaves lies in S^{Seg}_Φ, where ∂x is as introduced in Definition 5.9.*

Proof. We first recall a fitration of the map $f\colon \Delta^n \otimes \Lambda^2_1 \coprod_{\partial\Delta^n \otimes \Lambda^2_1} \partial\Delta^n \otimes \Delta^2 \to \Delta^n \otimes \Delta^2$ of simplicial sets given by the proof of [Lur09, Proposition 2.3.2.1]. Then we will use Lemma 5.8 to lift the fitration to one of $\tilde{f}$ and we will see that $\tilde{f}$ is a composition of Segal equivalences.

As in the proof of [Lur09, Proposition 2.3.2.1], we define an $n+1$-simplex $\sigma_{i,j}$ of $\Delta^n \times \Delta^2$, for each $0 \leq i \leq j < n$, determined by the map

$$s_{ij}\colon [n+1] \to [n] \times [2]$$

$$s_{ij}(k) = \begin{cases} (k,0) & \text{if } 0 \leq k \leq i, \\ (k-1,1) & \text{if } i+1 \leq k \leq j+1, \\ (k-1,2) & \text{if } j+2 \leq k \leq n+1. \end{cases}$$

If we define $F_0 = \Delta^n \otimes \Lambda^2_1 \coprod_{\partial\Delta^n \otimes \Lambda^2_1} \partial\Delta^n \otimes \Delta^2$ and set $F_{k+1} = F_k \cup \sigma_{0,j} \ldots \cup \sigma_{j,j}$, for $0 \leq j < n$, then we have a filtration

$$F_j \hookrightarrow F_j \cup \sigma_{0j} \hookrightarrow \ldots \hookrightarrow F_j \cup \sigma_{0,j} \ldots \cup \sigma_{j,j} = F_{j+1}$$

such that each of the inclusions is a pushout of an inner horn inclusion. For each $0 \leq i \leq j \leq n$, we write $\tau_{i,j}$ for the $(n+2)$-simplex of $\Delta^n \times \Delta^2$ determined by the map

$$t_{ij}\colon [n+2] \to [n] \times [2]$$

$$t_{ij}(k) = \begin{cases} (k,0) & \text{if } 0 \leq k \leq i, \\ (k-1,1) & \text{if } i+1 \leq k \leq j+1, \\ (k-2,2) & \text{if } j+2 \leq k \leq n+2. \end{cases}$$

If we define $F_{j+1} = F_j \cup \sigma_{0,j-n} \ldots \cup \sigma_{j-n,j-n}$, for $n \leq j \leq 2n$, then we have a filtration

$$F_j \cup \tau_{0,j-m} \hookrightarrow \ldots \hookrightarrow F_j \cup \sigma_{0,j-m} \ldots \cup \sigma_{j-m,j-m} = F_{j+1}$$

such that each of the inclusions is a pushout of an inner horn inclusion. The observation that $F_{n+1} = \Delta^n \times \Delta^2$ implies that $f\colon \Delta^n \otimes \Lambda^2_1 \coprod_{\partial\Delta^n \otimes \Lambda^2_1} \partial\Delta^n \otimes \Delta^2 \to \Delta^n \otimes \Delta^2$ is given by a composition of inclusions of the maps $f_k\colon F_{k-1} \to F_k$, for $1 \leq k \leq 2n+1$ which are given by pushouts of an inner horn inclusions. By identifying the category sSet of simplicial sets with the subcategory of $\mathrm{P}(\Delta)$ spanned by objects F sucht that $F([n])$ is a discrete simplicial set for every n, we regard all the morphisms f_k as morphisms in $\mathrm{P}(\Delta)$.

In the following we first want to see that $\tilde{f}$ is given by a pullback of f. From the observation above we know that f is a composition of maps f_k and we want to lift these maps to $\tilde{f}_k$ in $\mathrm{P}(\mathcal{V}^\vee)$.

Using the property that each map f_k is a pushout of an inner horn inclusion, Lemma 5.8 then implies that the morphisms $\tilde{f}_k$ in $\mathrm{P}(\mathcal{V}^\vee)$ are given by pushouts of Segal equivalences.

If $p\colon \mathcal{V}^\vee \to \Delta$ denotes the composite of the Cartesian fibrations $p^\vee\colon \mathcal{V}^\vee \to \Delta_\Phi$ and $\Delta_\Phi \to \Delta$ and $p^*\colon \mathrm{P}(\Delta) \to \mathrm{P}(\mathcal{V}^\vee)$ is the induced functor, then it follows from the definition that $p^*(\Delta^n \times \Delta^m) = p^*\Delta^n \otimes \Delta^m$. Since the functor p^* is a left adjoint, it preserves colimits and therefore there exists a commutative square

$$\begin{tikzcd} x \otimes \Lambda^2_1 \coprod_{\partial x \otimes \Lambda^2_1} \partial x \otimes \Delta^2 \arrow[r, "\tilde{f}"] \arrow[d] & x \otimes \Delta^2 \arrow[d] \\ p^*\Delta^n \otimes \Lambda^2_1 \coprod_{\partial p^*\Delta^n \otimes \Lambda^2_1} \partial p^*\Delta^n \otimes \Delta^2 \arrow[r] & p^*\Delta^n \otimes \Delta^2, \end{tikzcd} \tag{5.3}$$

for every $x \in \mathcal{V}^\vee$ lying over n, where the bottom horizontal map is $p^*(f)$. By the definition of the tensor product, the evaluation of the diagram 5.3 at an object $y \in \mathcal{V}^\vee$ lying over $[m]$ provides a commutative diagram in $\mathcal{S}$

$$\begin{tikzcd} x(y) \times \Lambda^2_1(m) \coprod_{\partial x(y) \times \Lambda^2_1(m)} \partial x(y) \times \Delta^2(m) \arrow[r, "\tilde{f}(y)"] \arrow[d] & x(y) \times \Delta^2(m) \arrow[d] \\ \Delta^n(m) \times \Lambda^2_1(m) \coprod_{\partial \Delta^n(m) \times \Lambda^2_1(m)} \partial \Delta^n(m) \times \Delta^2(m) \arrow[r] & \Delta^n(m) \times \Delta^2(m). \end{tikzcd}$$

Since $\mathcal{S}$ is an ∞-topos, pullbacks preserve colimits. Therefore, this commutative square is a pullback square if the following commutative squares

$$\begin{tikzcd} x(y) \times \Lambda^2_1(m) \arrow[r] \arrow[d] & x(y) \times \Delta^2(m) \arrow[d] \\ \Delta^n(m) \times \Lambda^2_1(m) \arrow[r] & \Delta^n(m) \times \Delta^2(m), \end{tikzcd} \quad \begin{tikzcd} \partial x(y) \times \Lambda^2_1(m) \arrow[r] \arrow[d] & x(y) \times \Delta^2(m) \arrow[d] \\ \partial \Delta^n(m) \times \Lambda^2_1(m) \arrow[r] & \Delta^n(m) \times \Delta^2(m) \end{tikzcd}$$

and

$$\begin{tikzcd} \partial x(y) \times \Delta^2(m) \arrow[r] \arrow[d] & x(y) \times \Delta^2(m) \arrow[d] \\ \partial \Delta^n(m) \times \Delta^2(m) \arrow[r] & \Delta^n(m) \times \Delta^2(m) \end{tikzcd}$$

are all pullback squares in $\mathcal{S}$. But this ist true because they are products of pullback squares. In particular, we have that the commutative square 5.3 is a pullback square.

As mentioned above $f\colon \Delta^n \otimes \Lambda^2_1 \coprod_{\partial \Delta^n \otimes \Lambda^2_1} \partial \Delta^n \otimes \Delta^2 \to \Delta^n \otimes \Delta^2$ is given by compositions of maps $f_k\colon F_{k-1} \to F_k$. We define $p_k\colon \tilde{F}_k \to p^*F_k$ to be the morphism given by the pullback of $x \otimes \Delta^2 \to p^*\Delta^n \otimes \Delta^2$ along $p^*F_k \to p^*\Delta^n \otimes \Delta^2$ and we write $\tilde{f}_k\colon \tilde{F}_{k-1} \to \tilde{F}_k$ for the pullback of p^*f_k along p_k. Since the diagram 5.3 is a pullback square, we have that $\tilde{f}$ is given by the composite of the morphisms $\tilde{f}_k$. Therefore, the proof is complete, if we can show that every morphism $\tilde{f}_k$ is a Segal equivalence.

Pullback along the morphism $p_k\colon \tilde{F}_k \to p^*F_k$ induces a commutative diagram

$$\begin{tikzcd}
& \tilde{G}^-_{k-1} \arrow[rr] \arrow[dl] \arrow[dd] & & \tilde{F}_{k-1} \arrow[dl] \arrow[dd] \\
\tilde{G}^+_k \arrow[rr] \arrow[dd] & & \tilde{F}_k \arrow[dd, "p_k"] & \\
& p^*L^-_{k-1} \arrow[rr] \arrow[dl] & & p^*F_{k-1} \arrow[dl, "p^*f_k"] \\
p^*L^+_k \arrow[rr] & & p^*F_k, &
\end{tikzcd} \tag{5.4}$$

where the bottom square is the image of the defining pushout square of the morphism f_k under the functor p^*, i.e. the map $L^-_{k-1} \to L^+_k$ is an inner horn inclusion. As mentioned above, the bottom square of the cube is a pushout diagram, because p^* admits a right adjoint. Using the fact that $\mathrm{P}(\mathcal{V}^\vee)$ is an ∞-topos and pullbacks in ∞-topoi preserve colimits, we obtain that the upper square of the cube diagram is also a pushout.

Therefore, the proof of the lemma is reduced to showing that the morphisms $\tilde{G}^-_{k-1} \to \tilde{G}^+_k$ are Segal equivalences. A closer inspection of the proof of [Lur09, Proposition 2.3.2.1] reveals that the morphism $L^-_{k-1} \to L^+_k$ in the cube above is an inner horn inclusion either of the form $\Lambda^{n+1}_i \to \Delta^{n+1}, 0 < i < n+1$, or $\Lambda^{n+2}_i \to \Delta^{n+2}, 0 < i < n+2$. It follows from the construction that $\tilde{G}^+_k \to p^*\Delta^m$ is given by the pullback square

$$\begin{tikzcd}
\tilde{G}^+_k \arrow[r] \arrow[d] & x \otimes \Delta^2 \arrow[d, "j"] \\
p^*\Delta^m \arrow[r, "g"'] & p^*\Delta^n \otimes \Delta^2,
\end{tikzcd}$$

for $m \in \{n+1, n\}$. If $\mathrm{pr}\colon p^*\Delta^n \otimes \Delta^2 \to p^*\Delta^n$ denotes the map induced by the projection $\Delta^n \times \Delta^2 \to \Delta^n$, then by evaluating the commutative diagram

$$\begin{tikzcd}
x \otimes \Delta^2 \arrow[r] \arrow[d] & x \arrow[d] \\
p^*\Delta^n \times \Delta^2 \arrow[r, "\mathrm{pr}"'] & p^*\Delta^n
\end{tikzcd}$$

at objects in $\mathcal{V}^\vee$ one easily checks that it is a pullback square. As a consequence the commutative square

$$\begin{tikzcd}
\tilde{G}^+_k \arrow[r] \arrow[d] & x \arrow[d] \\
p^*\Delta^m \arrow[r, "\mathrm{pr} g"'] & p^*\Delta^n
\end{tikzcd}$$

is a pullback square and, according to Lemma 5.8 and the proof of [Lur09, Proposition 2.3.2.1], it is induced by a p-Cartesian morphism $a_k \to x$ in $\mathcal{V}^\vee$ lying over $\Delta^m \to \Delta^n$ induced by a surjective map $[m] \to [n]$. In particular, the presheaf $\tilde{G}_k^+$ is representable by an object $a_k \in \mathcal{V}^\vee$. Since all the vertical squares in the diagram 5.4 are pullback squares by construction, the commutative square

$$\begin{tikzcd}
\tilde{G}_{k-1}^- \arrow[r] \arrow[d] & a_k \arrow[d] \\
p^*\Lambda_i^m \arrow[r] & p^*\Delta^m
\end{tikzcd}$$

is in particular a pullback square. Using the fact that pullbacks preserve colimits in ∞-topoi once more, we have that $\tilde{G}_{k-1}^-$ is given by $\Lambda_i^m a_k$. Hence, all morphisms $\tilde{G}_{k-1}^- \to \tilde{G}_k^+$ are of the form $\Lambda_i^m a_k \to a_k, 0 < i < m$, which are are Segal equivalences by Lemma 5.11. $\square$

Lemma 5.13. *For every object $F \in \mathrm{P}(\mathcal{V}^\vee)$, the map $F \otimes \Lambda_1^2 \to F \otimes \Delta^2$ in $\mathrm{P}(\mathcal{V}^\vee)$ is a Segal equivalence.*

Proof. Every presheaf is a colimit of representable presheaves and the tensor product preserves colimits in both variables by Remark 5.3. Furthermore the set of Segal equivalences is closed under colimits. Therefore, we only need to show the claim for representable presheaves, i.e. we want to see that the map $f\colon x \otimes \Lambda_1^2 \to x \otimes \Delta^2$ is a Segal equivalence of presheaves, for every object $x \in \mathcal{V}^\vee$. We prove this by induction on the length of the object x.

If $x \in \mathcal{V}^\vee$ has length 0, then it is equivalent to the terminal object and f coincides with the inner horn inclusion $\Lambda_1^2 \to \Delta^2$ regarded as a map in $\mathrm{P}(\mathcal{V}^\vee)$. Let us now assume that the statement is true for all objects of length smaller than n. If x has length n, then the presheaf ∂x is a colimit of presheaves represented by faces of x which are of length smaller than n. Since the tensor product preserves colimits, the induction hypothesis implies that $g\colon \partial x \otimes \Lambda_1^2 \to \partial x \otimes \Delta^2$ is a Segal equivalence. It follows from this that the map $f_1\colon x \otimes \Lambda_1^2 \to x \otimes \Lambda_1^2 \coprod_{\partial x \otimes \Lambda_1^2} \partial x \otimes \Delta^2$ given by the pushout of g is also a Segal equivalence, as is the map $f_2\colon x \otimes \Lambda_1^2 \coprod_{\partial x \otimes \Lambda_1^2} \partial x \otimes \Delta^2 \to x \otimes \Delta^2$ by Lemma 5.12. As is a composite of Segal equivalences f_1 and f_2, the map f is then a Segal equivalence itself. $\square$

The lemma below is a special case of [Lur09, Definition 2.3.2.4]. We provide the proof which is a slight variation of [Lur09, Proposition 2.3.2.1], because we are going to use the filtration defined here in the next lemma.

Lemma 5.14. *For every n and $0 < k < n$, the map*

$$(\Lambda_k^n \times \Delta^1) \coprod_{\Lambda_k^n \times \partial \Delta^1} (\Delta^n \times \partial \Delta^1) \hookrightarrow \Delta^n \times \Delta^1$$

is a composite of pushouts of inner horn inclusions.

Proof. For the sake of simplicity, we write F_0 for $(\Lambda_k^n \times \Delta^1) \coprod_{\Lambda_k^n \times \partial \Delta^1} (\Delta^n \times \partial \Delta^1)$. The idea of the proof is to show that the map $F_0 \to \Delta^n \times \Delta^1$ can be realized as a composite of maps which are pushouts along inner horn inclusions.

For $j \in \{0, \dots, n-1\} \setminus \{k\}$, let σ_j denote the n-simplex of $\Delta^n \times \Delta^1$ determined by the map

$$f_j \colon [n] \to [n] \times [1]$$

$$f_j(i) = \begin{cases} (i,0) & \text{if } 0 \leq i \leq j \\ (i,1) & \text{if } j+1 \leq i \leq n. \end{cases}$$

We see that the n-simplex σ_j is given the set of vertices $V(\sigma_j) = \{(0,0), \dots (j,0), (j+1,1), \dots, (n,1)\}$ and every face of σ_j (i.e. non-empty subset of $V(\sigma_j)$) lies in F_0 except the one determined by the set $V(\sigma_j) \setminus \{(k,0)\}$, if $k < j$, and $V(\sigma_j) \setminus \{(k,1)\}$, if $k > j$. This observation implies that we can define

$$F_{j+1} = F_0 \cup \sigma_0 \cup \dots \cup \sigma_j,$$

for $j \in \{0, \dots, n-1\} \setminus \{k\}$, and the inclusion map $F_i \hookrightarrow F_j$, for $i, j \in \{0, \dots, n\} \setminus \{k+1\}$ and $i < j$, is given by compositions of pushouts of inner horn inclusions.

Set $G_0 = F_n$, if $k \neq n-1$ and $G_0 = F_{n-1}$ otherwise. It follows from the construction that G_0 contains all the nondegenerated m-simplices of $\Delta^n \times \Delta^1$ for $m \leq n$. Now, we want to add all the missing nondegenerated $n+1$-simplices of $\Delta^n \times \Delta^1$ to G_0. For this reason let τ_j, $0 \leq j \leq n$, denote the $n+1$-simplex of $\Delta^n \times \Delta^1$ determined by the map

$$g_j \colon [n+1] \to [n] \times [1]$$

$$g_j(i) = \begin{cases} (i,0) & \text{if } 0 \leq i \leq j \\ (i-1,1) & \text{if } j+1 \leq i \leq n+1. \end{cases}$$

The $n+1$-simplex τ_j is determined by the set of vertices $V(\tau_j) = \{(0,0), \dots (j,0), (j,1), \dots, (n,1)\}$ and all the faces of τ_j lie in G_0 except the ones given by $V(\tau_j) \setminus \{(k,0)\}$, if $k < j$, and $V(\tau_j) \setminus \{(k,1)\}$, if $k \geq j$. Therefore, if we define

$$G_{j+1} = G_0 \cup \tau_0 \cup \dots \cup \tau_j,$$

for $0 \leq j < n$, then the inclusion map $G_i \hookrightarrow G_j$, $0 \leq i < j \leq n+1$, is given by a composite of pushouts of inner horn inclusions. In particular, we have that $G_{n+1} = \Delta^n \times \Delta^1$ and the map $(\Lambda^n_k \times \Delta^1) \coprod_{\Lambda^n_k \times \partial\Delta^1} (\Delta^n \times \partial\Delta^1) = F_0 \hookrightarrow G_{n+1} = \Delta^n \times \Delta^1$ is given by a composite of pushouts of inner horn inclusions. $\square$

Lemma 5.15. *Let $p \colon \mathcal{V}^\vee \to \Delta$ be the composite of the Cartesian fibrations $p^\vee \colon \mathcal{V}^\vee \to \Delta_\Phi$ and $\Delta_\Phi \to \Delta$. If x is an object in $\mathcal{V}^\vee$ such that $p(x) = n$, then, for every $0 < k < n$, the map*

$$\tilde{f} \colon (\Lambda^n_k x \otimes \Delta^1) \coprod_{\Lambda^n_k x \otimes \partial\Delta^1} (x \otimes \partial\Delta^1) \to x \otimes \Delta^1$$

in $\mathrm{P}(\mathcal{V}^\vee)$ is a Segal equivalence.

Proof. The idea of the proof is similar to that of Lemma 5.12. By Lemma 5.14, the map $f \colon (\Lambda^n_k \times \Delta^1) \coprod_{\Lambda^n_k \times \partial\Delta^1} (\Delta^n \times \partial\Delta^1) \hookrightarrow \Delta^n \times \Delta^1$ is a composite of pushouts of inner horn inclusions, for which we now write $F_{j-1} \to F_j$. We want to use Lemma 5.8 in order to lift those maps to $\tilde{F}_{j-1} \to \tilde{F}_j$ and show that they are given by pushouts of Segal equivalences.

Since p^* is a left adjoint, it preserves coproducts and there exists a commutative square

$$\begin{tikzcd}
(\Lambda^n_k x \otimes \Delta^1) \coprod_{\Lambda^n_k x \otimes \partial\Delta^1} (x \otimes \partial\Delta^1) \arrow[r, "\tilde{f}"] \arrow[d] & x \otimes \Delta^1 \arrow[d] \\
(p^*\Lambda^n_k \otimes \Delta^1) \coprod_{p^*\Lambda^n_k \otimes \partial\Delta^1} (p^*\Delta^n \otimes \partial\Delta^1) \arrow[r, "p^* f"] & p^*\Delta^n \otimes \Delta^1.
\end{tikzcd} \tag{5.5}$$

As in the proof of Lemma 5.12, one shows that this commutative diagram is a pullback square, because it is a one after evaluation at any object in $\mathcal{V}^\vee$.

We define $p_j \colon \tilde{F}_j \to p^* F_j$ to be the morphism given by the pullback of $x \otimes \Delta^1 \to p^*\Delta^n \otimes \Delta^1$ along $p^*F_j \to p^*\Delta^n \otimes \Delta^1$ and we denote the pullback of $p^* f_j$ along p_j by $\tilde{f}_j \colon \tilde{F}_{j-1} \to \tilde{F}_j$. Since the map $(\Lambda^n_k \times \Delta^1) \coprod_{\Lambda^n_k \times \partial\Delta^1} (\Delta^n \times \partial\Delta^1) \hookrightarrow \Delta^n \times \Delta^1$ is a composite of maps $F_{j-1} \to F_j$, the pullback square 5.5 implies that $\tilde{f}$ is given by a composite of the morphisms $\tilde{f}_j$. Using the same arguments of Lemma 5.12, one shows that all the morphisms $\tilde{f}_j$ are Segal equivalences. □

Proposition 5.16. *For every object x in $\mathcal{V}^\vee$ and every m, let x_{Seg} and Δ^m_{Seg} be as defined in Notation 4.21. The following maps are Segal equivalences:*

1. $x \otimes \Delta^m_{\mathrm{Seg}} \to x \otimes \Delta^m$
2. $x_{\mathrm{Seg}} \otimes \Delta^m \to x \otimes \Delta^m$

Proof. 1. We prove the first claim by an induction on m. The statement is void for $m = 0, 1$. For $m \geq 2$, we have a factorization $x \otimes \Delta^m_{\mathrm{Seg}} \to x \otimes \Lambda^m_i \to x \otimes \Delta^m$, where $0 < i < m$. Since Δ^m_{Seg} and Λ^m_i are colimits of k-simplices with $k < m$, the tensor product preserves colimits by Remark 5.3 and Segal equivalences are closed under colimits, the induction hypothesis implies that the first map is a Segal equivalence. Thus, by the 2-of-3 property, we only need to show that the second map $x \otimes \Lambda^m_i \to x \otimes \Delta^m$ is a Segal equivalence in $\mathrm{P}(\mathcal{V}^\vee)$.

[Lur09, Proposition 2.3.2.1] implies that morphisms in sSet of the form

$$(\Delta^i \times \Lambda^2_1) \coprod_{\partial\Delta^i \times \Lambda^2_1} (\partial\Delta^i \times \Delta^2) \hookrightarrow \Delta^i \times \Delta^2,$$

for $i \geq 0$, generate the class of inner anodyne maps and hence all inner horn inclusions. Since the tensor product preserves colimits, it therefore suffices to show that, for every i, the map $x \otimes (\Delta^i \times \Lambda^2_1) \coprod_{x \otimes (\partial\Delta^i \times \Lambda^2_1)} x \otimes (\partial\Delta^i \times \Delta^2) \to x \otimes (\Delta^i \times \Delta^2)$ is a Segal equivalence in $\mathrm{P}(\mathcal{V}^\vee)$. It follows from the definition of the tensor product that, for every $K, L \in \mathrm{P}(\Delta)$ and every $F \in \mathrm{P}(\mathcal{V}^\vee)$, we have $F \otimes (K \times L) = (F \otimes K) \otimes L$. This implies that we have to show that

$$(x \otimes \Delta^i) \otimes \Lambda^2_1 \coprod_{(x \otimes \partial\Delta^i) \otimes \Lambda^2_1} (x \otimes \partial\Delta^i) \otimes \Delta^2 \hookrightarrow (x \otimes \Delta^i) \otimes \Delta^2$$

is a Segal equivalence.

By Lemma 5.13, we know that the map $(x \otimes \Delta^i) \otimes \Lambda^2_1 \to (x \otimes \Delta^i) \otimes \Delta^2$ is a Segal equivalence.

Thus, by the 2-of-3 property, it is sufficent to verify that

$$(x \otimes \Delta^i) \otimes \Lambda^2_1 \to (x \otimes \Delta^i) \otimes \Lambda^2_1 \coprod_{(x\otimes\partial\Delta^i)\otimes\Lambda^2_1} (x \otimes \partial\Delta^i) \otimes \Delta^2$$

is a Segal equivalence for every i. But this is true because it is given by the pushout of $(x \otimes \partial\Delta^i) \otimes \Lambda^2_1 \to (x \otimes \partial\Delta^i) \otimes \Delta^2$, which is a Segal equivalence by Lemma 5.13.

2. We prove the second claim by induction on the length of the object $x \in \mathcal{V}^\vee$. We can assume that the length of x is $n \geq 2$, because otherwise the statement is obviously true. Since the map $x_{\mathrm{Seg}} \otimes \Delta^m \to x \otimes \Delta^m$ is given by the composition of the maps $x_{\mathrm{Seg}} \otimes \Delta^m \to \Lambda^n_k x \otimes \Delta^m$ and $\Lambda^n_k x \otimes \Delta^m \to x \otimes \Delta^m$, for $0 < k < n$, where the first map is a colimit of maps in $\mathrm{P}(\mathcal{V}^\vee)$ of the form $y_{\mathrm{Seg}} \otimes \Delta^m \to y \otimes \Delta^m$, where the length of $y \in \mathcal{V}^\vee$ is smaller than n. Then the induction hypothesis implies that the first map is a Segal equivalence, because the set of Segal equivalences is closed under colimits. By the 2-of-3 property, it suffices to show that $\Lambda^n_k x \otimes \Delta^m \to x \otimes \Delta^m$ is a Segal equivalence too. One observes that this map fits into a commutative square in $\mathrm{P}(\mathcal{V}^\vee)$

$$\begin{tikzcd}
\Lambda^n_k x \otimes \Delta^m_{\mathrm{Seg}} \arrow[r] \arrow[d] & x \otimes \Delta^m_{\mathrm{Seg}} \arrow[d] \\
\Lambda^n_k x \otimes \Delta^m \arrow[r] & x \otimes \Delta^m,
\end{tikzcd}$$

where the vertical maps are Segal equivalences by the first part of this proposition and the fact that the tensor product preserves colimits (Remark 5.3). By the 2-of-3 property, we only need to the verify that the upper horizontal map is a Segal equivalence and by using the fact that the tensor product preserves colimits once more, we see that it suffices to show that $\Lambda^n_k x \otimes \Delta^1 \to x \otimes \Delta^1$ is a Segal equivalence.

The map $\Lambda^n_k x \otimes \Delta^1 \to x \otimes \Delta^1$ is the composite of

$$\Lambda^n_k x \otimes \Delta^1 \to (\Lambda^n_k x \otimes \Delta^1) \coprod_{\Lambda^n_k x\otimes\partial\Delta^1} (x \otimes \partial\Delta^1)$$

and

$$(\Lambda^n_k x \otimes \Delta^1) \coprod_{\Lambda^n_k x\otimes\partial\Delta^1} (x \otimes \partial\Delta^1) \to x \otimes \Delta^1,$$

where the first map is a Segal equivalence, because it is a pushout of $\Lambda^n_k x \coprod \Lambda^n_k x \simeq \Lambda^n_k x \otimes \partial\Delta^1 \to x \otimes \partial\Delta^1 \simeq x \coprod x$, which is a coproduct of Segal equivalences. The claim follows from the 2-of-3 property because the second map is also a Segal equivalence by Lemma 5.15.

□

Corollary 5.17. *For objects $F \in \mathrm{P}(\mathcal{V}^\vee)$, $G \in \mathrm{P}(\Delta)$, $f \in S^{\mathrm{Seg}}_\Phi$ and $g \in S^{\mathrm{Seg}}_*$, the morphisms $\mathrm{id}_F \otimes g$ and $f \otimes \mathrm{id}_G$ lie $\overline{S}_\Phi$.*

Proof. Every presheaf $F \in \mathrm{P}(\mathcal{V}^\vee)$ is a colimit of representable presheaves and the tensor product preserves colimits by Remark 5.3. Furthermore, the strongly saturated set $\overline{S}_\Phi$ is closed under colimits by definition, so we can assume that F is represented by an object $x \in \mathcal{V}^\vee$. Similarly, we can assume that the presheaf G in the claim is represented by an object $[n] \in \Delta$. Then the claim coincides with the statement of Proposition 5.16. $\square$

5.3 Preserving Segal Equivalences $S_\Phi^{\lim}$

In this last section of the chapter we want prove that the tensor product defined in the first section preserves elements in the set $S_\Phi^{\lim}$. Once this is shown, it is easy to prove the main theorem of this chapter at the end at this section by using results of the previous section.

Lemma 5.18. *Suppose $\mathfrak{c}_I \in \Delta_\Phi$ is a corolla and $x \in \mathcal{V}^\vee$ is an object lying over $\mathfrak{c}_I$, then $x \otimes \Delta^1 \in \mathrm{P}(\mathcal{V}^\vee)$ is given by a pushout of representable presheaves.*

Proof. It is clear that we have a pushout diagram in $\mathrm{P}(\Delta)$

$$\begin{tikzcd} \Delta^1 \arrow[r, "f_2"'] \arrow[d, "f_1"'] & \Delta^2 \arrow[d, "g_2"] \\ \Delta^2 \arrow[r, "g_1"'] & \Delta^1 \times \Delta^1, \end{tikzcd} \tag{5.6}$$

where

1. the maps f_1 and f_2 are induced by $\{0\} \mapsto \{0\}$ and $\{1\} \mapsto \{2\}$,
2. the map g_1 is induced by $\{0\} \mapsto \{(0,0)\}$, $\{1\} \mapsto \{(0,1)\}$ and $\{2\} \mapsto \{(1,1)\}$ and
3. the map g_2 is induced by $\{0\} \mapsto \{(0,0)\}$, $\{1\} \mapsto \{(1,0)\}$ and $\{2\} \mapsto \{(1,1)\}$.

If $p\colon \mathcal{V}^\vee \to \Delta$ denotes the Cartesian fibration given by the composite of $p^\vee\colon \mathcal{V}^\vee \to \Delta_\Phi$ and $\Delta_\Phi \to \Delta$, then, since the functor p^* is a left adjoint, it carries the pushout square 5.6 to a pushout square in $\mathrm{P}(\mathcal{V}^\vee)$. The following commuative square

$$\begin{tikzcd} x \otimes \Delta^1 \arrow[r] \arrow[d] & x \arrow[d] \\ p^*\Delta^1 \otimes \Delta^1 \arrow[r] & p^*\Delta^1 \end{tikzcd}$$

is given by $p^*\Delta^1 \otimes \Delta^1 \simeq p^*(\Delta^1 \otimes \Delta^1) \to \Delta^1$, where the second map is induced by the projection to the second component, is a pullback diagram in $\mathrm{P}(\mathcal{V}^\vee)$, because it is a pullback square after evaluation at any object in $\mathcal{V}^{\vee,\mathrm{op}}$. If we extend this square to a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & x \otimes \Delta^1 \arrow[r] \arrow[d] & x \arrow[d] \\ p^*\Delta^2 \arrow[r, "p^* g_1"'] & p^*\Delta^1 \otimes \Delta^1 \arrow[r] & p^*\Delta^1, \end{tikzcd}$$

where the right square is the pullback from above, then we see that the composite of the lower horizontal maps is induced by the degeneracy map $s^0\colon [2] \to [1]$ given by the assignments $\{0,1\} \mapsto$

$\{0\}$ and $\{2\} \mapsto \{1\}$. Lemma 5.8 then implies that the upper horizontal map $F \to x$ in $\mathrm{P}(\mathcal{V}^\vee)$ of the big square is the image of the Yoneda embedding of a p-Cartesian lift of the degeneracy map s^0. Hence, the presheaf F is represented by an object $x^+ \in \mathcal{V}^\vee$.

The same argument implies that there exists an object $x^- \in \mathcal{V}^\vee$ such that the diagram in $\mathrm{P}(\mathcal{V}^\vee)$

$$\begin{tikzcd}
& x \arrow[rr] \arrow[dl] \arrow[dd] & & x^- \arrow[dl] \arrow[dd] \\
x^+ \arrow[rr] \arrow[dd] & & x \otimes \Delta^1 \arrow[dd] & \\
& p^*\Delta^1 \arrow[rr, "p^* f_2"] \arrow[dl, "p^* f_1"'] & & p^*\Delta^2 \arrow[dl, "p^* g_2"] \\
p^*\Delta^2 \arrow[rr, "p^* g_1"'] & & p^*\Delta^1 \times \Delta^1 &
\end{tikzcd}$$

commutes and all the vertical squares are pullback squares. Using the fact that pullbacks preserve colimits in ∞-topoi, we see that the upper square in the diagram a pushout diagram. $\square$

Remark 5.19. *Let $p^\vee \colon \mathcal{V}^\vee \to \Delta_\Phi$ denote the Cartesian fibration introduced in Definition 4.9 and let $\pi \colon \Delta_\Phi \to \Delta$ be the Cartesian fibration given by the canonical projection. If $\mathfrak{c}_I^+ \to \mathfrak{c}_I$ in Δ_Φ is the π-Cartesian lift of the degeneracy map $s^0 \colon [2] \to [1] \in \Delta$ at a corolla $\mathfrak{c}_I \in \Delta_\Phi$, then Definition 2.6 implies that we obtain $\mathfrak{c}_I^+$ by gluing the object $([1], J) \in \Delta_\Phi$ determined by $\mathrm{id} \colon J(0) = I(0) \to I(0) = J(1)$ to $\mathfrak{c}_I$ along $I(0)$. If $x \in \mathcal{V}^\vee$ is an object lying over $\mathfrak{c}_I$, then we can lift $\mathfrak{c}_I^+ \to \mathfrak{c}_I$ to a $p^\vee$-Cartesian morphism $x^+ \to x$. Using the notation of Remark 4.11, the object $x \in \mathcal{V}^\vee$ is of the form $(\mathfrak{c}_I, v)$ for some $v \in \mathcal{V}_{\langle 1 \rangle}$ and we can think of $x^+ \in \mathcal{V}^\vee$ to be given by labeling a corolla in $\mathfrak{c}_I^+$ by v, if the corolla corresponds to $\mathfrak{c}_I$, and by the unit $\mathbb{1}$ of the symmetric monoidal ∞-category $\mathcal{V}$ otherwise.*

A similar consideration shows that $\mathfrak{c}_I^- \in \Delta_\Phi$ is constructed from $\mathfrak{c}_I$ by gluing a single 1-corolla to $\mathfrak{c}_I$ along the root edge. The object $x^- \in \mathcal{V}^\vee$ lying over $\mathfrak{c}_I^-$ is the given by labeling the corolla corresponding to $\mathfrak{c}_I$ by v and the other corolla by the unit $\mathbb{1} \in \mathcal{V}$.

Before we prove Proposition 5.21, we first want to show that one can replace the set S_Φ^{lim} in Notation 4.21 by the set $S_\Phi''^{\,\mathrm{lim}}$ consisting of morphisms which are indexed by diagrams of a simpler form.

Lemma 5.20. *Let $S_\Phi''^{\,\mathrm{lim}}$ denote the strongly saturated set generated by the set of morphisms of the following form:*

- $\mathrm{colim}_{k \in K}\, q(k) \to q(\infty) \in S_\Phi^{\mathrm{lim}}$, *where $q \colon K^\triangleright \to \mathcal{V}^{\kappa,\vee}$ is given by $(\overline{g} * q_n)^{\mathrm{op}} \colon (L^\triangleleft * (\mathbf{n}+\mathbf{1}))^{\mathrm{op}} \to \mathcal{V}^{\kappa,\vee}$ such that $n \geq 0$, L is a κ-small discrete set and $\overline{g}, q_n$ are as defined in the proof of Proposition 4.20,*
- $\mathrm{colim}_{k \in K}\, q(k) \to q(\infty)$ *such that $q \colon K^\triangleright \to \mathcal{V}^{\kappa,\vee}_{\langle 1 \rangle} \subseteq \mathcal{V}^{\kappa,\vee}$ is a pushout square in $\mathcal{V}^{\kappa,\vee}$.*

If we write S_Φ for the strongly saturated set generated by elements in $S_\Phi''^{\,\mathrm{lim}} \coprod S_\Phi^{\mathrm{Seg}}$, then we have an equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})[{S_\Phi}^{-1}]$.

Proof. By definition, the strongly saturated set $\overline{S}_\Phi$ is generated by elements in $S_\Phi^{\lim} \coprod S_\Phi^{\mathrm{Seg}}$ and Proposition 4.20 says that $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})[\overline{S}_\Phi{}^{-1}]$. It follows that we only need to show that an object in $\mathrm{P}(\mathcal{V}^\vee)$ is local with respect to all morphisms in $S_\Phi^{\lim}$ if and only if it is local with respect to all morphisms in $S_\Phi''^{\lim}$.

First, let us recall that a presheaf $f \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ is local with respect to all elements in $S_\Phi^{\lim}$ if and only if, for every corolla $\mathfrak{c}_I \in \Delta_\Phi$, the functor $(\mathcal{V}^{\kappa,\vee,\mathrm{op}})_{\mathfrak{c}_I} \simeq \mathcal{V}^{\kappa,\mathrm{op}}_{\langle 1\rangle} \to \mathcal{S}^{n+1}_{/f(\bar{\mathfrak{e}})}$, $\bar{\mathfrak{e}} \in (\mathcal{V}^\vee)_{\mathfrak{e}} \simeq \{*\}$, induced by f preserves all κ-small limits. By Proposition [Lur09, 4.4.2.7], a functor preserves κ-small limits if and only if it preserves pullback squares and κ-small products. In other words, if $S_\Phi'^{\lim}$ is the set consisting of maps of the form $\mathrm{colim}_{k\in K}\, q(k) \to q(\infty)$, where $K = (\mathbf{n}+\mathbf{1}) * L$ and $L^\triangleright$ is either a pushout square or a κ-small coproduct diagram, then f is local with respect to elements in the subset $S_\Phi^{\lim}$ if and only if f is local with respect to elements in $S_\Phi'^{\lim}$. We now want to show that we can replace $S_\Phi'^{\lim}$ by the set $S_\Phi''^{\lim}$ as defined in the claim of the lemma.

For every ∞-category $\mathcal{C}$, any object $x \in \mathcal{C}$ and any weakly contractible simplicial set K, a functor $K^\triangleleft \to \mathcal{C}_{x/}$ is a limit diagram if and only if its composition with the forgetful functor $\mathcal{C}_{x/} \to \mathcal{C}$ is a limit diagram. Since index categories of pullback diagrams are weakly contractible, the induced functor $\mathcal{V}^{\kappa,\mathrm{op}}_{\langle 1\rangle} \to \mathcal{S}^{n+1}_{/f(\bar{\mathfrak{e}})}$ preserves pullbacks if and only if its composition with the forgetful functor does, which is equivalent to requiring f to be local with respect to morphisms of the form $\mathrm{colim}_{k\in K}\, q(k) \to q(\infty)$, where $q\colon K^\triangleright \to \mathcal{V}^{\kappa,\vee}_{\langle 1\rangle} \subseteq \mathcal{V}^{\kappa,\vee}$ is a pushout square in $\mathcal{V}^{\kappa,\vee}$. In conclusion, we have that a presheaf f lies in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ if and only if it is local with respect to maps in $S_\Phi''^{\lim}$, hence there is an equivalence of ∞-categories $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})[S_\Phi{}^{-1}]$. $\square$

Proposition 5.21. *For every two objects $G \in \mathrm{P}(\Delta)$ and $f \in S_\Phi''^{\lim}$, the morphism $f \otimes \mathrm{id}_G$ lies S_Φ.*

Proof. Since every presheaf $G \in \mathrm{P}(\Delta)$ is a colimit of representable presheaves and the tensor product preserves colimits by Remark 5.3, we can assume that G is represented by $[n] \in \Delta$ for some n. Furthermore, for every morphism $f\colon F \to F'$ in $\mathrm{P}(\mathcal{V}^\vee)$, we have a commutative square

$$\begin{tikzcd} F \otimes \Delta^n_{\mathrm{Seg}} \arrow[r] \arrow[d] & F' \otimes \Delta^n_{\mathrm{Seg}} \arrow[d] \\ F \otimes \Delta^n \arrow[r, "f\otimes \mathrm{id}_{\Delta^n}"] & F' \otimes \Delta^n, \end{tikzcd}$$

where the vertical maps lie in S_Φ by Corollary 5.17. By the 2-of-3 property, the bottom horizontal map lies in S_Φ if and only if the upper horizontal map does. Using the facts that the tensor product preserves colimits and that the set S_Φ is closed under colimits, the maps $f \otimes \mathrm{id}_{\Delta^n_{\mathrm{Seg}}}$ lie in S_Φ, for every $n \geq 1$, if it is true for $n = 1$.

Let us first assume that $f \in S_\Phi''^{\lim}$ is of the form $\mathrm{colim}_{k\in K}\, q(k) \to q(\infty)$, where $K = (L * (\mathbf{n}+\mathbf{1}))^{\mathrm{op}}$, L is a κ-small set and the diagram $q\colon K^\triangleright \to \mathcal{V}^{\kappa,\vee}$ is as defined in Notation 4.21. We wish to show that $\mathrm{colim}_{k\in K}(q(k) \otimes \Delta^1) \to q(\infty) \otimes \Delta^1$ is a colimit of morphisms lying in the strongly saturated set $S_\Phi''^{\lim}$. By Lemma 5.18, we can express each presheaf $q(k)\otimes\Delta^1$ as a pushout $x_k^- \coprod_x x_k^+$. The functor q carries every object in the set $\mathbf{n}+\mathbf{1} \subseteq K$ to the essential unique object $\bar{\mathfrak{e}} \in (\mathcal{V}^{\kappa,\vee})_{\mathfrak{e}}$ and therefore we have $q(i) \otimes \Delta^1 \simeq \Delta^1$ for $0 \leq i \leq n$. If we decompose Δ^1 as a trivial pushout $[0] \coprod_{[0]} \Delta^1$, then we can define the two assigments $q^+, q^-, q^0\colon \mathrm{Obj}(K) \to \mathrm{Obj}(\mathcal{V}^{\kappa,\vee})$ as follows

$$q^{+}(z) = \begin{cases} x_l^{+} & \text{if } z = l \in L \\ (\mathfrak{c}_1, \mathbb{1}) & \text{if } z \in \mathbf{n+1} \text{ and } 1 \leq z \leq n \\ \bar{\mathfrak{e}} & \text{otherwise} \end{cases}$$

$$q^{-}(z) = \begin{cases} x_l^{-} & \text{if } z = l \in L \\ (\mathfrak{c}_1, \mathbb{1}) & z \in \mathbf{n+1} \text{ and } z = 0 \\ \bar{\mathfrak{e}} & \text{otherwise.} \end{cases}$$

$$q^{0}(z) = \begin{cases} x_l & \text{if } z = l \in L \\ \bar{\mathfrak{e}} & \text{if } z \in \mathbf{n+1} \end{cases}$$

Let $p^{\vee}\colon \mathcal{V}^{\kappa,\vee} \to \Delta_{\Phi}$ denote the Cartesian fibration of Definition 4.9 given by projection. Note that the only non-identity morphisms in K are of the from $g_{il}\colon i \to l$, with $i \in \mathbf{n+1}$ and $l \in L$. Therefore, if q^{+} carries each of these maps g_{il} to morphisms in $\mathcal{V}^{\kappa,\vee}$ that are compatible with compositions, then q^{+} is actually a functor. It follows from the definition of $\mathfrak{c}_I^{+}$ that it is determined by the sequence of maps $I(0) \xrightarrow{\mathrm{id}} I(0) \to I(1)$. If $\mathfrak{c}_J$ denotes the corolla in Δ_{Φ} given by the unique map $* = J(0) \to J(1) = *$, then every $i \in |I(0)|$ induces a map $a_i\colon \mathfrak{c}_J \to \mathfrak{c}_I^{+}$. For $1 \leq i \leq n$, we can define $\tilde{g}_{il}^{+}\colon (\mathfrak{c}_J, \mathbb{1}) \to x_l^{+}$ (see Remark 4.11 for this notation) to be the $p^{\vee}$-Cartesian lift of a_i, since $x_l^{+} \in \mathcal{V}^{\kappa,\vee}$ lies over the object $\mathfrak{c}_I^{+} \in \Delta_{\Phi}$. Moreover, we can also lift the inclusion $\mathfrak{e} \to \mathfrak{c}_I^{+}$ of the root edge of $\mathfrak{c}_I^{+}$ to the $p^{\vee}$-Cartesian morphism $l_{0l}^{+}\colon \bar{\mathfrak{e}} \to x_l^{+}$. It is easy to see that we can extend the assigment q^{+} to a functor by defining the image of each morphism g_{il}, for $i \in \mathbf{n+1}$ and $l \in L$, as follows

$$q^{+}(g_{il}) = \begin{cases} \tilde{g}_{il}^{+}\colon (\mathfrak{c}_J, \mathbb{1}) \to x_l^{+} & \text{if } 1 \leq i \leq n \\ \tilde{g}_{0l}^{+}\colon \bar{\mathfrak{e}} \to x_l^{+} & \text{otherwise.} \end{cases}$$

Similarly, we extend q^{-} to a functor by requiring

$$q^{-}(g_{il}) = \begin{cases} \tilde{g}_{0l}^{-}\colon (\mathfrak{c}_J, \mathbb{1}) \to x_l^{-} & \text{if } i = 0 \\ \tilde{g}_{il}^{-}\colon \bar{\mathfrak{e}} \to x_l^{-} & \text{otherwise,} \end{cases}$$

where $\tilde{g}_{0l}^{-}$ is the $p^{\vee}$-Cartesian lift of the inclusion $g_{0l}^{-}\colon \mathfrak{c}_1 \to \mathfrak{c}_I^{+}$ of the root corolla of $\mathfrak{c}_I^{+}$ as defined in Remark 5.19 and $\tilde{g}_{il}^{-}$ is the $p^{\vee}$-Cartesian lift of the inclusions $g_{il}^{-}\colon \mathfrak{e} \to \mathfrak{c}_I^{+}$ of the i-th leaf edge of $\mathfrak{c}_I^{-}$. Last but not least, we extend the assignment q^{0} to a functor by defining

$$q^{0}(g_{il})\colon \bar{\mathfrak{e}} \to x_l$$

to be the $p^{\vee}$-Cartesian lift of the inclusion $\mathfrak{e} \to \mathfrak{c}_I$ of the i-th leaf edge of $\mathfrak{c}_I$, if $1 \leq i \leq n$ and the $p^{\vee}$-Cartesian lift of the inclusion of the root edge otherwise.

The definition of the functors q^{+}, q^{-} and q^{0} imply that we have a decomposition

$$\mathrm{colim}_{k \in K}(q(k) \otimes \Delta^1) = \mathrm{colim}_{z \in K}\, q^{+}(z) \coprod_{\mathrm{colim}_{z \in K}\, q^{0}(z)} \mathrm{colim}_{z \in K}\, q^{-}(z).$$

If we write x_{∞} for $q(\infty) \in \mathcal{V}^{\kappa,\vee}$, then, by applying Lemma 5.18 once more, we can decompose the presheaf $x_\infty \otimes \Delta^1 \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ as a pushout of presheaves

$$x_\infty^+ \coprod_{x_\infty} x_\infty^-.$$

Hence, the map $\mathrm{colim}_{k\in K}\, q(k) \to q(\infty)$ in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ is a pushout of the maps

$$\mathrm{colim}_{z\in K}\, q^+(z) \to x_\infty^+, \mathrm{colim}_{z\in K}\, q^-(z) \to x_\infty^- \text{ and } \mathrm{colim}_{z\in K}\, q^0(z) \to x_\infty.$$

Since the strongly saturated set S_Φ is closed under colimits and the map $\mathrm{colim}_{z\in K}\, q^0(z) \to x_\infty$ lies in $S_\Phi''^{\lim}$ by definition, we only need to show that the two other maps also lie in S_Φ.

The commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_{z\in K} (q^+z)_{\mathrm{Seg}} \arrow[r] \arrow[d] & (x_\infty^+)_{\mathrm{Seg}} \arrow[d] \\
\mathrm{colim}_{z\in K}\, q^+(z) \arrow[r] & x_\infty^+,
\end{tikzcd}$$

where the vertical maps lie in S_Φ^{Seg}, implies that in order to show that the bottom horizontal map lies in S_Φ, it suffices to verify that upper horizontal map lies in $S_\Phi''^{\lim}$ by showing that this map is a colimit of maps in $S_\Phi''^{\lim}$ below.

First we are going to show that the objects $(q^+z)_{\mathrm{Seg}}$ are given by pushouts. For $z \in K$, the definition of $(q^+z)_{\mathrm{Seg}}$ implies that it can be written as a pushout

$$(q^+z)_{\mathrm{Seg}} = \begin{cases} \coprod_{1\le i\le n}(\mathfrak{c}_J, \mathbb{1}) \coprod_{\coprod_{1\le i\le n}\bar{\mathfrak{e}}} x_l & \text{if } z = l \in L \\ (\mathfrak{c}_J, \mathbb{1}) & \text{if } z \in \mathbf{n+1} \setminus \{0\} \\ \bar{\mathfrak{e}} & \text{otherwise,} \end{cases}$$

where the first pushout object in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ is induced by $\coprod_{1\le i\le n}\bar{\mathfrak{e}} \to \coprod_{1\le i\le n}(\mathfrak{c}_J, \mathbb{1})$ and $\coprod_{1\le i\le n}\bar{\mathfrak{e}} \to x_l$ are induced by $p^\vee$-Cartesian lifts of inclusion of the root edge $\mathfrak{e} \to \mathfrak{c}_1$ and the inclusions of the leaf edges $\mathfrak{e} \to \mathfrak{c}_I$, respectively. Similarly, we can write $(x_\infty^+)_{\mathrm{Seg}}$ as a pushout

$$(x_\infty^+)_{\mathrm{Seg}} \simeq \coprod_{1\le i\le n}(\mathfrak{c}_J, \mathbb{1}) \coprod_{\coprod_{1\le i\le n}\bar{\mathfrak{e}}} x_\infty.$$

If $\emptyset$ denotes the initial object in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$, then, for $z = i$ and $1 \le i \le n$, we can write $(q^+i)_{\mathrm{Seg}} = (\mathfrak{c}_J, \mathbb{1})$ as the trivial pushout

$$(q^+i)_{\mathrm{Seg}} = \emptyset \coprod \dots \coprod (\mathfrak{c}_J, \mathbb{1}) \coprod \dots \coprod \emptyset \coprod_{\emptyset \coprod \cdots \coprod \bar{\mathfrak{e}} \coprod \cdots \coprod \emptyset} \bar{\mathfrak{e}},$$

where $\emptyset \coprod \dots \coprod (\mathfrak{c}_J, \mathbb{1}) \coprod \dots \coprod \emptyset$ and $\emptyset \coprod \dots \coprod \bar{\mathfrak{e}} \coprod \dots \coprod \emptyset$ are n-fold coproducts whose i-th components are given by $(\mathfrak{c}_J, \mathbb{1})$ and $\bar{\mathfrak{e}}$, respectively. Similarly, we write $(q^+0)_{\mathrm{Seg}} = \bar{\mathfrak{e}}$ for the trivial pushout

$$\coprod_{1\le i\le n} \emptyset \coprod_{\emptyset} \bar{\mathfrak{e}}.$$

This decomposition of objects allows us to define functors $q_{1i}, q_{2i}, q_3 \colon K \to \mathcal{V}^{\kappa,\vee}$, where $1 \leq i \leq n$, as follows

$$q_{1i}(z) = \begin{cases} (\mathfrak{c}_J, \mathbb{1}) & \text{if } z = l \in L \\ (\mathfrak{c}_J, \mathbb{1}) & \text{if } z = i \in \mathbf{n+1} \\ \emptyset & \text{otherwise} \end{cases}$$

$$q_{2i}(z) = \begin{cases} \bar{\mathfrak{e}} & \text{if } z = l \in L \\ \bar{\mathfrak{e}} & \text{if } z = i \in \mathbf{n+1} \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$q_3(z) = \begin{cases} x_l & \text{if } z = l \in L \\ \bar{\mathfrak{e}} & \text{if } z \in \mathbf{n+1}. \end{cases}$$

It follows from the definition that the map $\mathrm{colim}_{x\in K}(q^+x)_{\mathrm{Seg}} \to (x^+_\infty)_{\mathrm{Seg}}$ is given by a pushout of the following maps in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$

$$\coprod_{1\leq i\leq n} (\mathrm{colim}_{z\in K}\, q_{1i}(z)) \to (\mathfrak{c}_J, \mathbb{1})), \quad \coprod_{1\leq i\leq n} (\mathrm{colim}_{z\in K}\, q_2(z)) \to \bar{\mathfrak{e}})$$

and

$$\mathrm{colim}_{z\in K}\, q_3(z) \to x_\infty.$$

Since the third map is obviously lies in $S''^{\,\lim}_\Phi$, it suffices to show that the morphims $\mathrm{colim}_{z\in K}\, q_{1i}(z) \to (\mathfrak{c}_J, \mathbb{1})$, $1 \leq i \leq n$, and $(\mathrm{colim}_{z\in K}(q_2(z)) \to \bar{\mathfrak{e}})$ are elements of equivalences. But this follows from the observation that $\mathrm{colim}_{z\in K}\, q_{1i}(z)$ and $\mathrm{colim}_{z\in K}\, q_2(z)$ are given by constant pushout diagrams.

In order to complete the proof we have to show that the maps of the form $\mathrm{colim}_{k\in K}(q(k) \otimes \Delta^1) \to q(\infty) \otimes \Delta^1$ lie in $S''^{\,\lim}_\Phi$, where K is weakly contractible. As in the previous part of the proof, the following equivalences hold for every $k \in K$:

$$q(k) \otimes \Delta^1 \simeq x^+_k \coprod_{x_l} x^-_k \ \text{ and } \ q(k) \otimes \Delta^1 \simeq x^+_\infty \coprod_{x_\infty} x^-_\infty.$$

Hence, the map $\mathrm{colim}_{k\in K}(q(k) \otimes \Delta^1) \to q(\infty) \otimes \Delta^1$ is given by a pushout of the following maps

$$\mathrm{colim}_{k\in K}\, x^+_k \to x^+_\infty,\ \mathrm{colim}_{k\in K}\, x^-_k \to x^-_\infty \ \text{ and } \ \mathrm{colim}_{k\in K}\, x_l \to x_\infty.$$

A similar but simpler argument as above shows that all these maps lie $S''^{\,\lim}_\Phi$. $\square$

We are now able to prove the main result of this chapter.

Proof of Theorem 5.5. By Lemma 5.20, we have that $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ is equivalent to $\mathrm{P}(\mathcal{V}^{\kappa,\vee})[S_\Phi^{-1}]$. According to [Lur09, Proposition 5.2.7.12], the universal property of localization provides the existence of a functor

$$\otimes_{\mathrm{Seg}} \colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \times \mathrm{P}_{\mathrm{Seg}}(\Delta) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee),$$

if and only if $f \otimes g$ lies in S_Φ, where $f \in S_\Phi$ and $g \in S^{\mathrm{Seg}}_*$. In the proof of this theorem, we use the notation $\otimes_{\mathrm{Seg}}$ for the functor whose existence we wish to provide in order to distinguish it from the tensor product $\otimes$ of presheaf categories defined in Definition 5.2. Since S_Φ is the strongly saturated set generated by S_Φ^{Seg} and $S''^{\lim}_\Phi$, the claim follows from Corollary 5.17 and Proposition 5.21 which imply that $f \otimes g \in S_\Phi$, if $f \in S_\Phi^{\mathrm{Seg}}$ and $g \in S_*^{\mathrm{Seg}}$, and $f \otimes \mathrm{id}_G \in S_\Phi$, for every $f \in S''^{\lim}_\Phi$ and $G \in \mathrm{P}(\Delta)$.

In order to complete the proof, we show that $\otimes_{\mathrm{Seg}}$ preserves colimits in each variable. If we write $L\colon \mathrm{P}(\mathcal{V}^\vee) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ for the localization functor, then, for a pair $(F, G) \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \times \mathrm{P}_{\mathrm{Seg}}(\Delta)$, one can identify the object $F \otimes_{\mathrm{Seg}} G$ with the image of the presheaf $F \otimes G$ under the localization functor L. In particular, if $q\colon K^{\triangleright} \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ is a colimit diagram, then, for every $G \in \mathrm{P}(\Delta)$, the Segal presheaf $(\mathrm{colim}_{k\in K}\, q(k)) \otimes_{\mathrm{Seg}} G \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ is given by $L((\mathrm{colim}_{k\in K}\, q(k)) \otimes G)$. Since the localization functor L is a left adjoint and the tensor product $\otimes$ preserves colimits in each variable by Remark 5.3, there is an equivalence $L((\mathrm{colim}_{k\in K}\, q(k)) \otimes G) \simeq \mathrm{colim}_{k\in K}\, L(q(k) \otimes G)$ which implies that $(\mathrm{colim}_{k\in K}\, q(k)) \otimes_{\mathrm{Seg}} G$ is equivalent to $\mathrm{colim}_{k\in K} (q(k) \otimes_{\mathrm{Seg}} G)$. Analogously one shows that $\otimes_{\mathrm{Seg}}$ preserves colimits diagrams in $\mathrm{P}_{\mathrm{Seg}}(\Delta)$. $\square$

Chapter 6

Completeness, Fully Faithfulness and Essential Surjectivity

6.1 Fully Faithfulness and Essential Surjectivity

Definition 6.1. ([GH15, 5.1.13],[GH15, Definition 5.2.1]) Let $\mathcal{V}$ be a symmetric monoidal ∞-category and let E^1 be the $\mathcal{V}$-enriched ∞-category as introduced in Definition 3.11. We call a map $E^1 \to \mathcal{C}$ in $\mathrm{Alg}_*(\mathcal{V})$ an *equivalence in* $\mathcal{C}$ and we call the Kan complex $\iota_1\mathcal{C} = \mathrm{Map}_{\mathrm{Alg}_*(\mathcal{V})}(E^1, \mathcal{C})$ the *space of equivalences* of $\mathcal{C}$. Let $E^\bullet$ be the cosimplicial object in $\mathrm{Cat}_\infty^{\mathcal{V}}$ as defined in Definition 3.12. We write $\iota\mathcal{C}$ for the geometric realization $|\iota_\bullet\mathcal{C}| = |\,\mathrm{Map}(E^\bullet, \mathcal{C})|$ which we call the *classifying space of equivalences.*

Notation 6.2. *Recall that the inclusion* $\{*\} \hookrightarrow \Phi$ *of the terminal object induces a functor* $u\colon \Delta \simeq \Delta_* \to \Delta_\Phi$. *Let* $u_!\colon \mathrm{coCart}^{*,\mathrm{gen}}_{\mathrm{Seg}} \rightleftarrows \mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}} : u^*$ *denote the adjunction induced by* u. *If* $\mathcal{O}$ *is a* $\mathcal{V}$*-enriched* Φ*-*∞*-operad, then we write* $\iota\mathcal{O}$ *for* $\iota u^*\mathcal{O}$.

Let X be an ∞-groupoid and let $x, y \in X$. It follows from Definition 2.25 that an object $\mathfrak{c}(x;y) \in \Delta^{\mathrm{op}}_{\Phi,X}$ has to lie over the corolla $\mathfrak{c} \in \Delta^{\mathrm{op}}_{\Phi}$ which in turn is given by $u([1]) \in \Delta_\Phi$, i.e. $\mathfrak{c} = ([1], * \xrightarrow{\mathrm{id}} *)$.

Definition 6.3. Let $\mathcal{V}$ be a symmetric monoidal ∞-category with unit $\mathbb{1}$ and let $\mathcal{O}\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \mathcal{V}$ be a $\mathcal{V}$-enriched Φ-∞-operad. Let $u_!$ denote the left adjoint functor mentioned in the notation above. Given an object $\mathfrak{c}(x;y) \in \Delta^{\mathrm{op}}_{\Phi,X}$, we call a morphism $\mathbb{1} \to \mathcal{O}(x;y)$ in $\mathcal{V}$ an *equivalence in* $\mathcal{O}$, if it can be extened to a functor of $\mathcal{V}$-enriched Φ-∞-operads $u_!E^1 \to \mathcal{O}$ lying over the morphism $\{0,1\} \to X$ in $\mathcal{S}$ determined by $0 \mapsto x$ and $1 \mapsto y$.

Definition 6.4. Let s^0 denote the unique map $E^1 \to E^0$ in $\mathrm{Alg}_*(\mathcal{V})$. A morphism in $\mathrm{Alg}_\Phi(\mathcal{V})$ is called *local equivalence*, if it lies in the strongly saturated class (see Definition 4.19) of maps generated by $u_!s^0$.

Notation 6.5. *For the rest of this chapter, unless mentioned otherwise,* $\mathcal{V}$ *will be assumed to be a presentable symmetric monoidal* ∞*-category.*

Proposition 6.6. *The ∞-category $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ (see Definition 3.14) is presentable and is given by a localization of $\mathrm{Alg}_{\Phi}(\mathcal{V})$ with respect to the class of local equivalences.*

Proof. By Theorem 4.22, the ∞-category $\mathrm{Alg}_{\Phi}(\mathcal{V})$ is equivalent to $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$, which is presentable by Proposition 4.20. Since being complete is a property of the underlying $\mathcal{V}$-enriched ∞-category, [GH15, Proposition 5.4.2] implies that a $\mathcal{V}$-enriched Φ-∞-operad $\mathcal{O}$ is complete if and only if its underlying $\mathcal{V}$-enriched ∞-category $u^*\mathcal{O}$ is $\{s^0\}$-local for $s^0\colon E^1 \to E^0$. By adjunction, this means that $\mathcal{O}$ is $\{u_!(s^0)\}$-local. Thus, by [Lur09, Proposition 5.5.4.15], the inclusion $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_{\Phi}(\mathcal{V})$ has a left adjoint. Hence, the ∞-category $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ is a localization of $\mathrm{Alg}_{\Phi}(\mathcal{V})$ with respect to local equivalences. $\square$

Definition 6.7. Let $\mathcal{V}$ be an arbitrary symmetric monoidal ∞-category. Let $\mathcal{O}$ be a Φ-∞-operad lying over $X \in \mathcal{S}$. A morphism $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_{\Phi}(\mathcal{V})$ is called *fully faithful* if, for every object $\mathfrak{c}_I(x_1, \dots, x_n; x) \in \Delta^{\mathrm{op}}_{\Phi,X}$ lying over a corolla $\mathfrak{c}_I$ with $|I| = \mathbf{n}$, the morphism $\mathcal{O}(x_1, \dots, x_n; x) \to \mathcal{P}(f(x_1), \dots, f(x_n); f(x))$ (see Definition 3.10) is an equivalence in $\mathcal{V}_{\langle 1 \rangle}$.

Remark 6.8. *Let $f\colon \mathcal{O} \to \mathcal{P}$ be a morphism in $\mathrm{Alg}_{\Phi}(\mathcal{V})$ and let $\tilde{f}\colon \tilde{\mathcal{O}} \to \tilde{\mathcal{P}}$ denote its image in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ under the equivalence $\mathrm{Alg}_{\Phi}(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ provided by Theorem 4.22. Suppose $\tilde{f}$ lies over the map $f_0\colon X = \tilde{\mathcal{O}}(\bar{\mathfrak{e}}) \to \mathrm{P}(\bar{\mathfrak{e}}) = Y$ in $\mathcal{S}$, then, for a corolla $\mathfrak{c}_I \in \Delta_{\Phi}$ and $|I| = \mathbf{n}$, Example 4.26 implies that there is an equivalence*

$$\{(x_1, \dots, x_n, x)\} \times_{X^{n+1}} \tilde{\mathcal{O}}(v) \simeq \mathrm{Map}_{\mathcal{V}}(\mathcal{O}(\mathfrak{c}_I(x_1, \dots, x_n; x)), v)$$

for every object in $v \in (\mathcal{V}^{\vee})_{\mathfrak{c}_I}$. Therefore, the morphism $f\colon \tilde{\mathcal{O}} \to \tilde{\mathcal{P}}$ is fully faithful if and only if the map $\{(x_1, \dots, x_n, x)\} \times_{X^{n+1}} \tilde{\mathcal{O}}(v) \to \{(f_0(x_1), \dots, f_0(x_n), f_0(x))\} \times_{Y^{n+1}} \tilde{\mathcal{P}}(v)$ induced by $\tilde{f}$ is an equivalence for every corolla $\mathfrak{c}_I \in \Delta_{\Phi}$ and every $v \in (\mathcal{V}^{\vee})_{\mathfrak{c}_I}$. In other words, the map f is fully faithful if and only if the commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{O}}(v) \arrow[r, "\tilde{f}(v)"] \arrow[d] & \tilde{\mathcal{P}}(v) \arrow[d] \\ X^{n+1} \arrow[r, "f_0^{n+1}"'] & Y^{n+1} \end{tikzcd}$$

is a pullback square for every $v \in (\mathcal{V}^{\vee})_{\mathfrak{c}_I}$.

Lemma 6.9. *If $p\colon \mathrm{Alg}_{\Phi}(\mathcal{V}) \to \mathcal{S}$ denotes the Cartesian fibration as defined in Definition 3.9, then a morphism $f\colon \mathcal{O} \to \mathcal{P}$ of $\mathcal{V}$-enriched Φ-∞-operads lying over $f_0\colon X \to Y$ is fully faithful, if and only if f is a p-Cartesian lift of f_0.*

Proof. We can factorize the morphism f in $\mathrm{Alg}_{\Phi}(\mathcal{V})$ into a morphism $g\colon \mathcal{O} \to f_0^*(\mathcal{P})$ lying over X followed by a p-Cartesian morphism $h\colon f_0^*(\mathcal{P}) \to \mathcal{P}$ lying over f_0. If f is fully faithful, then we wish to show that g is an equivalence in $\mathrm{Alg}_{\Phi}(\mathcal{V})$, which is equivalent to requiring g to be an equivalence in $\mathrm{Alg}_{\Phi,X}(\mathcal{V})$. By [GH15, Theorem A.5.3], the morphism g in $\mathrm{Alg}_{\Phi,X}(\mathcal{V})$ is an equivalence if and only if the map $g(A)\colon \mathcal{O}(A) \to f_0^*\mathcal{P}(A)$ is an equivalence in $\mathcal{V}$, for every object $A \in \Delta^{\mathrm{op}}_{\Phi,X}$. Since $\mathcal{O}$ and $\mathcal{P}$ are $\mathcal{V}$-enriched Φ-∞-operads and the fibre of $\mathcal{V}$ over $\mathfrak{e}$ is contractible, the objects $\mathcal{O}(A)$ and $f_0^*\mathcal{P}(A)$ are determined by objects of the form $\mathcal{O}(x_1, \dots, x_n; x)$ and $f_0^*\mathcal{P}(x_1, \dots, x_n; x)$ in $\mathrm{Cr}^*_{\Phi}\,\mathcal{V}$ which lie over some corolla

$\mathfrak{c}_I \in \Delta_\Phi^{\mathrm{op}}$ with $|I| = \mathbf{n}$. By assumption, the map f is fully faithful, therefore the equivalences $\mathcal{O}(x_1, \ldots, x_n; x) \to \mathcal{P}(f(x_1), \ldots, f(x_n); f(x)) = f_0^*\mathcal{P}(x_1, \ldots, x_n; x)$ imply that g is an equivalence.

Conversely, if f is a p-Cartesian lift of $f_0\colon X \to Y$, then $\mathcal{O} \simeq f_0^*(\mathcal{P})$ and hence $\mathcal{O}(x_1, \ldots, x_n; x) \to \mathcal{P}(f(x_1), \ldots, f(x_n); f(x))$ is an equivalence in $\mathrm{Cr}_\Phi^* \mathcal{V}$ for every object $\mathfrak{c}_I(x_1, \ldots, x_n; x) \in \Delta_{\Phi,X}^{\mathrm{op}}$. □

In the theory of simplicial operads, a morphism is called essentially surjective if the induced morphism on homotopy categories is an essentially surjective functor. The following definition is the Φ-operadic adaption of this classical definition.

Definition 6.10. Let $\mathcal{V}$ be an arbitrary symmetric monoidal ∞-category. A morphism $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$ is called *essentially surjective*, if the induced functor $u^*f\colon u^*\mathcal{O} \to u^*\mathcal{P}$ of the underlying $\mathcal{V}$-enriched ∞-categories is essentially surjective, which means that for every object $x \in \iota_0\mathcal{P}$ (see Definition 3.13) there exists an equivalence (i.e. a functor of $\mathcal{V}$-enriched ∞-categories $E^1 \to u^*\mathcal{P}$) connecting x with an object lying in the image of $\iota_0 f$ in $\iota_0\mathcal{P}$.

By unwinding the definition above, we see that [GH15, Lemma 5.3.4] implies that f is essentially surjective if and only if the induced map $\pi_0(\iota f)\colon \pi_0(\iota\mathcal{O}) \to \pi_0(\iota\mathcal{P})$ of sets is surjective.

Notation 6.11. *We write* FFES *for the class of morphisms in* $\mathrm{Alg}_\Phi(\mathcal{V})$ *which are fully faithful and essentially surjective.*

Lemma 6.12. *A fully faithful and essentially surjective morphism in* $\mathrm{Alg}_\Phi(\mathcal{V})$ *between complete objects is an equivalence.*

Proof. Let $f\colon \mathcal{O} \to \mathcal{P}$ be a fully faithful and essentially surjective morphism in $\mathrm{Alg}_\Phi(\mathcal{V})$. It follows from Definition 6.7 and Definition 6.10 that the map $u^*(f)$ between the underlying $\mathcal{V}$-enriched ∞-categories $u^*\mathcal{O}$ and $u^*\mathcal{P}$ is fully faithful and essentially surjective. By [GH15, Definition 5.3.8], the map $u^*(f)$ is an equivalence. This implies that the Cartesian fibration $p\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathcal{S}$ carries f to an equivalence $p(f)$ in $\mathcal{S}$. Since the map f is then a p-Cartesian lift of the equivalence $p(f)$ by Lemma 6.9, it has to be an equivalence as well. □

The following two lemmata are Φ-operadic versions of [GH15, Proposition 5.1.15] and [GH15, Proposition 5.3.9], respectively.

Lemma 6.13. *Let $\mathcal{O}$ be a $\mathcal{V}$-enriched Φ-∞-operad. Suppose there exist equivalences in $\mathcal{O}$ (i.e. maps $u_!E^1 \to \mathcal{O}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$) which are induced by $f\colon \mathbb{1} \to \mathcal{O}(y; z)$ and $f_i\colon \to \mathcal{O}(a_i; b_i)$, for $1 \le i \le n$. Then the following assertions hold:*

1. *For every object $\mathfrak{c}_I(b_1, \ldots, b_n; c) \in \Delta_{\Phi,X}^{\mathrm{op}}$ lying over a corolla $\mathfrak{c}_I$, the composite*

$$
\begin{aligned}
&\mathcal{O}(b_1, \ldots, b_n; c) \to (\mathbb{1}, \ldots, \mathbb{1}, \mathcal{O}(b_1, \ldots, b_n; c)) \\
&\overset{(f_1,\ldots,f_n;\mathrm{id})}{\to} (\mathcal{O}(a_1, b_1), \ldots, \mathcal{O}(a_n, b_n), \mathcal{O}(b_1, \ldots, b_n; c)) \\
&\simeq \mathcal{O}([2], J, a_1, a_2, \ldots b_{n-1}, b_n, c) \to \mathcal{O}(a_1, \ldots, a_n; c)
\end{aligned}
\tag{6.1}
$$

is an equivalence in $\mathcal{V}$, where $([2], J) \in \Delta_\Phi^{\mathrm{op}}$ is determined by the sequence $I(0) \overset{\mathrm{id}}{\to} I(1) \to I(2) = *$ and the last map is given by the corolla $\mathfrak{c}_I$.

2. *For every object $\mathfrak{c}_I(x_1, \ldots, x_n; y) \in \Delta^{\mathrm{op}}_{\Phi,X}$ lying over a corolla $\mathfrak{c}_I$, the composite*

$$\begin{aligned}\mathcal{O}(x_1, \ldots, x_n; y) \to (\mathcal{O}(x_1, \ldots, x_n; y), \mathbb{1}) \xrightarrow{f} (\mathcal{O}(x_1, \ldots, x_n; y), \mathcal{O}(y; z)) \\ \simeq \mathcal{O}([2], J, x_1, \ldots, x_n, y; z) \to \mathcal{O}(x_1, \ldots, x_n; z),\end{aligned} \tag{6.2}$$

is an equivalence in $\mathcal{V}$, where $([2], J)$ is determined by the sequence $I(0) \to I(1) \simeq * \xrightarrow{\mathrm{id}} * \simeq I(2)$ and the first map is given by the corolla $\mathfrak{c}_I$.

Proof. We will prove the first statement, the second claim can be shown in a similar manner. If we write $\prod_{1 \leq i \leq n} f_i^* \colon \mathcal{O}(b_1, \ldots, b_n; c) \to \mathcal{O}(a_1, \ldots, a_n; c)$ for the composition of the maps in 6.1, then it is clear that it can written as a composite of maps

$$\mathrm{id} \times \ldots \times f_i^* \times \ldots \mathrm{id} \colon \mathcal{O}(a_1, \ldots, a_{i-1}, b_i, b_{i+1}, \ldots, b_n; c) \to \mathcal{O}(a_1, \ldots, a_{i-1}, a_i, b_{i+1}, \ldots, b_n; c), \tag{6.3}$$

for $1 \leq i \leq n$. Therefore, it suffices to prove that, for $1 \leq i \leq n$, the map in 6.3 has an inverse. We will prove this for $i = 1$, the other cases can shown analogously.

By Definition 6.3, the morphism $f_1 \colon \mathbb{1} \to \mathcal{O}(a_1, b_1)$ can be regarded as a morphism in the underlying $\mathcal{V}$-enriched ∞-category $u^*\mathcal{O}$ and it can be extended to an equivalence $\tilde{f}_1 \colon E^1 \to u^*\mathcal{O}$. If the morphism $f_1^{-1} \colon \mathbb{1} \to \mathcal{O}(b_1, a_1)$ denotes the inverse map, then the inverse of $f_1^* \times \mathrm{id} \times \ldots \times \mathrm{id}$ is clearly given by $(f_1^{-1})^* \times \mathrm{id} \times \ldots \times \mathrm{id}$. □

Proposition 6.14. *Morphisms in $\mathrm{Alg}_\Phi(\mathcal{V})$ which are fully faithful and essentially surjective satisfy the 2-of-3 property.*

Proof. Let $f \colon \mathcal{O} \to \mathcal{P}$ and $g \colon \mathcal{P} \to \mathcal{Q}$ be two morphisms in $\mathrm{Alg}_\Phi(\mathcal{V})$ lying over $X \to Y$ and $Y \to Z$, respectively.

1. If f and g are fully faithful and essentially surjective, then it follows directly from the definition that their composite $g \circ f$ is fully faithful. Since being essentially surjective is a property of the underlying $\mathcal{V}$-enriched ∞-categories, [GH15, Proposition 5.3.9] implies that $g \circ f$ is also essentially surjective.

2. If $g \circ f$ and g are fully faithful and essentially surjective, then they are in particular p-Cartesian morphisms by Lemma 6.9. [Lur09, Proposition 2.4.1.7] implies that the map f is also p-Cartesian which is equivalent to being fully faithful by Lemma 6.9 again. By [GH15, Proposition 5.3.9], f is also essentially surjective.

3. If $g \circ f$ and f are fully faithful and essentially surjective, then [GH15, Proposition 5.3.9] implies that g is essentially surjective. For fully faithfulness, let $\mathfrak{c}(y_1, \ldots, y_n; y) \in \Delta^{\mathrm{op}}_{\Phi,Y}$ be an object lying over a corolla $\mathfrak{c} \in \Delta^{\mathrm{op}}_\Phi$. We want to show that the natural map $\mathcal{P}(y_1, \ldots, y_n; y) \to \mathcal{Q}(g(y_1), \ldots, g(y_n); g(y))$ is an equivalence in $\mathcal{V}$. The essential surjectivity of f provides the existence of objects $x_1, \ldots, x_n, x \in X$ and morphisms $h \colon \mathbb{1} \to \mathcal{P}(f(x); y))$ and $h_i \colon \mathbb{1} \to \mathcal{P}(y_i; f(x_i))$ in $\mathcal{V}_{\langle 1 \rangle}$, for $1 \leq i \leq n$, which induce equivalences $u_! E^1 \to \mathcal{P}$ in $\mathcal{P}$. By Lemma 6.13, the morphisms h and h_i, $1 \leq i \leq n$, induce a commuta-

tive diagram, where the vertical morphisms are equivalences in $\mathcal{V}_{\langle 1\rangle}$:

$$\begin{tikzcd}
\mathcal{P}(f(x_1),\dots,f(x_n);f(x)) \arrow[r] \arrow[d] & \mathcal{Q}(g\circ f(x_1),\dots,g\circ f(x_n);g\circ f(x)) \arrow[d] \\
\mathcal{P}(y_1,\dots,y_n;y) \arrow[r] & \mathcal{Q}(g(y_1),\dots,g(y_n);g(y)).
\end{tikzcd} \tag{6.4}$$

By the 2-of-3 property of equivalences, we only need to verify that the upper horizontal map is also an equivalence. By applying 2-of-3 property once again, the claim follows from the fact that the map in question is the right diagonal map of the commutative diagram

$$\begin{tikzcd}
\mathcal{O}(x_1,\dots,x_n;x) \arrow[rr] \arrow[dr] & & \mathcal{P}(f(x_1),\dots,f(x_n);f(x)) \arrow[dl] \\
& \mathcal{Q}(g\circ f(x_1),\dots,g\circ f(x_n);g\circ f(x)), &
\end{tikzcd} \tag{6.5}$$

where the horizontal map and the left diagonal map are equivalences in $\mathcal{V}_{\langle 1\rangle}$ by the assumption of fully faithfulness of f and $g\circ f$.

□

Definition 6.15. Let $\mathrm{Alg}_\Phi(\mathcal{V})\to\mathcal{S}$ be the Cartesian fibration defined in Definition 3.9. We write $l\colon \mathrm{Alg}_\Phi^{\mathrm{Set}}(\mathcal{V})\to\mathrm{Alg}_\Phi(\mathcal{V})$ for the functor given by the pullback diagram

$$\begin{tikzcd}
\mathrm{Alg}_\Phi^{\mathrm{Set}}(\mathcal{V}) \arrow[r, "l"] \arrow[d] & \mathrm{Alg}_\Phi(\mathcal{V}) \arrow[d] \\
\mathrm{Set} \arrow[r, hook] & \mathcal{S},
\end{tikzcd}$$

where the bottom horizontal map is the canonical inclusion map.

For an ∞-category $\mathcal{C}$ and a class of morphisms W in $\mathcal{C}$ satisfying certain conditions, let us recall the definition of the localization of $\mathcal{C}$ with respect to W as introduced in [Lur, Construction 4.1.3.1].

Definition 6.16. Let $\mathcal{C}$ be an ∞-category. A *system* on $\mathcal{C}$ is defined to be a collection of morphisms $W\subseteq \mathrm{Hom}_{\mathrm{sSet}}(\Delta^1,\mathcal{C})$ which contains all equivalences and is stable under homotopy and composition. We write $\mathrm{Sys}(\mathcal{C})$ for the partially ordered set of all systems on $\mathcal{C}$ and we write $\mathcal{W}\,\mathrm{Cat}_\infty\to\mathrm{Cat}_\infty$ for the Cartesian fibration associated to the functor $\mathrm{Cat}_\infty^{\mathrm{op}}\to\mathrm{Cat}_\infty$ induced by the assignment $\mathcal{C}\mapsto\mathrm{Sys}(\mathcal{C})$.

The definition of $\mathcal{W}\,\mathrm{Cat}_\infty$ implies that an object in $\mathcal{W}\,\mathrm{Cat}_\infty$ is a pair $(\mathcal{C},W)$, where $\mathcal{C}$ is an ∞-category and W is a system on $\mathcal{C}$. It follows from the construction that the mapping space $\mathrm{Map}_{\mathcal{W}\,\mathrm{Cat}_\infty}((\mathcal{C},W),(\mathcal{C}',W'))$between two objects $(\mathcal{C},W)$ and $(\mathcal{C}',W')$ in $\mathcal{W}\,\mathrm{Cat}_\infty$ can be identified with the summand of the mapping space $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{C}')$ of Cat_∞ spanned by functors $f\colon\mathcal{C}\to\mathcal{C}'$ such that $f(W)\subseteq W'$.

The Cartesian fibration $\mathcal{W}\,\mathrm{Cat}_\infty\to\mathrm{Cat}_\infty$ admits a section G determined by $G(\mathcal{C})=(\mathcal{C},W)$, where W denotes the collection of all equivalences in $\mathcal{C}$. By [Lur, Proposition 4.1.3.2], the functor $G\colon \mathrm{Cat}_\infty \to \mathcal{W}\mathrm{Cat}_\infty$ is a right adjoint. The left adjoint to G will be denoted by $(\mathcal{C}, W) \mapsto \mathcal{C}[W^{-1}]$.

The following theorem can be shown in a manner similar to [GH15, Theorem 5.3.17].

Theorem 6.17. *The map $l\colon \mathrm{Alg}_\Phi^{\mathrm{Set}}(\mathcal{V}) \to \mathrm{Alg}_\Phi(\mathcal{V})$ in Definition 6.15 induces an equivalence of ∞-categories*

$$\mathrm{Alg}_\Phi^{\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}] \to \mathrm{Alg}_\Phi(\mathcal{V})[\mathrm{FFES}^{-1}].$$

6.2 Completeness

Recall that $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ denotes the full subcategory of the ∞-category $\mathrm{Alg}_\Phi(\mathcal{V})$ spanned by the complete objects (see Definition 3.14). By Proposition 6.6, this ∞-category is given by localizing $\mathrm{Alg}_\Phi(\mathcal{V})$ with respect to local equivalences. The goal of this section is to verify Theorem 6.34 which asserts that a morphism in $\mathrm{Alg}_\Phi(\mathcal{V})$ is a local equivalence if and only if it is fully faithful and essentially surjective. In particular, we then obtain an equivalence of ∞-categories

$$\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \simeq \mathrm{Alg}_\Phi(\mathcal{V})[\mathrm{FFES}^{-1}].$$

Here, we follow the idea of [GH15, 5.4-5.6] and introduce pseudo equivalences in Definition 6.21. These are morphisms which admit an inverse up to a natural equivalence. We show in Proposition 6.22 and Theorem 6.27 that pseudo equivalences are fully faithful and essentially surjective as well as local equivalences. As in the enriched ∞-categorical case, these observations allow us to define a new completion functor $\widehat{(-)}\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ in Proposition 6.32, which finally lead us to the proof of the desired equivalence.

As a step to the proof of the equivalence between fully faithful, essentially surjective functors and local equivalences, we check the necessary condition that they are the same in $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$:

Proposition 6.18. *A map between complete $\mathcal{V}$-enriched Φ-∞-operads is fully faithful and essentially surjective if and only if it is an (local) equivalence.*

Proof. Since every local equivalence between complete $\mathcal{V}$-enriched Φ-∞-operads is an equivalence, the "if" part of the claim is obvious. For the "only if" part, let $\mathcal{O} \in \mathrm{Alg}_{\Phi,\mathrm{cp},X}(\mathcal{V})$ and $\mathcal{P} \in \mathrm{Alg}_{\Phi,\mathrm{cp},Y}(\mathcal{V})$. If $f\colon \mathcal{O} \to \mathcal{P}$ is a fully faithful and essentially surjective morphism in $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$, then it induces a fully faithful and essentially surjective map $u^*\mathcal{O} \to u^*\mathcal{P}$ between the associated underlying $\mathcal{V}$-enriched ∞-categories. By [GH15, Definition 5.3.7], f induces an equivalence $\iota\mathcal{O} \simeq \iota\mathcal{P}$ (see Definition 6.1). Definition 3.13, Definition 6.3 and [GH15, Lemma 5.1.2] imply that $X \simeq \iota_0\mathcal{O} \simeq \iota_0\mathcal{P} \simeq Y$. Hence, we can assume that f is a morphism in $\mathrm{Alg}_{\Phi,\mathrm{cp},X}(\mathcal{V})$. [GH15, Theorem A.5.3] implies that if f is a pointwise equivalence, then f is already an equivalence in $\mathrm{Alg}_{\Phi,\mathrm{cp},X}(\mathcal{V})$ and, therefore, also in $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$. This means that it suffices to show that the map $f(A)\colon \mathcal{O}(A) \to \mathcal{P}(A)$ is an equivalence, for every object $A \in \Delta^{\mathrm{op}}_{\Phi,X}$. Since $\mathcal{O}$ and $\mathcal{P}$ are $\mathcal{V}$-enriched Φ-∞-operads and the fibre $\mathcal{V}_{\mathfrak{c}}$ is contractible, the objects $\mathcal{O}(A)$ and $\mathcal{P}(A)$ are determined by objects of the form $\mathcal{O}(x_1, \ldots, x_n; x)$ and $\mathcal{P}(x_1, \ldots, x_n; x)$. The claim then follows from the assumption that f is fully faithful. $\square$

Notation 6.19. *Let $\mathcal{O}$ be an object in $\mathrm{Alg}_\Phi(\mathcal{V})$. For every $n \geq 0$, we will write $\mathcal{O} \otimes E^n$ for the object in $\mathrm{Alg}_\Phi(\mathcal{V})$ corresponding to $\tilde{\mathcal{O}} \otimes E^{\mathcal{S}}_{\{0,\ldots,n\}} \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$, where $\tilde{\mathcal{O}}$ is the image of $\mathcal{O}$ under the equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ and $E^{\mathcal{S}}_{\{0,\ldots,n\}} \in \mathrm{P}_{\mathrm{Seg}}(\Delta)$.*

Definition 6.20. Let $f, g\colon \mathcal{O} \to \mathcal{P}$ be two morphisms in $\mathrm{Alg}_\Phi(\mathcal{V})$ and let $d^0, d^1\colon E^0 \to E^1$ denote the face maps. We define a *natural equivalence* from f to g to be a morphism $h\colon \mathcal{O} \otimes E^1 \to \mathcal{P}$ such that $h \circ (\mathrm{id} \otimes d^1) \simeq f$ and $h \circ (\mathrm{id} \otimes d^0) \simeq g$. We say that f and g are *naturally equivalent*, if there exists a natural equivalence between f and g.

Definition 6.21. For a morphism $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$, a *pseudo-inverse* of f is a morphism $g\colon \mathcal{P} \to \mathcal{O}$ such that there exists a natural equivalences ϕ from $\mathrm{id}_\mathcal{O}$ to $g \circ f$ and ψ from $f \circ g$ to $\mathrm{id}_\mathcal{P}$. We call a morphism f a *pseudo equivalence*, if it has a pseudo-inverse g and we call the triple (f, g, ϕ, ψ) a *pseudo equivalence datum.*

The following proposition is the Φ-operadic interpretation of [GH15, Proposition 5.5.3].

Proposition 6.22. *If $f\colon \mathcal{O} \to \mathcal{P}$ is a pseudo equivalence, then it is fully faithful and essentially surjective.*

Proof. Let (f, g, ϕ, ψ) be the pseudo equivalence datum associated to the pseudo equivalence f. For a point $y \in \iota_0\mathcal{P}$, the natural equivalence ψ provides an equivalence $fg(y) \simeq y$. Thus, f is esentially surjective.

For the fully faithfulness of f, suppose $\mathfrak{c}(x_1, \ldots, x_n; x) \in \Delta^{\mathrm{op}}_{\Phi,X}$. We want to show that the map $f'\colon \mathcal{O}(x_1, \ldots, x_n; x) \to \mathcal{P}(f(x_1), \ldots, f(x_n); f(x))$ induced by f is an equivalence in $\mathcal{V}_{\langle 1\rangle}$. Let $g'\colon \mathcal{P}(f(x_1), \ldots, f(x_n); f(x)) \to \mathcal{O}(gf(x_1), \ldots, gf(x_n); gf(x))$ denote the map induced by g. Then the natural equivalence ϕ provides a morphism $\phi'\colon \mathcal{O}(gf(x_1), \ldots, gf(x_n); gf(x)) \to \mathcal{O}(x_1, \ldots, x_n; x)$ rendering the following diagram commutative

$$\begin{tikzcd}
\mathcal{P}(f(x_1), \ldots, f(x_n); f(x)) \arrow[r, "g'"] & \mathcal{O}(gf(x_1), \ldots, gf(x_n); gf(x)) \arrow[d, "\phi'"] \\
\mathcal{O}(x_1, \ldots, x_n; x) \arrow[u, "f'"] \arrow[r, "\mathrm{id}"'] & \mathcal{O}(x_1, \ldots, x_n; x).
\end{tikzcd}$$

This means that $\phi' g' f' \simeq \mathrm{id}$. Therefore, it suffices to show that $\phi' g'$ admits a left inverse. If $f''\colon \mathcal{O}(gf(x_1), \ldots, gf(x_n); gf(x)) \to \mathcal{P}(fgf(x_1), \ldots, fgf(x_n); fgf(x))$ denotes the map induced by f, then the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(gf(x_1), \ldots, gf(x_n); gf(x)) \arrow[r, "f''"] & \mathcal{P}(fgf(x_1), \ldots, fgf(x_n); fgf(x)) \arrow[d, "\psi'"] \\
\mathcal{P}(f(x_1), \ldots, f(x_n); f(x)) \arrow[u, "g'"] \arrow[r, "\mathrm{id}"'] & \mathcal{P}(f(x_1), \ldots, f(x_n); f(x)),
\end{tikzcd}$$

where $\psi'\colon \mathcal{P}(fgf(x_1), \ldots, fgf(x_n); fgf(x)) \to \mathcal{P}(f(x_1), \ldots, f(x_n); f(x))$ denotes the equivalence induced by the natural equivalence ψ. Hence, we have $\psi' f'' g' \simeq \mathrm{id}$. Since ϕ' is induced by a natural equivalence, it has an inverse ϕ'^{-1} which implies that $\psi' f'' \phi'^{-1}$ is a left inverse of $\phi' g'$. $\square$

Corollary 6.23. *Pseudo equivalences between complete objects are equivalences in* $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$.

Proof. By Proposition 6.22, pseudo equivalences in $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ are fully faithful and essentially surjective and, by Proposition 6.18, these maps are equivalences in $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$. $\square$

Similar to [GH15, Lemma 5.5.7], we have the following result:

Lemma 6.24. *If $f\colon \mathcal{C} \to \mathcal{D}$ is a pseudo equivalence in* $\mathrm{P}_{\mathrm{Seg}}(\Delta) = \mathrm{Seg}_* \simeq \mathrm{Alg}_*(\mathcal{S})$*, then it induces a pseudo equivalence $\mathcal{O}^f\colon \mathcal{O}^{\mathcal{D}} \to \mathcal{O}^{\mathcal{C}}$ in* $\mathrm{Alg}_\Phi(\mathcal{V})$*, for every object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$.*

Proof. If (f, g, ϕ, ψ) denotes the pseudo equivalence datum associated to f, then, by Corollary 5.6, the natural equivalences $\phi\colon \mathcal{C} \otimes E^1 \to \mathcal{C}$ and $\psi\colon \mathcal{D} \otimes E^1 \to \mathcal{D}$ in $\mathrm{Alg}_*(\mathcal{S})$ induce maps $\mathcal{O}^{\mathcal{C}} \to \mathcal{O}^{\mathcal{C}\otimes E^1} \simeq (\mathcal{O}^{\mathcal{C}})^{E^1}$ and $\mathcal{O}^{\mathcal{D}} \to \mathcal{O}^{\mathcal{D}\otimes E^1} \simeq (\mathcal{O}^{\mathcal{D}})^{E^1}$ in $\mathrm{Alg}_*(\mathcal{S})$, respectively. If $\phi^f\colon \mathcal{O}^{\mathcal{C}} \otimes E^1 \to \mathcal{O}^{\mathcal{C}}$ and $\psi^f\colon \mathcal{O}^{\mathcal{D}} \otimes E^1 \to \mathcal{O}^{\mathcal{D}}$ denote the corresponding adjoint maps, then ϕ^f and ψ^f are natural equivalences and one readily checks that $(\mathcal{O}^f, \mathcal{O}^g, \phi^f, \psi^f)$ is a pseudo equivalence datum associated to $\mathcal{O}^f$. □

Lemma 6.25. *For every complete object $\mathcal{O} \in \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$, the map $\mathcal{O}^{s_0}\colon \mathcal{O} \simeq \mathcal{O}^{E^0} \to \mathcal{O}^{E^1}$ induced by the obvious map $s^0\colon E^1 \to E^0$ in $\mathrm{Alg}_*(\mathcal{S})$ is an equivalence in $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$.*

Proof. By [GH15, Definition 5.5.6], the map s_0 is a pseudo equivalence in $\mathrm{Alg}_*(\mathcal{S})$ and the lemma above implies that $\mathcal{O}^{s_0}$ is a pseudo equivalence in $\mathrm{Alg}_\Phi(\mathcal{V})$. Since $\mathcal{O}^{E^0} \simeq \mathcal{O}$ is complete by assumption, it suffices to show that $\mathcal{O}^{E^1}$ is also complete by Corollary 6.23.

The adjunctions $(u_!, u^*)$ and $(- \otimes E^1, (-)^{E^1})$ provide a chain of equivalences

$$\iota_0(\mathcal{O}^{E^1}) = \mathrm{Map}(E^0, u^*(\mathcal{O}^{E^1})) \simeq \mathrm{Map}(u_! E^0 \otimes E^1, \mathcal{O}) \simeq \mathrm{Map}(u_! E^1, \mathcal{O}) \simeq \mathrm{Map}(E^1, u^*\mathcal{O}).$$

Similarly, we have the equivalence $\iota_1(\mathcal{O}^{E^1}) \simeq \mathrm{Map}(E^1 \otimes E^1, u^*\mathcal{O})$. It follows that the map $\iota_0(\mathcal{O}^{E^1}) \to \iota_1(\mathcal{O}^{E^1})$ can be identified with $\mathrm{Map}(\mathrm{id}_{E^1} \otimes s^0, u^*\mathcal{O})\colon \mathrm{Map}(E^1, u^*\mathcal{O}) \to \mathrm{Map}(E^1 \otimes E^1, u^*\mathcal{O})$. Since $\mathcal{O}$ is complete and $\mathrm{id}_{E^1} \otimes s^0$ is a local equivalence by [GH15, Lemma 5.4.7], the claim follows from Proposition 6.6. □

Lemma 6.26. *For every object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, the map $\mathrm{id}_{\mathcal{C}} \otimes s^0\colon \mathcal{O} \otimes E^1 \to \mathcal{O} \otimes E^0 \simeq \mathcal{O}$ induced by $s^0\colon E^1 \to E^0$ is a local equivalence.*

Proof. In order to show that $\mathrm{id}_{\mathcal{C}} \otimes s^0$ is a local equivalence, we have to verify that the induced map

$$(\mathrm{id}_{\mathcal{C}} \otimes s^0)^*\colon \mathrm{Map}(\mathcal{O}, \mathcal{P}) \to \mathrm{Map}(\mathcal{O} \otimes E^1, \mathcal{P})$$

is an equivalence for every complete object $\mathcal{P}$. By adjunction, this is equivalent to requiring $\mathrm{Map}(\mathcal{O}, \mathcal{P}) \to \mathrm{Map}(\mathcal{O}, \mathcal{P}^{E^1})$ is an equivalence, which is true for every complete object $\mathcal{P}$ by the previous lemma. □

Theorem 6.27. *Every pseudo equivalence is a local equivalence.*

Proof. Let $f\colon \mathcal{O} \to \mathcal{P}$ be a pseudo equivalence in $\mathrm{Alg}_\Phi(\mathcal{V})$ and let (f, g, ϕ, ψ) be its corresponding pseudo equivalence datum. We want to show that for every complete object $\mathcal{Q}$, the map $f^*\colon \mathrm{Map}(\mathcal{P}, \mathcal{Q}) \to \mathrm{Map}(\mathcal{O}, \mathcal{Q})$ is an equivalence. It follows from Definition 6.21 that

$$f^* g^* \simeq (\mathrm{id} \otimes d^0)^* \phi^* \text{ and } \mathrm{id}^* \simeq (\mathrm{id} \otimes d^1)^* \phi^*.$$

Similarly, we have

$$g^* f^* \simeq (\mathrm{id} \otimes d^1)^* \psi^* \text{ and } \mathrm{id}^* \simeq (\mathrm{id} \otimes d^0)^* \psi^*.$$

Hence, we only need to show that the morphisms $(\mathrm{id} \otimes d^0)^*$ and $(\mathrm{id} \otimes d^1)^*$ are equivalent in $\mathcal{S}$.

One observes that, for $i \in \{0,1\}$, there exists a map $\sigma\colon \Delta^2 \to \mathrm{Alg}_*(\mathcal{S})$ exhibiting the equivalence $d^i \circ s^0 \simeq \mathrm{id}_{E^0}$. For every $\mathcal{O}' \in \mathrm{Alg}_\Phi(\mathcal{V})$, composing σ with $\mathrm{Map}(\mathcal{O}' \otimes -, \mathcal{Q})$ shows that

$$(\mathrm{id} \otimes s^0)^* \circ (\mathrm{id} \otimes d^i)^*\colon \mathrm{Map}(\mathcal{O}', \mathcal{Q}) \to \mathrm{Map}(\mathcal{O}', \mathcal{Q}) \tag{6.6}$$

is equivalent to the identity. Since $(\mathrm{id} \otimes s^0)$ is a local equivalence by Lemma 6.26, the map $(\mathrm{id} \otimes s^0)^*$ is an equivalence for every complete object $\mathcal{Q}$. Therefore, the maps $(\mathrm{id} \otimes d^0)^*$ and $(\mathrm{id} \otimes d^1)^*$ are equivalent, because both are right inverses of the equivalence $(\mathrm{id} \otimes s^0)^*$. $\square$

We will show that a construction similar to the one presented in [GH15, 5.5] provides a completion functor for $\mathcal{V}$-enriched Φ-∞-operads, which generalizes the completion functor introduced in [GH15, 5.5] and that given by Rezk [Rez01].

Definition 6.28. Given an object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, we write $\widehat{\mathcal{O}}$ for its geometric realization $|\mathcal{O}^{E^\bullet}|$.

Lemma 6.29. *For an object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, we have an equivalence $u^*(\mathcal{O}^{E^\bullet}) \simeq (u^*\mathcal{O})^{E^\bullet}$.*

Proof. Let $\mathcal{C}$ be an object in $\mathrm{Alg}_*(\mathcal{V})$, i.e. a $\mathcal{V}$-enriched ∞-category. The adjunctions $(u_!, u^*)$ and $(- \otimes E^\bullet, (-)^{E^\bullet})$ induce a chain of equivalences of mapping spaces

$$\mathrm{Map}_{\mathrm{Alg}_*(\mathcal{V})}(\mathcal{C}, u^*(\mathcal{O}^{E^\bullet})) \simeq \mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(u_!\mathcal{C}, \mathcal{O}^{E^\bullet}) \simeq \mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(u_!\mathcal{C} \otimes E^\bullet, \mathcal{O}).$$

It follows from the equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ and the definition of the tensor product in Definition 5.2 that $u_!\mathcal{C} \otimes E^\bullet$ is equivalent to $u_!(\mathcal{C} \otimes E^\bullet)$. Hence, the mapping space above is equivalent to

$$\mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(u_!(\mathcal{C} \otimes E^\bullet), \mathcal{O}) \simeq \mathrm{Map}_{\mathrm{Alg}_*(\mathcal{V})}(\mathcal{C}, (u^*\mathcal{O})^{E^\bullet}).$$

$\square$

Lemma 6.30. *For every $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, the canonical map $u^*|\mathcal{O}^{E^\bullet}| \to |u^*\mathcal{O}^{E^\bullet}|$ is an equivalence.*

Proof. The equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ provided by Theorem 4.22 allows us to identify the diagram $\mathcal{O}^{E^\bullet}\colon \Delta^{\mathrm{op}} \to \mathrm{Alg}_\Phi(\mathcal{V})$ with a diagram $\tilde{\mathcal{O}}^{E^\bullet}\colon \Delta^{\mathrm{op}} \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ whose colimit is given by the localization of the colimit of $\tilde{\mathcal{O}}^{E^\bullet}$ in the presheaf category $\mathrm{P}(\mathcal{V}^\vee)$.

We want to show that its colimit in $\mathrm{P}(\mathcal{V}^\vee)$ is already a Segal Φ-presheaf by verifying that it satisfies the two conditions of Definition 4.13. For every object $v \in (\mathcal{V}^{\kappa,\vee})_{\mathfrak{c}_I}$ lying over a corolla $\mathfrak{c}_I$ with $|I| = k$ and every morphism $[m] \to [n]$ in Δ^{op}, there is a commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{O}}^{E^m}(v) \arrow[r] \arrow[d] & \tilde{\mathcal{O}}^{E^n}(v) \arrow[d] \\
\tilde{\mathcal{O}}^{E^m}(\bar{\mathfrak{e}})^{k+1} \arrow[r] & \tilde{\mathcal{O}}^{E^n}(\bar{\mathfrak{e}})^{k+1}.
\end{tikzcd}$$

It follows from Lemma 6.24, Proposition 6.22 and Remark 6.8 that the functor $\tilde{\mathcal{O}}^{E^m} \to \tilde{\mathcal{O}}^{E^n}$ is fully faithful if and only if the square above is Cartesian. Since $\mathcal{S}$ is an ∞-topos, Theorem 3.26

implies that the following commutative square

$$\begin{tikzcd}
\tilde{\mathcal{O}}^{E^0}(v) \arrow[r] \arrow[d] & |\tilde{\mathcal{O}}^{E^\bullet}|(v) \arrow[d] \\
\tilde{\mathcal{O}}^{E^0}(\bar{\mathfrak{c}})^{k+1} \arrow[r] & |\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{c}})^{k+1}.
\end{tikzcd} \tag{6.7}$$

is Cartesian. The surjectivity of the bottom horizontal map on connected components and the pullback condition show that each fibre of $|\tilde{\mathcal{O}}^{E^\bullet}|(v) \to |\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{c}})^{k+1}$ is equivalent to a fibre of $\tilde{\mathcal{O}}^{E^0}(v) \to \tilde{\mathcal{O}}^{E^0}(\bar{\mathfrak{c}})^{k+1}$.

To check the first condition for being a Segal Φ-presheaf, let $q\colon K^{\triangleleft} \to (\mathcal{V}^{\vee,\mathrm{op}})_{\mathfrak{c}_I}$ be a small limit diagram. Let $\lim_{k\in K} \tilde{\mathcal{O}}^{E^0}(q(k))$ denote the limit of the diagram $K \to (\mathcal{V}^{\vee,\mathrm{op}})_{\mathfrak{c}_I} \to \mathcal{S}_{/\tilde{\mathcal{O}}^{E^0}(\bar{\mathfrak{c}})^{k+1}}$ and let $\lim_{k\in K} |\tilde{\mathcal{O}}^{E^\bullet}|(q(k))$ be defined analogously. In the induced commutative diagram in $\mathcal{S}$

$$\begin{tikzcd}
& |\tilde{\mathcal{O}}^{E^\bullet}|(\lim_{k\in K} q(k)) \arrow[rr] \arrow[dd] & & \lim_{k\in K} |\tilde{\mathcal{O}}^{E^\bullet}|(q(k)) \arrow[dd] \\
\tilde{\mathcal{O}}^{E^0}(\lim_{k\in K} q(k)) \arrow[ur] \arrow[rr, "\simeq"] \arrow[dd] & & \lim_{k\in K} \tilde{\mathcal{O}}^{E^0}(q(k)) \arrow[ur] \arrow[dd] & \\
& |\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{c}})^{k+1} \arrow[rr, "="'] & & |\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{c}})^{k+1} \\
\tilde{\mathcal{O}}^{E^0}(\bar{\mathfrak{c}})^{k+1} \arrow[ur] \arrow[rr, "="'] & & \tilde{\mathcal{O}}^{E^0}(\bar{\mathfrak{c}})^{k+1} \arrow[ur] &
\end{tikzcd}$$

the horizontal maps of the front square are equivalences by the assumption that $\tilde{\mathcal{O}}^{E^0} \simeq \tilde{\mathcal{O}}$ is a Segal Φ-presheaf. Since limits commute with pullbacks, the pullback square 6.7 implies that the commutative squares on both sides are also pullback squares. Thus, the back commutative square is a pullback, which is equivalent to saying that the map $|\tilde{\mathcal{O}}^{E^\bullet}|(\lim_{k\in K} q(k)) \to \lim_{k\in K} |\tilde{\mathcal{O}}^{E^\bullet}|(q(k))$ is an equivalence. This shows that $|\tilde{\mathcal{O}}^{E^\bullet}|$ satisfies the first condition of Definition 4.13.

To verify the second condition of Definition 4.13, we have to show that, for every object $v \in \mathcal{V}^{\vee,\mathrm{op}}$ lying over $([m], I)$, the canonical map $|\tilde{\mathcal{O}}^{E^\bullet}|(v) \to \lim_{\alpha\in(\Delta_\Phi^{\mathrm{el},\mathrm{op}})_{([m],I)/}} |\tilde{\mathcal{O}}^{E^\bullet}|(\alpha^* v)$ is an equivalence. This follows from the same arguments as above.

Hence, the colimit $|\tilde{\mathcal{O}}^{E^\bullet}|$ in $\mathrm{P}(\mathcal{V}^\vee)$ is already a Segal Φ-presheaf. If we interpret the forgetful functor $u^*\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_*(\mathcal{V})$ as functor between the corresponding presheaf categories, then it obviously preserves colimits and the observation above implies that the canonical map $u^*|\mathcal{O}^{E^\bullet}| \to |u^*\mathcal{O}^{E^\bullet}|$ is an equivalence. $\square$

Corollary 6.31. *For $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, the underlying $\mathcal{V}$-enriched ∞-category $u^*\widehat{\mathcal{O}}$ of $\widehat{\mathcal{O}}$ is given by the geometric realization $|(u^*\mathcal{O})^{E^\bullet}|$.*

Proof. Since the functor u^* preserves geometric realizations by the previous lemma and u^* commutes with $(-)^{E^\bullet}$ by Lemma 6.29, we have the following equivalences

$$u^*\widehat{\mathcal{O}} = u^*|\mathcal{O}^{E^\bullet}| \simeq |u^*(\mathcal{O}^{E^\bullet})| \simeq |(u^*\mathcal{O})^{E^\bullet}|.$$

$\square$

Proposition 6.32. *The assignment $\mathcal{O} \mapsto \widehat{\mathcal{O}}$ induces a left adjoint functor $\widehat{(-)}\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ whose right adjoint is given by the canonical inclusion. In particular, for every $\mathcal{O}$, the canonical map $\mathcal{O} \to \widehat{\mathcal{O}}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$ is a local equivalence.*

Proof. For $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, the underlying $\mathcal{V}$-enriched ∞-category $u^*\widehat{\mathcal{O}}$ is equivalent to $|(u^*\mathcal{O})^{E^\bullet}|$ by Corollar 6.31 and is complete by [GH15, Theorem 5.6.2]. It is clear from the definition that the assignment $\mathcal{O} \mapsto \widehat{\mathcal{O}}$ is natural in $\mathcal{O}$ and therefore induces a functor $\widehat{(-)}\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$. By [Lur09, Proposition 5.2.7.8], it is left adjoint to the canonical inclusion $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \subseteq \mathrm{Alg}_\Phi(\mathcal{V})$ if and only if, for every $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$ and every $\mathcal{P} \in \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$, the canonical map $\mathcal{O} \simeq \mathcal{O}^{E^0} \to |\mathcal{O}^{E^\bullet}| = \widehat{\mathcal{O}}$ induces an equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})}(\widehat{\mathcal{O}}, \mathcal{P}) \to \mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(\mathcal{O}, \mathcal{P}).$$

Since $\mathcal{P}$ is complete, it suffices to show that $\mathcal{O} \to \widehat{\mathcal{O}}$ is a local equivalence. Since the class of local equivalences is strongly saturated, it is closed under colimits. Therefore, $\mathcal{O} \to \widehat{\mathcal{O}}$ is a local equivalence, provided every map $\mathcal{O}^{E^n} \to \mathcal{O}^{E^m}$ induced by $[m] \to [n] \in \Delta$ is a local equivalence. According to [GH15, Definition 5.5.6], the map $E^m \to E^n$ is a pseudo equivalence in $\mathrm{P}_{\mathrm{Seg}}(\Delta) \simeq \mathrm{Alg}_*(\mathcal{S})$. Then Lemma 6.24 implies that $\mathcal{P}^{E^n} \to \mathcal{P}^{E^m}$ is a pseudo equivalence which has to be a local equivalence by Theorem 6.27. $\square$

Lemma 6.33. *Let $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$ and let $\tilde{\mathcal{O}} \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ denote the image of $\mathcal{O}$ under the equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$. Then the objects $\iota_0\mathcal{O}$ and $\mathcal{O}(\bar{\mathfrak{e}})$ are equivalent in $\mathcal{S}$. In particular, we obtain an equivalence $\iota_0 u^*|\mathcal{O}^{E^\bullet}| \simeq |\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{e}})$.*

Proof. For every object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, the equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ induces the following equivalences:

$$\iota_0\mathcal{O} = \mathrm{Map}_{\mathrm{Alg}_*(\mathcal{V})}(E^0, u^*\mathcal{O}) \simeq \mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(u_!E^0, \mathcal{O}) \simeq \mathrm{Map}_{\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)}(\bar{\mathfrak{e}}, \tilde{\mathcal{O}}) \simeq \tilde{\mathcal{O}}(\bar{\mathfrak{e}}).$$

This shows that ι_0 preserves colimits. Since u^* preserves geometric realizations by Lemma 6.30, we obtain equivalences $\iota_0 u^*|\mathcal{O}^{E^\bullet}| \simeq |\iota_0 u^*\mathcal{O}^{E^\bullet}| \simeq |\tilde{\mathcal{O}}^{E^\bullet}(\bar{\mathfrak{e}})|$. $\square$

Theorem 6.34. *A map in $\mathrm{Alg}_\Phi(\mathcal{V})$ is a local equivalence if and only if it is fully faithful and essentially surjective. Since the ∞-category $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$of complete Φ-∞-operads is given by a localization of $\mathrm{Alg}_\Phi(\mathcal{V})$ with respect to local equivalences, we obtain an equivalence of ∞-categories*

$$\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \simeq \mathrm{Alg}_\Phi(\mathcal{V})[\mathrm{FFES}^{-1}].$$

Proof. For every $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V})$, let $\gamma(\mathcal{O})$ denote the map $\mathcal{O} \to \widehat{\mathcal{O}}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$. We will first show that this map is fully faithful and essentially surjective. As explained below it is not hard to see that $\gamma(\mathcal{O})$ is essentially surjective. The essential part of the proof of this theorem consists of the verification of the fully faithfulness of the map $\gamma(\mathcal{O})$. Once this is shown, it is not hard to prove that the local equivalences in $\mathrm{Alg}_\Phi(\mathcal{V})$ are exactly the fully faithful and essentially surjective morphisms.

Corollary 6.31 and [GH15, Theorem 5.6.2] imply that the underlying map $u^*\gamma(\mathcal{O})\colon u^*\mathcal{O} \to u^*\widehat{\mathcal{O}}$ is essentially surjective, which proves that $\gamma(\mathcal{O})$ is essentially surjective. Let $\tilde{\gamma}(\mathcal{O})\colon \tilde{\mathcal{O}} \to \tilde{\widehat{\mathcal{O}}}$ denote the corresponding equivalence of Segal Φ-presheaves. According to Remark 6.8, $\tilde{\gamma}(\mathcal{O})$ is fully faithful if and only if the induced commutative square

$$\begin{tikzcd} \tilde{\mathcal{O}}(w) \arrow[r, "\tilde{\gamma}(\mathcal{O})(w)"] \arrow[d] & \tilde{\widehat{\mathcal{O}}}(w) \arrow[d] \\ X^{k+1} \arrow[r, "\tilde{\gamma}(\mathcal{O})_0^{k+1}"'] & Y^{k+1} \end{tikzcd} \tag{6.8}$$

is Cartesian for every object $w \in (\mathcal{V}^\vee)$ lying over some corolla with k leaves. The equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \simeq \mathrm{P}(\mathcal{V}^{\kappa,\vee})[\overline{S}_\Phi^{-1}]$ of ∞-categories provided by Proposition 4.20 implies that the Segal Φ-presheaf $\tilde{\widehat{\mathcal{O}}}$ is given by the localization of the colimit of the diagram $\tilde{\mathcal{O}}^{E^\bullet}\colon \Delta^{\mathrm{op}} \to \mathrm{P}(\mathcal{V}^{\kappa,\vee})$. Lemma 6.24 and Proposition 6.22 guarantee that every morphism $[m] \to [n] \in \Delta^{\mathrm{op}}$ induces a fully faithful map $\tilde{\mathcal{O}}^{E^m} \to \tilde{\mathcal{O}}^{E^n}$. Thus, by Remark 6.8 again, the induced commutative square

$$\begin{tikzcd} \tilde{\mathcal{O}}^{E^m}(w) \arrow[r] \arrow[d] & \tilde{\mathcal{O}}^{E^n}(w) \arrow[d] \\ (\tilde{\mathcal{O}}^{E^m}(\bar{\mathfrak{e}}))^{k+1} \arrow[r] & (\tilde{\mathcal{O}}^{E^n}(\bar{\mathfrak{e}}))^{k+1} \end{tikzcd} \tag{6.9}$$

is Cartesian. If we define $\varphi\colon \Delta^{\mathrm{op}} \times \Delta^1 \to \mathcal{S}$ to be the natural transformation such that $\varphi_{|\{[n]\}\times\Delta^1}$ is given by $\tilde{\mathcal{O}}^{E^n}(w) \to (\tilde{\mathcal{O}}^{E^n}(\bar{\mathfrak{e}}))^{k+1}$ for every $[n] \in \Delta$, then the pullback square 6.9 implies that φ is Cartesian (see Definition 3.25).

Let $\varphi^\triangleright\colon (\Delta^{\mathrm{op}})^\triangleright \times \Delta^1 \to \mathcal{S}$ denote the extension of φ determined by requiring $\varphi^\triangleright|_{\{\infty\}\times\Delta^1}$ to be the morphism $|\tilde{\mathcal{O}}^{E^\bullet}|(w) \to |\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{e}})^{k+1}$ in $\mathcal{S}$. Since the ∞-category Δ^{op} is sifted by [Lur09, Lemma 5.5.8.4], geometric realization commutes with finite products by [Lur09, Lemma 5.5.8.11]. Hence, Lemma 6.33 together with the definition of $\iota_0\widehat{\mathcal{O}}$ implies that $|\tilde{\mathcal{O}}^{E^\bullet}|(\bar{\mathfrak{e}}) \simeq \iota_0\widehat{\mathcal{O}} \simeq Y$ and

$$|\tilde{\mathcal{O}}^{E^n}|(\bar{\mathfrak{e}})^{k+1} \simeq Y^{k+1}.$$

Since $\mathcal{S}$ is an ∞-topos, Theorem 3.26 implies that the natural transformation $\varphi^\triangleright$ induces a pullback square

$$\begin{tikzcd} \tilde{\mathcal{O}}^{E^m}(w) \arrow[r] \arrow[d] & |\tilde{\mathcal{O}}^{E^\bullet}|(w) \arrow[d] \\ (\tilde{\mathcal{O}}^{E^m}(\bar{\mathfrak{e}}))^{k+1} \arrow[r] & Y^{k+1}, \end{tikzcd} \tag{6.10}$$

for every $[m] \in \Delta^{\mathrm{op}}$, because $|\tilde{\mathcal{O}}^{E^\bullet}|(w) = \mathrm{colim}_{[m]\in\Delta^{\mathrm{op}}}\,\tilde{\mathcal{O}}^{E^m}(w)$ and $Y^{k+1} \simeq |\tilde{\mathcal{O}}^{E^n}|(\bar{\mathfrak{e}})^{k+1} = \mathrm{colim}_{[m]\in\Delta^{\mathrm{op}}}(\tilde{\mathcal{O}}^{E^m}(\bar{\mathfrak{e}}))^{k+1}$. By choosing $m = 0$, we have $\tilde{\mathcal{O}}(w) \simeq \tilde{\mathcal{O}}^{E^0}(w)$ and $(\tilde{\mathcal{O}}^{E^0}(\bar{\mathfrak{e}}))^{k+1} \simeq$

$(\iota_0\mathcal{O})^{k+1} \simeq X^{k+1}$ and the equivalence $\mathrm{Alg}_\Phi(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ implies $|\tilde{\mathcal{O}}^{E^\bullet}|(w) \simeq \tilde{\tilde{\mathcal{O}}}(w)$. Thus, diagram 6.8 coincides with the pullback square 6.10 and $\gamma(\mathcal{O})$ is fully faithful.

Every map $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_\Phi(\mathcal{V})$ induces a commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r, "f"] \arrow[d] & \mathcal{P} \arrow[d] \\ \widehat{\mathcal{O}} \arrow[r, "\widehat{f}"'] & \widehat{\mathrm{P}}, \end{tikzcd}$$

where the vertical maps are local equivalences by Proposition 6.32 as well as fully faithful and essentially surjective by the proof above. Since fully faithful and essentially surjective maps satisfy the 2-of-3 property by Propositon 6.14, the map f is fully faithful and essentially surjective if and only if $\widehat{f}$ is so. Proposition 6.32 implies that the map $\widehat{f}$ is a map between complete objects and, by Proposition 6.18, it is fully faithful if and only if it is an equivalence, which is equivalent to requiring $\widehat{f}$ to be a local equivalence. Hence, by applying 2-of-3 property again, we obtain that f is fully faithful and essentially surjective if and only if f is a local equivalence. This observation implies that the localization of $\mathrm{Alg}_\Phi(\mathcal{V})$ with respect to the class of fully faithful and essentially surjective morphisms is equivalent to its full subcategory $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ spanned by complete objects. □

We obtain the following corollary immediately from Proposition 6.32 and Theorem 6.34.

Corollary 6.35. *The inclusion of complete objects*

$$\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_\Phi(\mathcal{V})$$

has a left adjoint functor which exhibits $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ *as the localization of* $\mathrm{Alg}_\Phi(\mathcal{V})$ *with respect to the class of fully faithful and essentially surjective morphisms.*

Following the idea of the proof of [GH15, Theorem 5.6.6], by changing the universe, we can show that the statement of Corollary 6.35 is still valid even if the ∞-category $\mathcal{V}$ is not presentable. Before we can prove the theorem, we need to verify the following lemma:

Lemma 6.36. *If* $f\colon \mathcal{V} \to \mathcal{W}$ *is a fully faithful functor between arbitrary symmetric monoidal* ∞*-categories, then the induced functor* $\mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_\Phi(\mathcal{W})$ *of* ∞*-categories is fully faithful.*

Proof. The fully faithful functor f obviously induces a fully faithful functor $\mathrm{Cr}^*_\Phi(f)\colon \mathrm{Cr}^*_\Phi(\mathcal{V}) \to \mathrm{Cr}^*_\Phi(\mathcal{W})$ between objects in $\mathrm{coCart}^{\Phi,\mathrm{gen}}_{\mathrm{Seg}}$ which in turn induces a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_\Phi(\mathcal{V}) \arrow[rr] \arrow[dr] & & \mathrm{Alg}_\Phi(\mathcal{W}) \arrow[dl] \\ & \mathcal{S}, & \end{tikzcd}$$

where the diagonal maps are the Cartesian fibrations as defined in Definition 3.9. Let us write $f_*\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_\Phi(\mathcal{W})$ for the horizontal map. We want to see that f_* is fully faithful i.e., for every two objects $F, G \in \mathrm{Alg}_\Phi(\mathcal{V})$, we have to show that $\mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(F, G) \to$

$\mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{W})}(f_*F, f_*G)$ is an equivalence. Suppose F, G are objects in $\mathrm{Alg}_\Phi(\mathcal{V})$ lying over $X, Y \in \mathcal{S}$, then we have a commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{V})}(F,G) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Alg}_\Phi(\mathcal{W})}(f_*F, f_*G) \arrow[d] \\
\mathrm{Map}_{\mathcal{S}}(X,Y) \arrow[r, "="'] & \mathrm{Map}_{\mathcal{S}}(X,Y)
\end{tikzcd}$$

and we only need to show that it is Cartesian. According to [Lur09, Proposition 2.4.4.2], the fibres of the vertical maps over every $\alpha \in \mathrm{Map}_{\mathcal{S}}(X,Y)$ are given by $\mathrm{Map}_{\mathrm{Alg}_{\Phi,X}(\mathcal{V})}(F, \alpha^*G)$ and $\mathrm{Map}_{\mathrm{Alg}_{\Phi,X}(\mathcal{W})}(f_*F, \alpha^*f_*G)$, respectively, where α^*G and α^*f_*G are induced by the Cartesian lifts of α. Since $f_*\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_\Phi(\mathcal{W})$ preserves Cartesian morphisms, we have $\alpha^*f_*G \simeq f_*\alpha^*G$ and $\mathrm{Map}_{\mathrm{Alg}_{\Phi,X}(\mathcal{W})}(f_*F, \alpha^*f_*G) \simeq \mathrm{Map}_{\mathrm{Alg}_{\Phi,X}(\mathcal{W})}(f_*F, f_*\alpha^*G)$. Because the ∞-categories $\mathrm{Alg}_{\Phi,X}(\mathcal{V})$ and $\mathrm{Alg}_{\Phi,X}(\mathcal{W})$ are respective full subcategories of

$$\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathrm{Cr}^*_\Phi(\mathcal{V})) \times_{\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \Delta^{\mathrm{op}}_\Phi)} \{\pi_X\} \quad \text{and} \quad \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathrm{Cr}^*_\Phi(\mathcal{W})) \times_{\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \Delta^{\mathrm{op}}_\Phi)} \{\pi_X\},$$

where $\pi_X\colon \Delta^{\mathrm{op}}_{\Phi,X} \to \Delta^{\mathrm{op}}_\Phi$ is as defined in Definition 2.25, we only need to verify that the induced functor $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathrm{Cr}^*_\Phi(\mathcal{V})) \to \mathrm{Fun}(\Delta^{\mathrm{op}}_{\Phi,X}, \mathrm{Cr}^*_\Phi(\mathcal{W}))$ is fully faithful. Using [GHN15, Lemma 5.2] this is true, because the functor $\mathrm{Cr}^*_\Phi(\mathcal{V}) \to \mathrm{Cr}^*_\Phi(\mathcal{W})$ is fully faithful. ☐

Theorem 6.37. *Let $\mathcal{V}$ be a (possibly large) symmetric monoidal ∞-category which is not necessarily presentable. The inclusion*

$$\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_\Phi(\mathcal{V})$$

has a left adjoint functor which exhibits $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ as the localization of $\mathrm{Alg}_\Phi(\mathcal{V})$ with respect to the class of fully faithful and essentially surjective morphisms.

Proof. If we write $\widehat{\mathrm{P}}(\mathcal{V})$ for the ∞-category of presheaves of large spaces on $\mathcal{V}$, then by [Lur, Corollary 4.8.1.12], there exists a symmetric monoidal structure on $\widehat{\mathrm{P}}(\mathcal{V})$ such that the Yoneda embedding $\mathcal{V} \to \widehat{\mathrm{P}}(\mathcal{V})$ is symmetric monoidal. If $\widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V}))$ denotes the very large ∞-category of $\widehat{\mathrm{P}}(\mathcal{V})$-enriched Φ-∞-operads, then the corollary above provides an adjunction

$$\widehat{L}\colon \widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V})) \rightleftarrows \widehat{\mathrm{Alg}}_{\Phi,\mathrm{cp}}(\widehat{\mathrm{P}}(\mathcal{V})),$$

where the left adjoint functor $\widehat{L}$ is the localization functor and the right adjoint functor is the canonical inclusion.

By the previous lemma, the fully faithful Yoneda embedding $\mathcal{V} \to \widehat{\mathrm{P}}(\mathcal{V})$ induces a fully faithful functor $\mathrm{Alg}_\Phi(\mathcal{V}) \hookrightarrow \widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V}))$. Since being complete is a property of the underlying objects in $\mathrm{Alg}_*(\mathcal{V})$, this functor restricts to a fully faithful functor $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \hookrightarrow \widehat{\mathrm{Alg}}_{\Phi,\mathrm{cp}}(\widehat{\mathrm{P}}(\mathcal{V}))$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Alg}_{\Phi}(\mathcal{V}) \arrow[d, hook] \\
\widehat{\mathrm{Alg}}_{\Phi,\mathrm{cp}}(\widehat{\mathrm{P}}(\mathcal{V})) \arrow[r, hook] & \widehat{\mathrm{Alg}}_{\Phi}(\widehat{\mathrm{P}}(\mathcal{V})).
\end{tikzcd}$$

The functor $u\colon \Delta \simeq \Delta_* \to \Delta_\Phi$ induced by the obvious operator morphism $* \to \Phi$ provides a commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\Phi}(\mathcal{V}) \arrow[r, "u^*"] \arrow[d, hook] & \mathrm{Alg}_{*}(\mathcal{V}) \arrow[d, hook] \\
\widehat{\mathrm{Alg}}_{\Phi}(\widehat{\mathrm{P}}(\mathcal{V})) \arrow[r, "u^*"] & \widehat{\mathrm{Alg}}_{*}(\widehat{\mathrm{P}}(\mathcal{V})).
\end{tikzcd} \tag{6.11}$$

In order to complete the proof it suffices to show that the localization functor $\widehat{L}$ restricts to a localization functor $L\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$.

Let $\widehat{E}^\bullet$ denote the cosimplicial object given by $E^{\widehat{\mathrm{P}}(\mathcal{V}),n} = E^{\widehat{\mathrm{P}}(\mathcal{V})}_{\{1,\dots,n\}}$. According to Lemma 6.30 and Lemma 6.33, the functors u^* and ι_0 preserve geometric realizations. Hence, for every object $\mathcal{O} \in \mathrm{Alg}_\Phi(\mathcal{V}) \subseteq \widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V}))$, there is an equivalvence

$$\iota_0\widehat{L}(\mathcal{O}) = \iota_0 u^*|\mathcal{O}^{\widehat{E}^\bullet}| \simeq |\operatorname{Map}_{\widehat{\mathrm{Alg}}_*(\widehat{\mathrm{P}}(\mathcal{V}))}(\widehat{E}^\bullet, u^*\mathcal{O})|.$$

Since the symmetric monoidal Yoneda embedding $\mathcal{V} \to \widehat{\mathrm{P}}(\mathcal{V})$ preserves the unit of $\mathcal{V}$, the object $\widehat{E}^n \in \widehat{\mathrm{Alg}}_*(\widehat{\mathrm{P}}(\mathcal{V}))$ can be identified with $E^n \in \mathrm{Alg}_*(\mathcal{V}) \subseteq \widehat{\mathrm{Alg}}_*(\widehat{\mathrm{P}}(\mathcal{V}))$. This implies that there is an equivalence

$$|\operatorname{Map}_{\widehat{\mathrm{Alg}}_*(\widehat{\mathrm{P}}(\mathcal{V}))}(\widehat{E}^\bullet, u^*\mathcal{O})| \simeq |\operatorname{Map}_{\mathrm{Alg}_*(\mathcal{V})}(E^\bullet, u^*\mathcal{O}))| = \iota\mathcal{O},$$

which shows that $\iota_0\widehat{L}(\mathcal{O}) \simeq \iota\mathcal{O}$ is essentially small.

Since $\mathrm{Alg}_\Phi(\mathcal{V}) \hookrightarrow \widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V}))$ is fully faithful and the local equivalence $\mathcal{O} \to \widehat{L}(\mathcal{O})$ is fully faithful by Theorem 6.34, we have that $\widehat{L}(\mathcal{O})(x_1, \dots, x_n; x)$ lies in the essential image of $\mathcal{V}$ in $\widehat{\mathrm{P}}(\mathcal{V})$, for every $\mathfrak{c}(x_1, \dots, x_n; x) \in \Delta_\Phi^{\mathrm{op}}$. Therefore, the object $\widehat{L}(\mathcal{O})$ lies in the essential image of $\mathrm{Alg}_\Phi(\mathcal{V})$ in $\widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V}))$. In other words, the localization functor $\widehat{L}\colon \widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V})) \to \widehat{\mathrm{Alg}}_{\Phi,\mathrm{cp}}(\widehat{\mathrm{P}}(\mathcal{V}))$ restricts to a functor $L\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \widehat{\mathrm{Alg}}_{\Phi,\mathrm{cp}}(\widehat{\mathrm{P}}(\mathcal{V}))$ whose image also lies in $\mathrm{Alg}_\Phi(\mathcal{V}) \subseteq \widehat{\mathrm{Alg}}_\Phi(\widehat{\mathrm{P}}(\mathcal{V}))$. Hence, we obtain a functor $L\colon \mathrm{Alg}_\Phi(\mathcal{V}) \to \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ and the fully faithful embedding $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \hookrightarrow \widehat{\mathrm{Alg}}_{\Phi,\mathrm{cp}}(\widehat{\mathrm{P}}(\mathcal{V}))$ implies that L is left adjoint to the inclusion $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_\Phi(\mathcal{V})$. □

Using the results provided in this chapter it is now easy to show the following proposition:

Proposition 6.38. *The tensor product* $\otimes\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \times \mathrm{P}_{\mathrm{Seg}}(\Delta) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee)$ *introduced in Theorem 5.5 induces a tensor product on the complete objects*

$$\otimes\colon \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}) \times \mathrm{Cat}_\infty \to \mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V}),$$

which preserves colimits in each variable.

Proof. Theorem 4.22 and Theorem 3.24 provide the equivalences $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \simeq \mathrm{Alg}_{\Phi}(\mathcal{V})$ and $\mathrm{P}_{\mathrm{Seg}}(\Delta) = \mathrm{Seg}_{*} \simeq \mathrm{Alg}_{*}(\mathcal{S})$. Since both ∞-categories $\mathrm{Alg}_{*,\mathrm{cp}}(\mathcal{S}) \simeq \mathrm{Cat}_{\infty}$ and $\mathrm{Alg}_{\Phi,\mathrm{cp}}(\mathcal{V})$ are induced by the localization with respect to local equivalences, we only need to show that the tensor product preserves local equivalences in both variables. Local equivalences in $\mathrm{Alg}_{\Phi}(\mathcal{V})$ and $\mathrm{Alg}_{*}(\mathcal{S})$ are generated by the morphisms $E^{\mathcal{V}}_{\{0,1\}} \to E^{\mathcal{V}}_{\{0\}}$ in $\mathrm{Alg}_{*}(\mathcal{V})$ and $E^{\mathcal{S}}_{\{0,1\}} \to E^{\mathcal{S}}_{\{0\}}$ in $\mathrm{Alg}_{*}(\mathcal{S})$, respectively. Hence, we have to consider the following two cases.

Given an object $\mathcal{P} \in \mathrm{Alg}_{\Phi}(\mathcal{V})$, we first show that the map $\mathcal{P} \otimes E^{\mathcal{S}}_{\{0,1\}} \to \mathcal{P} \otimes E^{\mathcal{S}}_{\{0\}}$ is a local equivalence in $\mathrm{Alg}_{\Phi}(\mathcal{V})$. By adjunction, this is equivalent to requiring $\mathcal{O} \simeq \mathcal{O}^{E^{\mathcal{S}}_{\{0\}}} \to \mathcal{O}^{E^{\mathcal{S}}_{\{0,1\}}}$ to be an equivalence for every complete object $\mathcal{O}$. This follows from Lemma 6.25.

Given an object $\mathcal{C} \in \mathrm{Alg}_{*}(\mathcal{S})$, we want to see that $(u_! E^{\mathcal{V}}_{\{0,1\}}) \otimes \mathcal{C} \to (u_! E^{\mathcal{V}}_{\{0\}}) \otimes \mathcal{C}$ is a local equivalence in $\mathrm{Alg}_{\Phi}(\mathcal{V})$. By the discussion after Definition 3.14, we have $E^{\mathcal{V}}_{\{0,1\}} \simeq F_* E^{\mathcal{S}}_{\{0,1\}}$ and $E^{\mathcal{V}}_{\{0\}} \simeq F_* E^{\mathcal{S}}_{\{0\}}$. Using the fact that $(-) \otimes \mathcal{C}$ is a left adjoint functor, the map in question is of the form $u_! F_*(E^{\mathcal{S}}_{\{0,1\}} \otimes \mathcal{C}) \to u_! F_*(E^{\mathcal{S}}_{\{0\}} \otimes \mathcal{C})$. Lemma 6.26 then implies that $E^{\mathcal{S}}_{\{0,1\}} \otimes \mathcal{C} \to E^{\mathcal{S}}_{\{0\}} \otimes \mathcal{C}$ is a local equivalence in $\mathrm{Alg}_{*}(\mathcal{S})$, hence the left adjoint $u_! F_*$ carries it to a local equivalence in $\mathrm{Alg}_{\Phi}(\mathcal{V})$. $\square$

Chapter 7

Comparison between $\mathcal{F}$- and Ω-Presheaf Models

In Chapter 4 we introduced the Φ-presheaf model for enriched ∞-operads. In this chapter we use the dendroidal category Ω to define an Ω-presheaf model, which is another approach to enriched ∞-operads. The objects are called Ω-presheaves and they can be regarded as the enriched version of dendroidal Segal spaces. As in the case of Φ-presheaves, the enrichment is achieved by labelling the vertices of the trees in Ω by objects in a symmetric monoidal ∞-category. The main result of this chapter is the following theorem which implies that the Φ-presheaf model is equivalent to the Ω-presheaf model.

Theorem 7.1. *Let $\mathcal{V}$ be as in Notation 7.2 and let $\mathcal{V}^\vee \to \Delta_{\mathcal{F}}$ be as in Definition 4.9. There is an equivalence of ∞-categories*

$$\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^\vee).$$

The proof of the theorem will be provided at the end of this chapter and it indicates that Ω should be thought of as the dendroidal counterpart of $\Delta_{\mathcal{F}}$. If Ω is replaced by its planar variant Ω_π, then the same constructions for the Ω-presheaf model provides an Ω_π-presheaf model. Recall that Ord denotes the operator category of ordered finite sets. Then, after replacing Ω by Ω_π, the proof of the theorem above shows that the Ord- and Ω_π-presheaf models are equivalent. Since there is no dendroidal counterpart of any other operator category, these comparison results are the best we can get.

This chapter is divided into three sections. In the first section we introduce Segal Ω-presheaves in Definition 7.6 in a similar way as Segal Φ-presheaves. Then, using similar arguments in the proof of the presentability of the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^\vee)$ of Segal Φ-presheaves, we obtain in Proposition 7.10 that the ∞-category of Segal Ω-presheaves is also given by an accessible localization. In particular, it is then presentable. Using the category $\Delta^1_{\mathcal{F}}$ introduced at the end of the first chapter, we define the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ in Definition 7.5 and show that it is also given by an accessible localization in Proposition 7.12. At the end of this section several subcategories of $\mathcal{V}_\Omega^\vee$ and $\mathcal{V}^{1,\vee}$ as well as presheaf categories on them are introduced in Definition 7.13 and Definition 7.14, respectively.

In this section we provide some important results which are used for the proof of the Theorem 7.1 in the next section. First we show in Lemma 7.17 that the inclusion of the full subcategory

$\mathcal{V}^{1,\vee} \subseteq \mathcal{V}^{\vee}$ induces an equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \xrightarrow{\simeq} \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$. This observation indicates that the ∞-category $\mathcal{V}^{1,\vee}$, whose objects can be regarded as trees (not forests) with labelled vertices, already contains all the necessary information to encode compositions of multimorphisms. Then we prove in Lemma 7.21 that the functor $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ defined in the last section of Chapter 2 induces a functor $\tau^{\kappa,*}_{\mathcal{V}}$ which carries the set $S^{\lim}$ to $S^{\lim}_{\mathcal{F},1}$. This result will then imply that τ induces the comparison functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega}) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ as we will see in Proposition 7.34 of the next section.

The most important observation of this Section 7.2 is Lemma 7.24 which says that $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$ are the ∞-categories of algebras for monads on $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_{\Omega})$, respectively. Since the ∞-categories $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_{\Omega})$ are equivalent by Lemma 7.23, it suffices to prove that the corresponding monads are equivalent in order to verify the equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$ in Theorem 7.1. This is the content of the last section.

7.1 The Ω-Presheaf Model

Notation 7.2. *Unless mentioned otherwise, for the whole chapter, $\mathcal{V}$ will be a presentable symmetric monoidal ∞-category such that $\mathcal{V}_{\langle 1\rangle}$ is κ-presentable for a cardinal κ and $\mathcal{V}$ admits a symmetric monoidal full subcategory $\mathcal{V}^{\kappa}$ as defined in Definition 4.5. The existence of such a $\mathcal{V}^{\kappa}$ is provided by Corollary 4.8.*

Definition 7.3. Let $p\colon \mathcal{F}_* \to \mathrm{Cat}_{\infty}$ be the functor associated to the ∞-operad $\mathcal{V}$ and let $\mathcal{V}_{\otimes} \to \mathcal{F}^{\mathrm{op}}_*$ denote its associated Cartesian fibration. Let $i^1\colon \Delta^1_{\mathcal{F}} \hookrightarrow \Delta_{\mathcal{F}}$ denote the canonical inclusion as defined in Definition 2.78. We write $p^{1,\vee}\colon \mathcal{V}^{1,\vee} \to \Delta^1_{\mathcal{F}}$ for the Cartesian fibration given by the following pullback diagram

$$\begin{tikzcd} \mathcal{V}^{1,\vee} \arrow[r] \arrow[d, "p^{1,\vee}"'] & \mathcal{V}_{\otimes} \arrow[d] \\ \Delta^1_{\mathcal{F}} \arrow[r, "\mathrm{Cr}^{\mathrm{op}}_{\mathcal{F}} \circ i^1"'] & \mathcal{F}^{\mathrm{op}}_*. \end{tikzcd}$$

We write $p^{\vee}_{\Omega}\colon \mathcal{V}^{\vee}_{\Omega} \to \Omega$ for the Cartesian fibration given by the following pullback diagram

$$\begin{tikzcd} \mathcal{V}^{\vee}_{\Omega} \arrow[r] \arrow[d, "p^{\vee}_{\Omega}"'] & \mathcal{V}_{\otimes} \arrow[d] \\ \Omega \arrow[r, "\mathrm{Cr}^{\mathrm{op}}"'] & \mathcal{F}^{\mathrm{op}}_*. \end{tikzcd} \tag{7.1}$$

It follows directly from Definition 4.9 that $\mathcal{V}^{1,\vee} \simeq \mathcal{V}^{\vee} \times_{\Delta_{\mathcal{F}}} \Delta^1_{\mathcal{F}}$.

Remark 7.4. *Since the fibre $(\mathcal{V}^{\vee}_{\Omega})_{C_n}$ is equivalent to $\mathcal{V}_{\langle 1\rangle}$ for any corolla $C_n \in \Omega$, the definition above allows us to think of an object in $\mathcal{V}^{\vee}_{\Omega}$ as a tree in Ω whose vertices (which can be identified with corollas in this tree) are labelled by objects of $\mathcal{V}_{\langle 1\rangle}$. Therefore, we will write (C_n, v) for an object in $\mathcal{V}^{\vee}_{\Omega}$ which lies over a corolla C_n and is labelled by $v \in \mathcal{V}_{\langle 1\rangle}$.*

Definition 7.5. Let $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ denote the essential image of $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ under the restriction functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \subseteq \mathrm{P}(\mathcal{V}^{\vee}) \to \mathrm{P}(\mathcal{V}^{1,\vee})$ induced by the canonical functor $\mathcal{V}^{1,\vee} \simeq \mathcal{V}^{\vee} \times_{\Delta_{\mathcal{F}}} \Delta^1_{\mathcal{F}} \to \mathcal{V}^{\vee}$.

In the following definition we introduce the Ω-version of a Segal Φ-presheaf as defined in Definition 4.13.

Definition 7.6. A presheaf $F\colon \mathcal{V}_{\Omega}^{\vee,\mathrm{op}} \to \mathcal{S}$ is called a *Segal* Ω*-presheaf*, if the following conditions are satisfied:

1. If η denotes the trivial tree in Ω and $\overline{\eta}$ denotes the essentially unique object in $(\mathcal{V}_{\Omega}^{\vee})_{\eta}$, then the functor $(\mathcal{V}_{\Omega}^{\vee,\mathrm{op}})_{C_n} \simeq \mathcal{V}_{\langle 1\rangle}^{\mathrm{op}} \to \mathcal{S}_{/f(\overline{\eta})^{n+1}}$ induced by the Cartesian lifts of the $n+1$ morphisms $\eta \to C_n$ preserves all small limits in $\mathcal{V}_{\langle 1\rangle}^{\mathrm{op}}$.
2. For $T \in \Omega$, recall that $(\Omega_{\mathrm{el}}^{\mathrm{op}})_{T/}$ denotes the category $\Omega_{\mathrm{el}}^{\mathrm{op}} \times_{\Omega_{\mathrm{in}}^{\mathrm{op}}} (\Omega_{\mathrm{in}}^{\mathrm{op}})_{T/}$. The presheaf F satisfies the Segal condition, i.e., for every object $v \in \mathcal{V}_{\Omega}^{\vee,\mathrm{op}}$ lying over T, the canonical map
$$F(v) \to \lim_{\alpha\in(\Omega_{\mathrm{el}}^{\mathrm{op}})_{T/}} F(\alpha^* v)$$
is an equivalence, where $\alpha^* v \to v$ is the $p_{\Omega}^{\vee}$-Cartesian lift of the inert map α (corresponding to the coCartesian morphism in $\mathcal{V}$).

We write $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega}^{\vee})$ for the full subcategory of presheaves spanned by Segal Ω-presheaves.

Remark 7.7. *Analogous to the case of Segal* Φ*-presheaves, we observe that the following assertion holds: If* $F \in \mathrm{P}(\mathcal{V}_{\Omega}^{\vee})$ *is a presheaf which satisfies condition 2 in the definition above, then it satisfies condition 1 if and only if, for every tree* $T \in \Omega$ *containing* n *edges and* $\overline{\eta} \in (\mathcal{V}_{\Omega}^{\vee})_{\eta}$*, the functor* $(\mathcal{V}_{\Omega}^{\vee,\mathrm{op}})_{T} \to \mathcal{S}_{/f(\overline{\eta})^{n}}$*, induced by* F *and the* n*-many* $p_{\Omega}^{\vee}$*-Cartesian lifts of the morphisms* $\eta \to T$*, preserves all small limits.*

We know from Proposition 4.20 that the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ is given by localizing $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ with respect to a set of morphisms $\overline{S}_{\Phi} = S_{\Phi}^{\lim} \coprod S_{\Phi}^{\mathrm{Seg}}$. It is easily checked that the arguments used there are robust enough to show that an object $F \in \mathrm{P}(\mathcal{V}_{\Omega}^{\vee})$ satisfies condition 1 for being a Segal Ω-presheaf in the definition above if and only if it is local with respect to every element in the strongly saturated set $S^{\lim}$ as defined below:

Definition 7.8. Define Q to be the set of all functors $(\overline{g} * q_n)^{\mathrm{op}}\colon (L^{\triangleleft} * (\mathbf{n+1}))^{\mathrm{op}} \to \mathcal{V}_{\Omega}^{\kappa,\vee}$ such that:

1. There is a corolla $C_n \in \Omega$ and the functor $\overline{g}\colon L^{\triangleleft} \to (\mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{op}})_{C_n}$ is a small limit diagram.
2. The functor $q_n\colon \mathbf{n+1} \to \mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{op}}$ is the constant functor at $\overline{\eta} \in (\mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{op}})_{\eta} \simeq \{*\}$.

By leaving the Yoneda embedding implicit, we define $S^{\lim}$ to be the strongly saturated set (see Definition 4.19) generated by the set

$$\{\mathrm{colim}_{k\in K}\, q(k) \to q(\infty), q \in Q\}$$

of morphisms of presheaves.

Moreover, the proof of Proposition 4.20 implies that F satisfies condition 2 for being a Segal Ω-presheaf if and only if it is local with respect to every element in the strongly saturated set S^{Seg} defined as follows.

Definition 7.9. For $T^{\mathcal{V}} \in \mathcal{V}_{\Omega}^{\kappa,\vee}$ lying over T, let $p(T^{\mathcal{V}})\colon {\Omega^{\mathrm{el,op}}}_{T/} \to \mathcal{V}_{\Omega}^{\kappa,\vee}$ denote the functor which carries $g \in {\Omega^{\mathrm{el,op}}}_{T/}$ to the object $g^*(T^{\mathcal{V}}) \in \mathcal{V}_{\Omega}^{\kappa,\vee}$ given by the $p_{\Omega}^{\kappa,\vee}$-Cartesian lift of g. Define S^{Seg} to be the strongly saturated set generated by the set

$$\{\mathrm{colim}_{S^{\mathcal{V}} \in p(T^{\mathcal{V}})} S^{\mathcal{V}} \to T^{\mathcal{V}}, T^{\mathcal{V}} \in \mathcal{V}_{\Omega}^{\kappa,\vee}\}.$$

of morphisms of presheaves.

Therefore, we obtain that a presheaf $F \in \mathrm{P}(\mathcal{V}_{\Omega}^{\vee})$ is a Segal Ω-presheaf if and only if it is local with respect to elements in $S^{\lim} \coprod S^{\mathrm{Seg}}$. More precisely, we have the following proposition.

Proposition 7.10. *Let $\overline{S}$ denote the strongly saturated set generated by $S^{\lim} \coprod S^{\mathrm{Seg}}$. A presheaf $F \in \mathrm{P}(\mathcal{V}_{\Omega}^{\vee})$ is a Segal Ω-presheaf if and only if it is local with respect to all elements in $\overline{S}$. Hence, there is an equivalence of ∞-categories*

$$\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega}^{\vee}) \simeq \mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})[\overline{S}^{-1}].$$

Since $\mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})[\overline{S}^{-1}]$ is an accessible localization of the presentable ∞-category $\mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})$, the ∞-categories $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega}^{\vee})$ and $\mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})[\overline{S}^{-1}]$ are presentable.

Notation 7.11. *If $T^{\mathcal{V}} \in \mathcal{V}_{\Omega}^{\vee}$, then we will often write $T_{\mathrm{Seg}}^{\mathcal{V}}$ for the object $\mathrm{colim}_{S^{\mathcal{V}} \in p(T^{\mathcal{V}})} S^{\mathcal{V}} \in \mathrm{P}(\mathcal{V}_{\Omega}^{\vee})$ where $\mathrm{colim}_{S^{\mathcal{V}} \in p(T^{\mathcal{V}})} S^{\mathcal{V}} \to T^{\mathcal{V}}$ is an element of S^{Seg}.*

Similarly, if $X \in \Delta_{\mathcal{F}}^1$, then we will often write X_{Seg} for the object $\mathrm{colim}_{E \to X \in \Delta^1_{\mathcal{F},\mathrm{el}}} E$ (where E is regarded as a presheaf via Yoneda embedding).

If we replace $\mathcal{V}_{\Omega}^{\kappa,\vee}$ by $\mathcal{V}^{1,\kappa,\vee} = \mathcal{V}^{\kappa,\vee} \times_{\Delta_{\mathcal{F}}} \Delta_{\mathcal{F}}^1$ in the arguments above, then we obtain the following result.

Proposition 7.12. *There exist strongly saturated sets $S_{\mathcal{F},1}^{\lim}, S_{\mathcal{F},1}^{\mathrm{Seg}}$ and $\overline{S}_{\mathcal{F},1}$ such that $\overline{S}_{\mathcal{F},1}$ is generated by the set $S_{\mathcal{F},1}^{\lim} \coprod S_{\mathcal{F},1}^{\mathrm{Seg}}$ and there is an equivalence of ∞-categories*

$$\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) \simeq \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})[\overline{S}_{\mathcal{F},1}^{-1}].$$

In particular, $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ is a presentable.

Definition 7.13. Let Ω_{el} and Ω_{in} be as defined in Definition 2.56. Let $\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}$ and $\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}$ be the ∞-categories given by the diagram

$$\begin{tikzcd}
\mathcal{V}_{\Omega}^{\vee,\mathrm{el}} \arrow[r] \arrow[d] & \mathcal{V}_{\Omega}^{\vee,\mathrm{in}} \arrow[r] \arrow[d] & \mathcal{V}_{\Omega}^{\vee} \arrow[d, "p_{\Omega}^{\vee}"] \\
\Omega_{\mathrm{el}} \arrow[r] & \Omega_{\mathrm{in}} \arrow[r] & \Omega
\end{tikzcd}$$

in which both squares are pullback squares induced by the canonical inclusions.

We write $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^{\vee,\mathrm{el}})$ for the essential image of $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^\vee)$ under the restriction functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^\vee) \subseteq \mathrm{P}(\mathcal{V}_\Omega^\vee) \to \mathrm{P}(\mathcal{V}_\Omega^{\vee,\mathrm{el}})$ induced by the canonical functor $\mathcal{V}_\Omega^{\vee,\mathrm{el}} \to \mathcal{V}_\Omega^\vee$.

The embedding of the full subcategory $\Omega_{\mathrm{el}} \to \Omega_{\mathrm{in}}$ induces a fully faithful functor $\mathcal{V}_\Omega^{\vee,\mathrm{el},\mathrm{op}} \to \mathcal{V}_\Omega^{\vee,\mathrm{in},\mathrm{op}}$. We write $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ for the full subcategory of $\mathrm{P}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ spanned by presheaves which are right Kan extensions of objects in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^{\vee,\mathrm{el}})$ along $\mathcal{V}_\Omega^{\vee,\mathrm{el},\mathrm{op}} \to \mathcal{V}_\Omega^{\vee,\mathrm{in},\mathrm{op}}$.

The $\Delta^1_{\mathcal{F}}$-variant of the above definition is the following:

Definition 7.14. Let $\mathcal{V}^{1,\vee,\mathrm{el}}$, $\mathcal{V}^{1,\vee,\mathrm{in}}$ be the ∞-categories given by the diagram

$$\begin{tikzcd}
\mathcal{V}^{1,\vee,\mathrm{el}} \arrow[r] \arrow[d] & \mathcal{V}^{1,\vee,\mathrm{in}} \arrow[r] \arrow[d] & \mathcal{V}^{1,\vee} \arrow[d, "p^{1,\vee}"] \\
\Delta^1_{\mathcal{F},\mathrm{el}} \arrow[r] & \Delta^1_{\mathcal{F},\mathrm{in}} \arrow[r] & \Delta^1_{\mathcal{F}}
\end{tikzcd}$$

in which both squares are pullback squares induced by the canonical inclusions. Note that all the vertical maps are Cartesian fibrations, because $p^{1,\vee}$ is one.

We write $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{el}})$ for the essential image of $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ under the restriction functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) \subseteq \mathrm{P}(\mathcal{V}^{1,\vee}) \to \mathrm{P}(\mathcal{V}^{1,\vee,\mathrm{el}})$ induced by the canonical functor $\mathcal{V}^{1,\vee,\mathrm{el},\mathrm{op}} \to \mathcal{V}^{1,\vee,\mathrm{op}}$.

The functor $\mathcal{V}^{1,\vee,\mathrm{el}} \to \mathcal{V}^{1,\vee,\mathrm{in}}$ is an inclusion of a full subcategory, because $\Delta^1_{\mathcal{F},\mathrm{el}} \to \Delta^1_{\mathcal{F},\mathrm{in}}$ is one. We write $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ for the full subcategory of $\mathrm{P}(\mathcal{V}^{1,\vee,\mathrm{in}})$ spanned by presheaves which are right Kan extensions of objects in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{el}})$ along the inclusion $\mathcal{V}^{1,\vee,\mathrm{el}} \to \mathcal{V}^{1,\vee,\mathrm{in}}$.

Lemma 7.15. *The ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ is the full subcategory of $\mathrm{P}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ spanned by presheaves $F\colon \mathcal{V}_\Omega^{\vee,\mathrm{in},\mathrm{op}} \to \mathcal{S}$ satisfying condition 1 of Definition 7.6 as well as the following condition:*

2'. Let $p_\Omega^{\vee,\mathrm{in}}\colon \mathcal{V}_\Omega^{\vee,\mathrm{in}} \to \Omega_{\mathrm{in}}$ denote the Cartesian fibration given by the pullback of $p_\Omega^\vee\colon \mathcal{V}_\Omega^\vee \to \Omega$. For $T \in \Omega$ and for every object $v \in \mathcal{V}_\Omega^{\vee,\mathrm{op}}$ lying over T, the canonical map

$$F(v) \to \lim_{\alpha \in (\Omega_{\mathrm{el}}^{\mathrm{op}})_{T/}} F(\alpha^* v)$$

is an equivalence, where $\alpha^* v \to v$ is the $p_\Omega^{\vee,\mathrm{in}}$-Cartesian lift of the inert map α.

Proof. By definition, a presheaf $F \in \mathrm{P}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ lies in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ if and only if it is given by a right Kan extension of its restriction $F_{|\mathcal{V}^{1,\vee,\mathrm{el}}}\colon \mathcal{V}^{1,\vee,\mathrm{el},\mathrm{op}} \to \mathcal{S}$. This means, $F \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_\Omega^{\vee,\mathrm{in}})$ if and only if it satisfies the first condition of Definition 7.6 and, for every object $v \in \mathcal{V}_\Omega^{\vee,\mathrm{op}}$, the canonical map

$$F(v) \to \lim_{v_i \in (\mathcal{V}_\Omega^{\vee,\mathrm{el},\mathrm{op}})_{v/}} F(v_i)$$

is an equivalence. Hence, for $v \in \mathcal{V}_\Omega^{\vee,\mathrm{op}}$ lying over $T \in \Omega$, to prove the lemma we have to provide a natural equivalence

$$\lim_{\alpha \in (\Omega_{\mathrm{el}}^{\mathrm{op}})_{T/}} F(\alpha^* v) \simeq \lim_{v_i \in (\mathcal{V}_\Omega^{\vee,\mathrm{el},\mathrm{op}})_{v/}} F(v_i).$$

Since $(\Omega_{\mathrm{el}}^{\mathrm{op}})_{T/} \simeq (\Omega_{\mathrm{el}/T})^{\mathrm{op}}$ and $(\mathcal{V}_\Omega^{\vee,\mathrm{el},\mathrm{op}})_{v/} \simeq ({\mathcal{V}_\Omega^{\vee,\mathrm{el}}}_{/v})^{\mathrm{op}}$, it suffices to show that the inclusion $\Omega_{\mathrm{el}/T} \hookrightarrow {\mathcal{V}_\Omega^{\vee,\mathrm{el}}}_{/v}$ which carries an object $f \in \Omega_{\mathrm{el}/T}$ to $f^*v \to v \in$ is cofinal. By [Lur09, Theorem 4.1.3.1], this map is cofinal if and only if the ∞-category $\Omega_{\mathrm{el}/T} \times_{{\mathcal{V}_\Omega^{\vee,\mathrm{el}}}_{/v}} ({\mathcal{V}_\Omega^{\vee,\mathrm{el}}}_{/v})_{g/}$ is weakly contractible for every $\tilde{g} \in \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}$. Since every edge in a tree lies in at most in two corollas, it is not difficult to see that the category $\Omega_{\mathrm{el}/T}$ is weakly contractible. We claim that the left fibration given by the projection $\Omega_{\mathrm{el}/T} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} (\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v})_{\tilde{g}/} \to \Omega_{\mathrm{el}/T}$ is a trivial fibration. By [Lur09, Lemma 2.1.3.4], this is true, if every fibre of this left fibration is contractible and this is what we want to verify.

Given an object $f\colon S \to T$ in $\Omega_{\mathrm{el}/T}$, the inclusion $\Omega_{\mathrm{el}/T} \hookrightarrow \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}$ allows us to identify f with an object $\tilde{f}\colon f^*v \to v$ in $\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}$. Hence, the fibre of $\Omega_{\mathrm{el}/T} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} (\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v})_{\tilde{g}/}$ over f is given by $\{\tilde{f}\} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} (\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v})_{\tilde{g}/}$ which can be identified with the mapping space $\mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}{}_{/v}}(\tilde{g}, \tilde{f})$, because $\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}$ is a full subcategory of $\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}$. Of course, the mapping space $\mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}{}_{/v}}(\tilde{g}, \tilde{f})$ is also given by $(\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v})_{/\tilde{f}} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} \{\tilde{g}\}$. Since the object f^*v is initial in the diagram $\tilde{f} * v$, [Lur09, Proposition 2.1.2.5] implies that $(\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v})_{/\tilde{f}} \simeq \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/\tilde{f}*v} \to \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/f^*v}$ is a trivial fibration. If $\tilde{g} \in \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}$ is given by the morphism $u \to v$, then the mapping space $\mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}{}_{/v}}(\tilde{g}, \tilde{f}) \simeq \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/f^*v} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} \{\tilde{g}\}$ fits into the following diagram

$$\begin{tikzcd}
\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/f^*v} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} \{\tilde{g}\} \arrow[r] \arrow[d] & \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/f^*v} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}} \{u\} \simeq \mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}}(u, f^*v) \arrow[r] \arrow[d] & \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/f^*v} \arrow[d] \\
\{\tilde{g}\} \arrow[r] & \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}} \{u\} \simeq \mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}}(u, v) \arrow[r] & \mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v},
\end{tikzcd}$$

where the right square is Cartesian. Since the big square is Cartesian by construction, [Lur09, Lemma 4.4.2.1] implies that the left square is also a pullback square. Suppose $\tilde{g}\colon u \to v$ lies over $g\colon R \to T$ in Ω_{in}. Then, by composing the left pullback square of the above diagram with the pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}}(u, f^*v) \arrow[r] \arrow[d] & \mathrm{Map}_{\Omega_{\mathrm{in}}}(R, S) \arrow[d] \\
\mathrm{Map}_{\mathcal{V}_{\Omega}^{\vee,\mathrm{in}}}(u, v) \arrow[r] & \mathrm{Map}_{\Omega_{\mathrm{in}}}(R, T)
\end{tikzcd}$$

induced by the Cartesian morphism $\tilde{f}$, we obtain that $\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/f^*v} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} \{\tilde{g}\}$ can be identified with $\mathrm{Map}_{\Omega_{\mathrm{in}}}(R, S) \times_{\mathrm{Map}_{\Omega_{\mathrm{in}}}(R,T)} \{g\}$, which is the trivial Kan complex because g factors through f uniquely. Thus, the left fibration $\Omega_{\mathrm{el}/T} \times_{\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v}} (\mathcal{V}_{\Omega}^{\vee,\mathrm{el}}{}_{/v})_{\tilde{g}/} \to \Omega_{\mathrm{el}/T}$ has contractible fibres and is therefore a trivial fibration. □

Remark 7.16. *Using similar arguments as above, we obtain corresponding results for the $\mathcal{V}^{1,\vee,\mathrm{in}}$-case: The ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ is the full subcategory of $\mathrm{P}(\mathcal{V}^{1,\vee,\mathrm{in}})$ spanned by presheaves $F\colon \mathcal{V}^{1,\vee,\mathrm{in},\mathrm{op}} \to \mathcal{S}$ satisfying condition 1 of Definition 4.13 as well as the following:*

2'. Let $p^{1,\vee,\mathrm{in}}\colon \mathcal{V}^{1,\vee,\mathrm{in}} \to \Delta^1_{\mathcal{F},\mathrm{in}}$ denote the Cartesian fibration given by the pullback of $p^\vee\colon \mathcal{V}^\vee \to \Delta_{\mathcal{F}}$. For $([m], I) \in \Delta^1_{\mathcal{F},\mathrm{in}}$ and for every object $v \in \mathcal{V}^{\vee,\mathrm{op}}$ lying over $([m], I)$, the canonical map

$$F(v) \to \lim_{\alpha \in (\Delta^{1,\mathrm{op}}_{\mathcal{F},\mathrm{el}})_{([m],I)/}} F(\alpha^* v)$$

is an equivalence, where $\alpha^* v \to v$ is the $p^{1,\vee,\mathrm{in}}$-Cartesian lift of the inert map α.

7.2 Preparation for the Comparison Result

Our first important result is Lemma 7.17 which implies that the inclusion of the full subcategory $\mathcal{V}^{1,\vee} \subseteq \mathcal{V}^{\vee}$ induces an equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \xrightarrow{\simeq} \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$. Thus, the comparison between $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$ reduces to the comparison between $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$. We want to use the functor $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ introduced in Section 2.6 to define a comparison functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega}) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$. For this reason, we need to verify that the functor $\tau^{\kappa,*}_{\mathcal{V}}\colon \mathrm{P}(\mathcal{V}^{\kappa,\vee}_{\Omega}) \to \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ carries the sets S^{lim} and S^{Seg} to $S^{\mathrm{lim}}_{\mathcal{F},1}$ and $S^{\mathrm{Seg}}_{\mathcal{F},1}$, respectively. We verify the first claim in Lemma 7.21 and the rest is then shown in Proposition 7.34 of the next section.

The next important aim of this section is to verify in Lemma 7.24 that $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$ are the ∞-categories of algebras for monads on $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_{\Omega})$, respectively. Using the equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_{\Omega})$ of Lemma 7.23, we then prove the equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$ claimed in Theorem 7.1 by showing that these monads are equivalent. This is the content of the next section.

Lemma 7.17. *The canonical functor $i^1\colon \mathcal{V}^{1,\vee} \simeq \mathcal{V}^{\vee} \times_{\Delta_{\mathcal{F}}} \Delta^1_{\mathcal{F}} \to \mathcal{V}^{\vee}$ induces a functor $i^{1,*}\colon \mathrm{P}(\mathcal{V}^{\vee}) \to \mathrm{P}(\mathcal{V}^{1,\vee})$ which restricts to an equivalence $i^{1,*}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \xrightarrow{\simeq} \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$.*

Proof. The inclusion functor $i^1\colon \mathcal{V}^{1,\vee} \hookrightarrow \mathcal{V}^{\vee}$ induces an adjunction of (large) presheaf categories

$$i^{1,*}\colon \mathrm{P}(\mathcal{V}^{\vee}) \rightleftarrows \mathrm{P}(\mathcal{V}^{1,\vee}) : i^1_*,$$

where the right adjoint i^1_* is given by right Kan extension. By definition of $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$, the functor $i^{1,*}$ restricts to $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$. In order to show that $i^{1,*}$ has an inverse given by the restriction of i^1_* to the full subcategory $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$, it suffices to show that the unit $\eta\colon \mathrm{id}_{\mathrm{P}(\mathcal{V}^{\vee})} \to i^1_* \circ i^{1,*}$ as well as the counit $\epsilon\colon i^{1,*} \circ i^1_* \to \mathrm{id}_{\mathrm{P}(\mathcal{V}^{1,\vee})}$ are equivalences on Segal presheaves.

Since $i^1\colon \mathcal{V}^{1,\vee} \to \mathcal{V}^{\vee}$ is given by a pullback of the inclusion of a full subcategory $\Delta^1_{\mathcal{F}} \hookrightarrow \Delta_{\mathcal{F}}$, the functor i^1 has to be fully faithful. This implies that i^1_* is also fully faithful and the counit $\epsilon\colon i^{1,*} \circ i^1_* \to \mathrm{id}_{\mathrm{P}(\mathcal{V}^{1,\vee})}$ is a natural equivalence.

For $F \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee})$, in order to show that the unit $\eta_F\colon F \to i^1_* \circ i^{1,*}F$ is an equivalence, it suffices to verify that the evaluations $F(v) \to i^1_* \circ i^{1,*}F(v) \simeq \lim_{(\mathcal{V}^{1,\vee}{}_{/v})^{\mathrm{op}}} F_{|\mathcal{V}^{1,\vee}}$ are equivalences for every $v \in \mathcal{V}^{\vee}$. If $p^{\vee}\colon \mathcal{V}^{\vee} \to \Delta_{\mathcal{F}}$ denotes the canonical projection and $p^{\vee}(v) = ([m], I)$, then, for $k \in I(m)$, the inclusion $\{k\} \hookrightarrow I(m)$ induces a map $\iota_k\colon ([m], I_k) \to ([m], I)$, where $([m], I_k)$ is given by

$$I(0)_k \to I(1)_k \to \ldots \to I(m-1)_k \to \{k\}$$

and $I(j)_k$ denotes the fibre of $I(j)$ over k. If $f_k\colon v_k \to v$ denotes the $p^{\vee}$-Cartesian lift of the map $\iota_k\colon ([m], I_k) \to ([m], I)$ at v, then the set $\{f_k\}_{1 \le k \le I(m)}$ can be regarded as a subcategory of $\mathcal{V}^{1,\vee}{}_{/v}$. We want to show that the inclusion $\{f_k\}_{1 \le k \le I(m)} \hookrightarrow \mathcal{V}^{1,\vee}{}_{/v}$ is cofinal. In the following we are going to argume in a very similar way as we did in the proof of Lemma 7.15.

By [Lur, Theorem 4.1.3.1], it suffices to show that the category

$$(\{f_k\}_{1 \le k \le I(m)})_{g/} = \{f_k\}_{1 \le k \le I(m)} \times_{\mathcal{V}^{1,\vee}{}_{/v}} (\mathcal{V}^{1,\vee}{}_{/v})_{g/}$$

is weakly contractible for every object $g\colon w \to v$ in ${\mathcal{V}^{1,\vee}}_{/v}$. The morphism $p^\vee(g)$ in $\Delta_{\mathcal{F}}$ has the form $([n], J) \to ([m], I)$ such that $([n], J) \in \Delta^1_{\mathcal{F}}$, i.e. $J(n) = \{*\}$. Therefore, for $l = p^\vee(g)(J(n)) \in I(m)$, the map $p^\vee(g)$ factorizes through $\iota_l\colon ([m], I_l) \to ([m], I)$ and we have $(\{f_l\}_{1\le k\le I(m)})_{g/} \simeq \{f_l\} \times_{{\mathcal{V}^{1,\vee}}_{/v}} ({\mathcal{V}^{1,\vee}}_{/v})_{g/}$. This ∞-category can be identified with the mapping space $\mathrm{Map}_{{\mathcal{V}^\vee}_{/v}}(g, f_l)$ because $\mathcal{V}^{1,\vee}$ is a full subcategory of $\mathcal{V}^\vee$. Of course, the mapping space $\mathrm{Map}_{{\mathcal{V}^\vee}_{/v}}(g, f_l)$ is also given by $({\mathcal{V}^{1,\vee}}_{/v})_{/f_l} \times_{{\mathcal{V}^{1,\vee}}_{/v}} \{g\}$. Since the object v_l is initial in the diagram $f_l * v$, [Lur09, Proposition 2.1.2.5] implies that $({\mathcal{V}^{1,\vee}}_{/v})_{/f_l} \simeq {\mathcal{V}^{1,\vee}}_{/f_l * v} \to {\mathcal{V}^{1,\vee}}_{/v_l}$ is a trivial fibration. Hence, the mapping space $\mathrm{Map}_{{\mathcal{V}^\vee}_{/v}}(g, f_l) \simeq {\mathcal{V}^{1,\vee}}_{/v_l} \times_{{\mathcal{V}^{1,\vee}}_{/v}} \{g\}$ fits into the following diagram

$$\begin{tikzcd}
{\mathcal{V}^{1,\vee}}_{/v_l} \times_{{\mathcal{V}^{1,\vee}}_{/v}} \{g\} \arrow[r] \arrow[d] & {\mathcal{V}^{1,\vee}}_{/v_l} \times_{\mathcal{V}^{1,\vee}} \{w\} \simeq \mathrm{Map}_{\mathcal{V}^\vee}(w, v_l) \arrow[r] \arrow[d, "f_{l,*}"] & {\mathcal{V}^{1,\vee}}_{/v_l} \arrow[d, "f_{l,*}"] \\
\{g\} \arrow[r] & {\mathcal{V}^{1,\vee}}_{/v} \times_{\mathcal{V}^{1,\vee}} \{w\} \simeq \mathrm{Map}_{\mathcal{V}^\vee}(w, v) \arrow[r] & {\mathcal{V}^{1,\vee}}_{/v},
\end{tikzcd} \tag{7.2}$$

where the right square is Cartesian. It follows from [Lur09, Lemma 4.4.2.1] that the left square is also a pullback square. By composing the left pullback square of the above diagram with the pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{V}^\vee}(w, v_l) \arrow[r] \arrow[d, "f_{l,*}"] & \mathrm{Map}_{\Delta_{\mathcal{F}}}(([n], J), ([m], I_l)) \arrow[d, "\iota_{l,*}"] \\
\mathrm{Map}_{\mathcal{V}^\vee}(w, v) \arrow[r] & \mathrm{Map}_{\Delta_{\mathcal{F}}}(([n], J), ([m], I)).
\end{tikzcd} \tag{7.3}$$

induced by the $p^\vee$-Cartesian morphism f_l, we obtain that ${\mathcal{V}^{1,\vee}}_{/v_l} \times_{{\mathcal{V}^{1,\vee}}_{/v}} \{g\}$ can be identified with $\mathrm{Map}_{\Delta_{\mathcal{F}}}(([n], J), ([m], I_l) \times_{\mathrm{Map}_{\Delta_{\mathcal{F}}}(([n],J),([m],I)} \{p^\vee(g)\}$, which is the trivial Kan complex because $p^\vee(g)$ factors through ι_l uniquely. Thus, $(\{f_l\}_{1\le k\le I(m)})_{g/}$ is equivalent to the trivial category and in particular weakly contractible.

The cofinality of the inclusion $\{f_k\}_{1\le k\le I(m)} \hookrightarrow {\mathcal{V}^{1,\vee}}_{/v}$ implies that

$$\lim_{({\mathcal{V}^{1,\vee}}_{/v})^{\mathrm{op}}} F_{|\mathcal{V}^{1,\vee}} \simeq \prod_{k=1}^{I(m)} F(v_k)$$

and $\eta_F(v)$ is given by the canonical map $F(v) \to \prod_{k=1}^{I(m)} F(v_k)$. Since F is a Segal $\mathcal{F}$-presheaf by assumption, it follows from the definition that the natural map $F(v) \to \lim_{\alpha\in(\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],I)/}} F(\alpha^* v)$ is an equivalence. One realizes that the category $(\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],I)/}$ is equivalent to the coproduct of the categories $(\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],I_k)/}$, for $1 \le k \le I(m)$, which implies that $F(v) \simeq \lim_{\alpha\in(\Delta^{\mathrm{el,op}}_{\mathcal{F}})_{([m],I)/}} F(\alpha^* v) \simeq \prod_{k=1}^{I(m)} F(v_k)$ and the unit $\eta\colon \mathrm{id}_{\mathrm{P}(\mathcal{V}^\vee)} \to i^1_* \circ i^{1,*}$ is indeed an equivalence on Segal $\mathcal{F}$-presheaves. $\square$

Remark 7.18. *If $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ is the functor as defined in Lemma 2.80, then it follows from the definition that $\mathrm{Cr} \circ \tau = \mathrm{Cr}_{\mathcal{F}} \circ i^1$. Thus, the Cartesian fibration $p^{1,\vee}\colon \mathcal{V}^{1,\vee} \to \Delta^1_{\mathcal{F}}$ as defined in*

Definition 7.3 is also given by the following pullback square

$$\begin{tikzcd}
\mathcal{V}^{1,\vee} \arrow[r, "\tau_{\mathcal{V}}"] \arrow[d, "p^{1,\vee}"'] & \mathcal{V}^{\vee}_{\Omega} \arrow[d, "p^{\vee}_{\Omega}"] \\
\Delta^1_{\mathcal{F}} \arrow[r, "\tau"'] & \Omega.
\end{tikzcd}$$

We will write $\tau_{\mathcal{V}}\colon \mathcal{V}^{1,\vee} \to \mathcal{V}^{\vee}_{\Omega}$ for the upper horizontal map.

The functor $\tau_{\mathcal{V}}\colon \mathcal{V}^{1,\vee} \to \mathcal{V}^{\vee}_{\Omega}$ induces an adjunction

$$\tau_{\mathcal{V}!}\colon \mathrm{P}(\mathcal{V}^{1,\vee}) \rightleftarrows \mathrm{P}(\mathcal{V}^{\vee}_{\Omega}) : \tau_{\mathcal{V}}{}^*,$$

whose left adjoint $\tau_{\mathcal{V}!}$ is induced by left Kan extension. The pullback square of the previous remark implies that, for every corolla $\mathfrak{c} \in \Delta^1_{\mathcal{F}}$, there is an equivalence $(\mathcal{V}^{1,\vee})_{\mathfrak{c}} \simeq (\mathcal{V}^{\vee}_{\Omega})_{\tau(\mathfrak{c})}$. Hence, if κ is the cardinal as in Notation 7.2, then $\mathcal{V}^{\kappa}$ is symmetric monoidal and the functor $\tau_{\mathcal{V}}$ restricts to a functor $\tau^{\kappa}_{\mathcal{V}}\colon \mathcal{V}^{1,\kappa,\vee} \to \mathcal{V}^{\kappa,\vee}_{\Omega}$.

Definition 7.19. We write

$$\tau^{\kappa}_{\mathcal{V},!}\colon \mathrm{P}(\mathcal{V}^{1,\kappa,\vee}) \rightleftarrows \mathrm{P}(\mathcal{V}^{\kappa,\vee}_{\Omega}) : \tau^{\kappa,*}_{\mathcal{V}}$$

for the adjunction of presheaf categories induced by $\tau^{\kappa}_{\mathcal{V}}\colon \mathcal{V}^{1,\kappa,\vee} \to \mathcal{V}^{\kappa,\vee}_{\Omega}$. The left adjoint $\tau^{\kappa}_{\mathcal{V},!}$ is given by left Kan extension.

Lemma 7.20. *For every symmetric monoidal ∞-category $\mathcal{V}$, the functor $\tau_{\mathcal{V}}\colon \mathcal{V}^{1,\vee} \to \mathcal{V}^{\vee}_{\Omega}$ induces an equivalence of ∞-categories $\tau_{\mathcal{V},\mathrm{el}}\colon \mathcal{V}^{1,\vee,\mathrm{el}} \to \mathcal{V}^{\vee,\mathrm{el}}_{\Omega}$.*

Proof. Definition 7.13 and Definition 7.14 imply that $\mathcal{V}^{\vee,\mathrm{el}}_{\Omega}$ and $\mathcal{V}^{1,\vee,\mathrm{el}}$ are given by the respective pullback diagrams

$$\begin{tikzcd}
\mathcal{V}^{\vee,\mathrm{el}}_{\Omega} \arrow[rr] \arrow[d] & & \mathcal{V}_{\otimes} \arrow[d] \\
\Omega_{\mathrm{el}} \arrow[r, hook] & \Omega \arrow[r, "\mathrm{Cr}"'] & \mathcal{F}^{\mathrm{op}}_*.
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{V}^{1,\vee,\mathrm{el}} \arrow[rr] \arrow[d] & & \mathcal{V}_{\otimes} \arrow[d] \\
\Delta^1_{\mathcal{F},\mathrm{el}} \arrow[r, hook] & \Delta^1_{\mathcal{F}} \arrow[r, "\mathrm{Cr}_{\mathcal{F}} \circ i"'] & \mathcal{F}^{\mathrm{op}}_*,
\end{tikzcd}$$

where $\mathcal{V}_{\otimes} \to \mathcal{F}^{\mathrm{op}}_*$ denotes the Cartesian fibration associated to the functor determined by the symmetric monoidal ∞-category $\mathcal{V}$. It follows from the definition of τ that $\mathrm{Cr}_{\mathcal{F}} \circ i = \mathrm{Cr} \circ \tau$ and that τ restricts to an equivalence $\Delta_{\mathcal{F}^1,\mathrm{el}} \simeq \Omega_{\mathrm{el}}$. Hence, the canonical map given by the pullbacks $\tau_{\mathcal{V},\mathrm{el}}\colon \mathcal{V}^{1,\vee,\mathrm{el}} \to \mathcal{V}^{\vee,\mathrm{el}}_{\Omega}$ is an equivalence. $\square$

Lemma 7.21. *The functor $\tau^{\kappa,*}_{\mathcal{V}}\colon \mathrm{P}(\mathcal{V}^{\kappa,\vee}_{\Omega}) \to \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ carries $S^{\lim}$ to $S^{\lim}_{\mathcal{F},1}$.*

Proof. Since the sets $S^{\lim}$ and $S^{\lim}_{\mathcal{F},1}$ are both strongly generated, it suffices to show that the functor $\tau^{\kappa,*}_{\mathcal{V}}$ carries all the generators of $S^{\lim}$ to elements in $S^{\lim}_{\mathcal{F},1}$. By Definition 7.8, the set $S^{\lim}$ is generated by a set of morphisms $g\colon \mathrm{colim}_{k\in K}\, q(k) \to q(\infty)$ of presheaves such that q is a functor of the from $q = (\overline{g} * q_n)^{\mathrm{op}}\colon K = (L^{\triangleleft} * (\mathbf{n+1}))^{\mathrm{op}} \to \mathcal{V}^{\kappa,\vee}_{\Omega}$, where $\overline{g}\colon L^{\triangleleft} \to \mathcal{V}^{\kappa,\vee,\mathrm{op}}_{\Omega}$ and $q_n\colon \mathbf{n+1} \to \mathcal{V}^{\kappa,\vee,\mathrm{op}}_{\Omega}$ are defined as in Definition 7.8.

Since $\tau_{\mathcal{V}}^{\kappa,*}$ is left adjoint to the functor $\tau^{\kappa}_{\mathcal{V},*}$ which is given by the right Kan extension, $\tau_{\mathcal{V}}^{\kappa,*}$ preserves colimits. Hence, we have that $\tau_{\mathcal{V}}^{\kappa,*}(g)$ is of the form $\mathrm{colim}_{k\in K}\,\tau_{\mathcal{V}}^{\kappa,*}q(k) \to \tau_{\mathcal{V}}^{\kappa,*}q(\infty)$ and we want to see that $\tau_{\mathcal{V}}^{\kappa,*}(g)$ lies in $S^{\lim}_{\mathcal{F},1}$. We will even show that $\tau_{\mathcal{V}}^{\kappa,*}(g)$ is a generator of $S^{\lim}_{\mathcal{F},1}$. In other words, we claim the existence of functors $\overline{g}', q_n'$ and a corolla $\mathfrak{c} \in \Delta^1_{\mathcal{F}}$ such that the following assertion hold:

1. The functor $\overline{g}' \colon L^{\triangleleft} \to (\mathcal{V}^{1,\kappa,\vee,\mathrm{op}})_{\mathfrak{c}}$ is a small limit diagram.

2. The functor $q_n' \colon \mathbf{n+1} \to \mathcal{V}^{1,\kappa,\vee,\mathrm{op}}$ is the constant functor at $\overline{\mathfrak{c}} \in (\mathcal{V}^{1,\kappa,\vee,\mathrm{op}})_{\mathfrak{c}}$.

3. We have $\tau_{\mathcal{V}}^{\kappa,*}q(k) = (\overline{g}' * q_n')^{\mathrm{op}}$, i.e. the functors $\overline{g}', q_n'$ render the following diagram

$$\begin{tikzcd}
(L^{\triangleleft} * (\mathbf{n+1}))^{\mathrm{op}} \arrow[r, "(\overline{g} * q_n)^{\mathrm{op}}"] \arrow[d, "\mathrm{id}"'] & \mathcal{V}_{\Omega}^{\kappa,\vee} \arrow[r] & \mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee}) \arrow[d, "\tau_{\mathcal{V}}^{\kappa,*}"] \\
(L^{\triangleleft} * (\mathbf{n+1}))^{\mathrm{op}} \arrow[r, "(\overline{g}' * q_n')^{\mathrm{op}}"] & \mathcal{V}^{1,\kappa,\vee} \arrow[r] & \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})
\end{tikzcd} \tag{7.4}$$

commutative, where the horizontal maps on the right hand side are Yoneda embeddings.

Since q maps every object $i \in \mathbf{n+1}$ to an object in $(\mathcal{V}_{\Omega}^{\kappa,\vee})_{\eta} \simeq \{\overline{\eta}\}$, q factorizes through $\mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{el}}$. By applying the previous lemma to the symmetric monoidal ∞-category $\mathcal{V}^{\kappa}$, we obtain an equivalence $\tau_{\mathcal{V},\mathrm{el}} \colon \mathcal{V}^{1,\kappa,\vee,\mathrm{el}} \xrightarrow{\simeq} \mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{el}}$. Therefore, the functors $\overline{g}' = \tau_{\mathcal{V},\mathrm{el}}^{-1} \circ \overline{g}$ and $q_n' = \tau_{\mathcal{V},\mathrm{el}}^{-1} \circ q_n$ exist and the definition of $\overline{g}, q_n$ implies that $\overline{g}', q_n'$ satisfy the two properties mentioned above.

In order to show that the functors $\overline{g}', q_n'$ render the diagram 7.4 commutative, we need to show that the presheaves

$$\tau_{\mathcal{V}}^{\kappa,*}(q(l)) = \mathrm{Map}_{\mathcal{V}_{\Omega}^{\kappa,\vee}}(\tau_{\mathcal{V}}^{\kappa}(-), q(l)) \quad \text{and} \quad \tau_{\mathcal{V}}^{*}(q(i)) = \mathrm{Map}_{\mathcal{V}_{\Omega}^{\kappa,\vee}}(\tau_{\mathcal{V}}^{\kappa}(-), q(i))$$

are representable in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ for every $l \in (L^{\triangleleft})^{\mathrm{op}}$ and $i \in (\mathbf{n+1})^{\mathrm{op}} \cong \mathbf{n+1}$. We verify the first equivalence; similar arguments also apply to the second case.

Let $C_n \in \Omega$ denote the corolla such that $\overline{g} \colon L^{\triangleleft} \to (\mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{op}})_{C_n}$ is a limit diagram. The equivalence $\tau_{\mathcal{V},\mathrm{el}} \colon \mathcal{V}^{1,\kappa,\vee,\mathrm{el}} \simeq \mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{el}}$ implies that there exists an object $x \in (\mathcal{V}^{1,\kappa,\vee})_{\tau^{-1}_{\mathcal{V},\mathrm{el}}(C_n)} \subseteq \mathcal{V}^{1,\kappa,\vee,\mathrm{el}}$ such that $\tau_{\mathcal{V},\mathrm{el}}(x) \simeq q(l) \in (\mathcal{V}_{\Omega}^{\kappa,\vee})_{C_n} \subseteq \mathcal{V}_{\Omega}^{\kappa,\vee,\mathrm{el}}$. The pullback square in Remark 7.18 implies that there exists a pullback square in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{V}^{1,\kappa,\vee}}(-,x) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{V}_{\Omega}^{\kappa,\vee}}(\tau_{\mathcal{V}}^{\kappa}(-), \tau_{\mathcal{V}}^{\kappa}(x)) \arrow[d] \\
\mathrm{Hom}_{\Delta^1_{\mathcal{F}}}(-, C_n) \arrow[r] & \mathrm{Hom}_{\Omega}(\tau(-), \tau(C_n)).
\end{tikzcd}$$

Since bottom horizontal map is an isomorphism of presheaves by [CHH16, Lemma 4.6], the upper horizontal map is also an equivalence and the presheaf $\tau_{\mathcal{V}}^{*}(q(l)) = \mathrm{Map}_{\mathcal{V}_{\Omega}^{\kappa,\vee}}(\tau_{\mathcal{V}}^{\kappa}(-), \tau_{\mathcal{V}}^{\kappa}(x))$ is representable. □

Lemma 7.22. *If $L_{\Omega} \colon \mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee}) \rightleftarrows \mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})[\overline{S}^{-1}] : \iota_{\Omega}$ denotes the accessible localization of $\mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})$ with respect to the set $\overline{S}$ (see Proposition 7.10), then the adjoint pair $(\tau^{\kappa}_{\mathcal{V},!}, \tau_{\mathcal{V}}^{\kappa,*})$ from Defini-*

tion 7.19 induces an adjunction

$$L_\Omega \circ \tau^\kappa_{\mathcal{V},!}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\kappa,\vee}) \simeq \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})[\overline{S}^{-1}_{\mathcal{F},1}] \rightleftarrows \mathrm{P}(\mathcal{V}^{\kappa,\vee}_\Omega)[\overline{S}^{-1}] \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\kappa,\vee}_\Omega) :\tau^{\kappa,*}_{\mathcal{V}} \circ \iota_\Omega.$$

Proof. The composite of the adjoint pairs (L_Ω, ι_Ω) and $(\tau^\kappa_{\mathcal{V},!}, \tau^{\kappa,*}_{\mathcal{V}})$ obviously induces an adjunction

$$L_\Omega \circ \tau^\kappa_{\mathcal{V},!}\colon \mathrm{P}(\mathcal{V}^{1,\kappa,\vee}) \rightleftarrows \mathrm{P}(\mathcal{V}^{\kappa,\vee}_\Omega)[\overline{S}^{-1}] :\tau^{\kappa,*}_{\mathcal{V}} \circ \iota_\Omega.$$

Therefore, we only need to verify that the image of $\tau^{\kappa,*}_{\mathcal{V}} \circ \iota_\Omega$ lies in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})[\overline{S}^{-1}_{\mathcal{F},1}]$, i.e. we have to show that $\tau^{\kappa,*}_{\mathcal{V}} \circ \iota_\Omega$ carries every $\overline{S}$-local object F in $\mathrm{P}(\mathcal{V}^{\kappa,\vee}_\Omega)$ to an $\overline{S}_{\mathcal{F},1}$-local object in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$. By adjunction, requiring that an object $\tau^{\kappa,*}_{\mathcal{V}} \circ \iota_\Omega(F)$ is $\overline{S}_{\mathcal{F},1}$-local is equivalent to requiring $\iota_\Omega F$ to be local with respect to all morphisms of the form $\tau^\kappa_{\mathcal{V},!}(f)$, for every $f \in \overline{S}_{\mathcal{F},1}$. We can assume that f lies in the generating set $S^{\lim}_{\mathcal{F},1} \coprod S^{\mathrm{Seg}}_{\mathcal{F},1}$.

Since $\tau^\kappa_{\mathcal{V},!}$ is a left adjoint, it preserves colimits and therefore carries the set $S^{\lim}_{\mathcal{F},1}$ to the set $S^{\lim}$. By using the colimit preserving property together with Lemma 7.20 we obtain that for every $f\colon v_{\mathrm{Seg}} \to v$ in $S^{\mathrm{Seg}}_{\mathcal{F},1}$, the morphism $\tau^\kappa_{\mathcal{V},!}f$ is given by $\tau^\kappa_{\mathcal{V},!}(v_{\mathrm{Seg}}) \simeq (\tau^\kappa_{\mathcal{V}}v)_{\mathrm{Seg}} \to v$, which lies in the set $S^{\mathrm{Seg}}_{\mathcal{F},1}$. Hence, the Segal Ω-sheaf F is local with respect to $\tau^\kappa_{\mathcal{V},!}f$. □

Lemma 7.23. *The functor $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ induces an equivalence of ∞-categories $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$.*

Proof. It follows easily from the definition that $\tau\colon \Delta^1_{\mathcal{F}} \to \Omega$ preserves the corresponding active-inert factorization systems. Therefore, it induces a functor $\tau_{\mathrm{in}}\colon \Delta^1_{\mathcal{F},\mathrm{in}} \to \Omega_{\mathrm{in}}$ by restriction and Definition 7.14 implies that there exists a map $\tau^{\mathcal{V}}_{\mathrm{in}}\colon \mathcal{V}^{1,\vee,\mathrm{in}} \to \mathcal{V}^{\vee,\mathrm{in}}_\Omega$ such that the diagram

$$\begin{tikzcd} \mathcal{V}^{1,\vee,\mathrm{in}} \arrow[r, "\tau^{\mathcal{V}}_{\mathrm{in}}"] \arrow[d] & \mathcal{V}^{\vee,\mathrm{in}}_\Omega \arrow[d] \\ \Delta^1_{\mathcal{F},\mathrm{in}} \arrow[r, "\tau_{\mathrm{in}}"'] & \Omega_{\mathrm{in}} \end{tikzcd}$$

is a pullback square. Lemma 7.15 and Remark 7.16 imply that the inclusions $\mathcal{V}^{\vee,\mathrm{el}}_\Omega \hookrightarrow \mathcal{V}^{\vee,\mathrm{in}}_\Omega$ and $\mathcal{V}^{1,\vee,\mathrm{el}} \hookrightarrow \mathcal{V}^{1,\vee,\mathrm{in}}$ induce functors $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{el}}_\Omega)$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}}) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{el}})$, respectively. Since $\tau^{\mathcal{V}}_{\mathrm{in}}$ restricts to the equivalence $\tau_{\mathcal{V},\mathrm{el}}\colon \mathcal{V}^{1,\vee,\mathrm{el}} \to \mathcal{V}^{\vee,\mathrm{el}}_\Omega$ of Lemma 7.20, we have a commutative diagram of Segal presheaves

$$\begin{tikzcd} \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega) \arrow[r, "\tau^{\mathcal{V},*}_{\mathrm{in}}"] \arrow[d] & \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}}) \arrow[d] \\ \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{el}}_\Omega) \arrow[r, "\simeq", "\tau^*_{\mathcal{V},\mathrm{el}}"'] & \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{el}}). \end{tikzcd}$$

By Definition 7.14 and Definition 7.13, objects in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ are given by right Kan extensions of objects in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{el}}_\Omega)$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{el}})$, respectively. Thus, [Lur09, Proposition 4.3.2.15] implies that the vertical maps are equivalences, which implies that the functor $\tau^{\mathcal{V},*}_{\mathrm{in}}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}})$ is also an equivalence by the 2-of-3 property. □

Lemma 7.24. *Let the functors* $\iota^1_{\mathrm{in}}\colon \mathcal{V}^{1,\vee,\mathrm{in}} \to \mathcal{V}^{1,\vee}$ *and* $\iota_{\mathrm{in}}\colon \mathcal{V}^{\vee,\mathrm{in}}_\Omega \to \mathcal{V}^\vee_\Omega$ *denote the functors given by pullback diagrams in Definition 7.14 and Definition 7.13, respectively.*

1. *There is a monadic adjunction*

$$L_\Omega \circ \iota_{\mathrm{in},!}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega) \rightleftarrows \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega) : \iota^*_{\mathrm{in}} \circ \iota_\Omega.$$

2. *There is a monadic adjunction*

$$L_{\mathcal{F},1} \circ \iota^1_{\mathrm{in},!}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}}) \rightleftarrows \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) : \iota^{1,*}_{\mathrm{in}} \circ \iota_{\mathcal{F},1}.$$

Proof. We will prove the first statement. The second claim can be shown analogously. It is clear that the functor $\iota^*_{\mathrm{in}}\colon \mathrm{P}(\mathcal{V}^\vee_\Omega) \to \mathrm{P}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$ restricts to a functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$. Hence, by composing with the adjunction $L_\Omega\colon \mathrm{P}(\mathcal{V}^\vee_\Omega) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega) : \iota_\Omega$, we have that $L_\Omega \circ \iota_{\mathrm{in},!}$ is left adjoint to $\iota^*_{\mathrm{in}} \circ \iota_\Omega$.

In order to show that the adjunction $L_\Omega \circ \iota_{\mathrm{in},!}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega) \rightleftarrows \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega) : \iota^*_{\mathrm{in}} \circ \iota_\Omega$ is monadic, we verify the conditions in [Lur, Theorem 4.7.4.5]. Since Ω_{in} and Ω have the same objects, it follows from the definition that the ∞-categories $\mathcal{V}^{\vee,\mathrm{in}}_\Omega$ and $\mathcal{V}^\vee_\Omega$ also have the same objects, which implies that $\iota^*_{\mathrm{in}} \circ \iota_\Omega$ detects equivalence. Since the ∞-category $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$ is presentable by Proposition 7.10, it is also cocomplete. In particular, every $\iota^*_{\mathrm{in}}\circ\iota_\Omega$-split object in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$ admits a colimit, where, by [Lur, Definition 4.7.3.2], an $\iota^*_{\mathrm{in}}\circ\iota_\Omega$-split simplicial object is a simplicial functor $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$ such that $\iota^*_{\mathrm{in}} \circ \iota_\Omega \circ X_\bullet$ extends to a functor $\tilde{X}_\bullet\colon \Delta^{\mathrm{op}}_{-\infty} \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$. In order to apply [Lur, Theorem 4.7.4.5], it only remains to verify that the functor $\iota^*_{\mathrm{in}} \circ \iota_\Omega$ preserves the colimit of every $\iota^*_{\mathrm{in}} \circ \iota_\Omega$-split simplicial object $X_\bullet$. For an $\iota^*_{\mathrm{in}} \circ \iota_\Omega$-split simplicial object $X_\bullet$, suppose X is the colimit of $\iota_\Omega \circ X_\bullet$ in the cocomplete category $\mathrm{P}(\mathcal{V}^\vee_\Omega)$. Then $L_\Omega(X)$ is the colimit of $X_\bullet$ in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$. Since $\iota^*_{\mathrm{in}}\colon \mathrm{P}(\mathcal{V}^\vee_\Omega) \to \mathrm{P}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$ admits a right adjoint ι^*_{in}, the functor ι^*_{in} preserves colimits and $\iota^*_{\mathrm{in}}(X)$ is the colimit of $\iota^*_{\mathrm{in}} \circ \iota_\Omega \circ X_\bullet$ in $\mathrm{P}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$. By [Lur, Remark 4.7.3.3], the functor $\tilde{X}_\bullet$ can also be regarded as a colimit diagram in $\mathrm{P}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$ and therefore the colimit of $\tilde{X}_\bullet|_{\Delta^{\mathrm{op}}} = X_\bullet$ given by $\tilde{X}_{-\infty}$ can be identified with $\iota^*_{\mathrm{in}}(X)$. The fact that $\tilde{X}_{-\infty}$ lies in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$ implies that X lies in $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$. Therefore, we have an equivalence $L_\Omega(X) \simeq X$ which induces equivalences $\iota^*_{\mathrm{in}} \circ \iota_\Omega(X) \simeq \iota^*_{\mathrm{in}} \circ \iota_\Omega(L_\Omega(X))$. Since the counit of the localization adjunction is a natural equivalence, we have $\iota^*_{\mathrm{in}} \circ \iota_\Omega(L_\Omega(X)) \simeq \iota^*_{\mathrm{in}}(X) \simeq \tilde{X}_{-\infty}$, which completes the proof that the functor $\iota^*_{\mathrm{in}} \circ \iota_\Omega$ preserves colimits of $\iota^*_{\mathrm{in}} \circ \iota_\Omega$-split simplicial objects. ☐

Notation 7.25. *For the sake of brevity, we will write* F_Ω *and* $F_{\mathcal{F},1}$ *for the left adjoint functors* $L_\Omega \circ \iota_{\mathrm{in},!}$ *and* $L_{\mathcal{F},1} \circ \iota^1_{\mathrm{in},!}$*, respectively.*

7.3 Proof of the Comparison Result

This section is devoted to the proof of the existence of an equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$ as claimed in Theorem 7.1. As a first step we show in Proposition 7.33 that the functor $\tau^{\kappa,*}_{\mathcal{V}}\colon \mathrm{P}(\mathcal{V}^{\kappa,\vee}_\Omega) \to \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ induced by τ carries the set S^{Seg} to the set $\overline{S}_{\mathcal{F},1}$. Using this it is not hard to provide an adjunction $\tau^*_{\mathcal{V}}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega) \rightleftarrows \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) : \tau_{\mathcal{V},*}$ in Proposition 7.35. This is the key result to show that the ∞-categories $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ and $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$ of algebras for monads on $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee,\mathrm{in}}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee,\mathrm{in}}_\Omega)$ are equivalent at the end of this section.

Definition 7.26. Let $T \in \Omega$ be a tree with at least two vertices. We write $\partial^{\mathrm{ext}}T$ for the *external boundary* of T which is the union of all the *external faces* of T. More precisely, the presheaf $\partial^{\mathrm{ext}}T$ is defined to be the colimit of the composition $\mathrm{Sub}(T) \to \Omega \to \mathcal{P}(\Omega)$, where $\mathrm{Sub}(T)$ denotes the full subcategory of $(\Omega_{\mathrm{in}})_{/T}$ on the proper subtrees of T and the last map is the Yoneda embedding.

Let $p^{\kappa,\vee}\colon \mathcal{V}_\Omega^{\kappa,\vee} \to \Omega$ denote the Cartesian fibration which corresponds to the same functor as the coCartesian fibration $\mathcal{V}_\Omega^\kappa \to \Omega$ and let $p_\Omega^{\kappa,\vee,*}\colon \mathrm{P}(\Omega) \to \mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$ be the functor induced by $p^{\kappa,\vee}$. Given an object $T^{\mathcal{V}} \in \mathcal{V}_\Omega^{\kappa,\vee}$ lying over $T \in \Omega$, we define the presheaf $\partial^{\mathrm{ext}}T^{\mathcal{V}}$ by the following pullback in $\mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$:

$$\begin{tikzcd}
\partial^{\mathrm{ext}}T^{\mathcal{V}} \arrow[r] \arrow[d] & T^{\mathcal{V}} \arrow[d] \\
p_\Omega^{\kappa,\vee,*}\partial^{\mathrm{ext}}T \arrow[r] & p_\Omega^{\kappa,\vee,*}T,
\end{tikzcd}$$

where the right vertical map is the adjunction unit $T^{\mathcal{V}} \to p_\Omega^{\kappa,\vee,*}p_{\Omega,!}^{\kappa,\vee}T^{\mathcal{V}} \simeq p_\Omega^{\kappa,\vee,*}T$.

Lemma 7.27. *Let $T^{\mathcal{V}} \in \mathcal{V}_\Omega^{\kappa,\vee}$ be an object lying over $T \in \Omega$. The presheaf $\partial^{\mathrm{ext}}T^{\mathcal{V}}$ is given by the colimit of the composite $\mathrm{Sub}(T) \to \mathcal{V}_\Omega^{\kappa,\vee} \to \mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$, where the first functor carries an object $\alpha\colon S \to T \in \mathrm{Sub}(T)$ to the presheaf $\alpha^*T^{\mathcal{V}}$ given by the $p_\Omega^{\kappa,\vee}$-Cartesian lift of α and the second functor is the Yoneda embedding.*

Proof. Let $\alpha\colon S \to T \in \mathrm{Sub}(T)$ and let $S^{\mathcal{V}} \to T^{\mathcal{V}}$ be a $p_\Omega^{\kappa,\vee}$-Cartesian lift of α at $T^{\mathcal{V}}$. Lemma 5.8 states that there exists a pullback diagram in $\mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$:

$$\begin{tikzcd}
S^{\mathcal{V}} \arrow[r] \arrow[d] & T^{\mathcal{V}} \arrow[d] \\
p_\Omega^{\kappa,\vee,*}S \arrow[r] & p_\Omega^{\kappa,\vee,*}T,
\end{tikzcd} \tag{7.5}$$

where the vertical maps are the adjunction units $S^{\mathcal{V}} \to p_\Omega^{\kappa,\vee,*}p_{\Omega,!}^{\kappa,\vee}S^{\mathcal{V}} \simeq S$ and $T^{\mathcal{V}} \to p_\Omega^{\kappa,\vee,*}p_{\Omega,!}^{\kappa,\vee}T^{\mathcal{V}} \simeq p_\Omega^{\kappa,\vee,*}T$, respectively.

Since $p_\Omega^{\kappa,\vee,*}$ is left adjoint to the functor $p_{\Omega,*}^{\kappa,\vee}$ given by right Kan extension, it preserves colimits and the presheaf $p_\Omega^{\kappa,\vee,*}(\partial^{\mathrm{ext}}T)$ can be identified with $\mathrm{colim}_{S\to T\in\mathrm{Sub}(T)}\, p_\Omega^{\kappa,\vee,*}(S)$ in $\mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$. Let $F \in \mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$ denote the colimit of the composite $\mathrm{Sub}(T) \to \mathcal{V}_\Omega^{\kappa,\vee} \to \mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$ described in the statement of the lemma. The observation above implies that the presheaf $\alpha^*T^{\mathcal{V}}$ is equivalent to the presheaf $S^{\mathcal{V}}$ given by the pullback square 7.5. Since $\mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$ is an ∞-topos, colimits are preserved by pullbacks and we obtain a pullback square

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & T^{\mathcal{V}} \arrow[d] \\
p_\Omega^{\kappa,\vee,*}\partial^{\mathrm{ext}}T \arrow[r] & p_\Omega^{\kappa,\vee,*}T.
\end{tikzcd}$$

By definition, the presheaf F coincides with $\partial^{\mathrm{ext}}T^{\mathcal{V}}$. $\square$

Let $S \to T$ be an element in $\mathrm{Sub}(T)$. Its $p_\Omega^{\kappa,\vee}$-Cartesian lift $S^{\mathcal{V}} \to T^{\mathcal{V}}$ induces a pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{V}_\Omega^\vee}(R^{\mathcal{V}}, S^{\mathcal{V}}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{V}_\Omega^\vee}(R^{\mathcal{V}}, T^{\mathcal{V}}) \arrow[d] \\
\mathrm{Hom}_\Omega(R, S) \arrow[r] & \mathrm{Hom}_\Omega(R, T)
\end{tikzcd}$$

for every object $R^{\mathcal{V}} \in \mathcal{V}_\Omega^\vee$ lying over R. Since a morphism in Ω is uniquely determined by the underlying map of the corresponding sets of edges by Lemma 2.61 and $S \to T$ is inert, the bottom horizontal map is an injection of sets. Hence, the upper horizontal map is an inclusion of connected components and the map $S^{\mathcal{V}} \to T^{\mathcal{V}}$ induces an inclusion of representable presheaves.This implies that the presheaf $\partial^{\mathrm{ext}} T^{\mathcal{V}}$ can be regarded as a subpresheaf of $T^{\mathcal{V}}$.

Lemma 7.28. *For an object $T^{\mathcal{V}} \in \mathcal{V}_\Omega^{\kappa,\vee}$ lying over a tree $T \in \Omega$ with at least two vertices, let $(\partial^{\mathrm{ext}} T^{\mathcal{V}})_{\mathrm{Seg}}$ denote the colimit of the functor*

$$\mathrm{Sub}(T) \to \mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee}),$$

which carries $\alpha \in \mathrm{Sub}(T)$ to $(\alpha^* T^{\mathcal{V}})_{\mathrm{Seg}}$ as defined in Notation 7.11. There is an equivalence of presheaves $(\partial^{\mathrm{ext}} T^{\mathcal{V}})_{\mathrm{Seg}} \simeq T^{\mathcal{V}}_{\mathrm{Seg}}$.

Proof. Let T_{Seg} denote the colimit of the canonical inclusion $(\Omega_{\mathrm{el}})_{/T} \to \Omega \to \mathrm{P}(\Omega)$ and let $(\partial^{\mathrm{ext}} T)_{\mathrm{Seg}} \in \mathrm{P}(\Omega)$ denote the colimit of the functor $\mathrm{Sub}(T) \to \mathrm{P}(\Omega)$ which assigns each object $\alpha\colon S \to T$ in $\mathrm{Sub}(T)$ to the presheaf S_{Seg}. We obtain a commutative diagram in $\mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$:

$$\begin{tikzcd}
(\partial^{\mathrm{ext}} T^{\mathcal{V}})_{\mathrm{Seg}} \arrow[r] \arrow[d] & T^{\mathcal{V}}_{\mathrm{Seg}} \arrow[r] \arrow[d] & T^{\mathcal{V}} \arrow[d] \\
p_\Omega^{\kappa,\vee,*}(\partial^{\mathrm{ext}} T)_{\mathrm{Seg}} \arrow[r] & p_\Omega^{\kappa,\vee,*} T_{\mathrm{Seg}} \arrow[r] & p_\Omega^{\kappa,\vee,*} T.
\end{tikzcd}$$

Since pullbacks preserve colimits in an ∞-topos and the functor $p_\Omega^{\kappa,\vee,*}$ is a left adjoint, Lemma 5.8 implies that the right hand square is Cartesian. The same argument reveals that the big square is Cartesian too. By [Lur09, Lemma 4.4.2.1], we then obtain that the left hand square is Cartesian. The bottom left horizontal map is an equivalence by [CHH16, Lemma 5.8], therefore, the upper left horizontal map is an equivalence $(\partial^{\mathrm{ext}} T^{\mathcal{V}})_{\mathrm{Seg}} \simeq T^{\mathcal{V}}_{\mathrm{Seg}}$ in $\mathrm{P}(\mathcal{V}_\Omega^{\kappa,\vee})$. $\square$

In order to define the objects below in a clearer way, we work with (Segal presheaves on) $\Delta_{\mathcal{F}}$ instead of $\Delta^1_{\mathcal{F}}$; this makes no difference due to Lemma 7.17.

For the convenience of the reader, let us recall the following two results from [CHH16].

Lemma 7.29. *[CHH16, 5.10] Let X be an object in $\Delta_{\mathcal{F}}$ of length $n \geq 2$. The map $X_{\mathrm{Seg}} \to \Lambda^n_{n-1} X$ (see Notation 4.21.3 and Definition 5.9) in $\mathrm{P}(\Delta_{\mathcal{F}})$ is given by a composite of pushouts of morphisms of the form $\Lambda^m_{m-1} Y \to Y$, where $Y \in \Delta_{\mathcal{F}}$ has length $m < n$.*

Proposition 7.30. *[CHH16, 5.6] Let $T \in \Omega$ be a tree with at least two vertices. The map $\tau^*(\partial^{\mathrm{ext}} T) \to \tau^* T$ is given by a composite of maps $f_n\colon F_{n-1} \to F_n$ in $\mathrm{P}(\Delta^1_{\mathcal{F}})$ which are pushouts of morphisms of the form $\coprod_{s_n} \Lambda^n_{n-1} X \to \coprod_{s_n} X$ (see Definition 5.9), where $X \in \Delta^1_{\mathcal{F}}$ and s_n is a set.*

Lemma 7.31. *Let $X^{\mathcal{V}}$ be an object in $\mathcal{V}^{\kappa,\vee}$ lying over $X \in \Delta_{\mathcal{F}}$ of length n. The canonical map $\Lambda^n_{n-1}X^{\mathcal{V}} \to X^{\mathcal{V}}$ in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$ lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$.*

Proof. Since the map $X^{\mathcal{V}}_{\mathrm{Seg}} \to X^{\mathcal{V}}$ is given by the compsite of $\tilde{f}\colon X^{\mathcal{V}}_{\mathrm{Seg}} \to \Lambda^n_{n-1}X^{\mathcal{V}}$ and $\Lambda^n_{n-1}X^{\mathcal{V}} \to X^{\mathcal{V}}$, the 2-of-3 property implies that it suffices to show that $\tilde{f}$ lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$. We will prove the claim by an induction on the length of X.

If X is of length 2, then it is clear that $X^{\mathcal{V}}_{\mathrm{Seg}} \to \Lambda^2_1X^{\mathcal{V}}$ lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$, hence, the 2-of-3 property implies that $\Lambda^2_1X^{\mathcal{V}} \to X^{\mathcal{V}}$ also lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$. Now suppose X has length $n > 2$ and $\Lambda^n_{n-1}Y^{\mathcal{V}} \to Y^{\mathcal{V}}$ lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$, for every $Y^{\mathcal{V}} \in \mathcal{V}^{\kappa,\vee}$ lying over an object $Y \in \Delta_{\mathcal{F}}$ of length smaller than n.

Let f denotes the map $X_{\mathrm{Seg}} \to \Lambda^n_{n-1}X$ in $\mathrm{P}(\Delta_{\mathcal{F}})$, Lemma 5.8 and the fact that pullbacks in an ∞-topos commutes with colimits provides the existence of the following commutative diagram

$$\begin{tikzcd}
X^{\mathcal{V}}_{\mathrm{Seg}} \arrow[r, "\tilde{f}"] \arrow[d] & \Lambda^n_{n-1}X^{\mathcal{V}} \arrow[r] \arrow[d] & X^{\mathcal{V}} \arrow[d] \\
p^{\kappa,\vee,*}X_{\mathrm{Seg}} \arrow[r, "p^{\kappa,\vee,*}(f)"'] & p^{\kappa,\vee,*}(\Lambda^n_{n-1}X) \arrow[r] & p^{\kappa,\vee,*}(X)
\end{tikzcd} \tag{7.6}$$

in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$, where the big square as well as the right hand square are Cartesian. It follows from [Lur09, Lemma 4.4.2.1] that the left hand square is a Cartesian too. By Lemma 7.29, the map $f\colon X_{\mathrm{Seg}} \to \Lambda^n_{n-1}X$ in $\mathrm{P}(\Delta_{\mathcal{F}})$ is given by a composite of maps of presheaves $f_k\colon F_{k-1} \to F_k$. Let $\tilde{F}_k \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ and $\tilde{f}_k\colon \tilde{F}_{k-1} \to \tilde{F}_k$ be determined by the diagram consisting of pullback squres in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$:

$$\begin{tikzcd}
\tilde{F}_{k-1} \arrow[r, "\tilde{f}_k"] \arrow[d] & \tilde{F}_k \arrow[r] \arrow[d] & X^{\mathcal{V}} \arrow[d] \\
p^{\kappa,\vee,*}(F_{k-1}) \arrow[r, "f_k"'] & p^{\kappa,\vee,*}(F_k) \arrow[r] & p^{\kappa,\vee,*}(X).
\end{tikzcd}$$

The right hand Cartesian square of the diagram 7.6 implies that the map $\tilde{f}$ is given by a composite of the maps $\tilde{f}_k$. Therefore, to prove the lemma, we only need to verify that each map $\tilde{f}_k\colon \tilde{F}_{k-1} \to \tilde{F}_k$ lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$.

Lemma 7.29 also implies that each map $f_k\colon F_{k-1} \to F_k$ is given by a pushout of morphisms of the form $\Lambda^m_{m-1}Y \to Y$, where $Y \in \Delta_{\mathcal{F}}$ has length $m < n$. Therefore, by pullback we obtain a commutative cube diagram

$$\begin{tikzcd}
 & G(Y)^- \arrow[rr] \arrow[dl] \arrow[dd] & & \tilde{F}_{k-1} \arrow[dl, "\tilde{f}_k"] \arrow[dd] \\
G(Y)^+ \arrow[rr, crossing over] \arrow[dd] & & \tilde{F}_k & \\
 & p^{\kappa,\vee,*}(\Lambda^m_{m-1}Y) \arrow[rr] \arrow[dl] & & p^{\kappa,\vee,*}(F_{k-1}) \arrow[dl, "f_k"] \\
p^{\kappa,\vee,*}(Y) \arrow[rr] & & p^{\kappa,\vee,*}(F_k), \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$, where all the vertical squares are Cartesian and the bottom square is a pushout square due to the fact that $p^{\kappa,\vee,*}$ is left adjoint to $p^{\kappa,\vee}_*$. By construction, the left hand vertical square can be extended to a diagram

$$\begin{tikzcd}
G(Y)^- \arrow[r] \arrow[d] & G(Y)^+ \arrow[r] \arrow[d] & X^{\mathcal{V}} \arrow[d] \\
p^{\kappa,\vee,*}(\Lambda^n_{n-1}Y) \arrow[r] & p^{\kappa,\vee,*}(Y) \arrow[r] & p^{\kappa,\vee,*}(X),
\end{tikzcd}$$

where both squares are Cartesian. Lemma 5.8 implies that the presheaf $G(Y)^+ \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$ is represented by an object $Y^{\mathcal{V}} \in \mathcal{V}^{\kappa,\vee}$. Since pullbacks commutes with colimits in an ∞-topos and since the big square of the above diagram is a pullback diagram by [Lur09, Lemma 4.4.2.1], Lemma 5.8 also implies that $G(Y)^- \simeq \Lambda^n_{n-1}Y^{\mathcal{V}} \in \mathrm{P}(\mathcal{V}^{\kappa,\vee})$. Using the fact that pullbacks preserves colimits in ∞-topos once again, we see that the upper horizontal square of the cube diagram above is given by a pushout in $\mathrm{P}(\mathcal{V}^{\kappa,\vee})$:

$$\begin{tikzcd}
\Lambda^m_{m-1}Y^{\mathcal{V}} \arrow[r] \arrow[d] & \tilde{F}_{k-1} \arrow[d, "\tilde{f}_k"] \\
Y^{\mathcal{V}} \arrow[r] & \tilde{F}_k.
\end{tikzcd}$$

Since the right vertical map lies in $S^{\mathrm{Seg}}_{\mathcal{F}}$ by induction hypothesis, the pushout square above shows that this is also true for $\tilde{f}_k$. $\square$

Lemma 7.32. *Let $T^{\mathcal{V}} \in \mathcal{V}^{\kappa,\vee}_{\Omega}$ be an object lying over a tree $T \in \Omega$ with at least two vertices. The canonical map $\tau^{\kappa,*}_{\mathcal{V}}\partial^{\mathrm{ext}}T^{\mathcal{V}} \to \tau^{\kappa,*}_{\mathcal{V}}T^{\mathcal{V}}$ lies in $\overline{S}_{\mathcal{F},1}$ (see Proposition 7.12).*

Proof. Since the set $\overline{S}_{\mathcal{F},1}$ is strongly saturated by definition, it suffices to show that the map $\tau^{\kappa,*}_{\mathcal{V}}\partial^{\mathrm{ext}}T^{\mathcal{V}} \to \tau^{\kappa,*}_{\mathcal{V}}T^{\mathcal{V}}$ is given by a composite of pushouts of maps in $S^{\mathrm{Seg}}_{\mathcal{F},1} \subseteq \overline{S}_{\mathcal{F},1}$.

As in Remark 7.18, there is a pullback square

$$\begin{tikzcd} \mathcal{V}^{1,\kappa,\vee} \arrow[r, "\tau^{\kappa}_{\mathcal{V}}"] \arrow[d, "p^{1,\kappa,\vee}"'] & \mathcal{V}^{\kappa,\vee}_{\Omega} \arrow[d, "p^{\kappa,\vee}_{\Omega}"] \\ \Delta^1_{\mathcal{F}} \arrow[r, "\tau"'] & \Omega. \end{tikzcd}$$

The induced equivalence $\tau^{\kappa,*}_{\mathcal{V}} \circ p^{\kappa,\vee,*}_{\Omega} \simeq p^{1,\kappa,\vee,*} \circ \tau^*$ then implies that the right adjoint functor $\tau^{\kappa,*}_{\mathcal{V}}$ carries this pullback square

$$\begin{tikzcd} (\partial^{\mathrm{ext}} T^{\mathcal{V}})_{\mathrm{Seg}} \arrow[r] \arrow[d] & T^{\mathcal{V}}_{\mathrm{Seg}} \arrow[d] \\ p^{\kappa,\vee,*}_{\Omega}(\partial^{\mathrm{ext}} T)_{\mathrm{Seg}} \arrow[r] & p^{\kappa,\vee,*}_{\Omega} T_{\mathrm{Seg}} \end{tikzcd}$$

in $\mathrm{P}(\mathcal{V}^{\kappa,\vee}_{\Omega})$ of Lemma 7.28 to the following pullback square in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$:

$$\begin{tikzcd} \tau^{\kappa,*}_{\mathcal{V}} \partial^{\mathrm{ext}} T^{\mathcal{V}} \arrow[r, "\tau^{\kappa,*}_{\mathcal{V}}(f)"] \arrow[d] & \tau^{\kappa,*}_{\mathcal{V}} T^{\mathcal{V}} \arrow[d] \\ p^{1,\kappa,\vee,*}\tau^* \partial^{\mathrm{ext}} T \arrow[r] & p^{1,\kappa,\vee,*}\tau^* T. \end{tikzcd} \tag{7.7}$$

By Proposition 7.30, we know that the map $\partial^{\mathrm{ext}} T \to T$ in $\mathrm{P}(\Omega)$ is given by the composite of maps $f_n \colon F_{n-1} \to F_n$ for $n \geq 1$. Let $F^{\mathcal{V}}_{n-1} \in \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ and $\tilde{f}_n \colon F^{\mathcal{V}}_{n-1} \to F^{\mathcal{V}}_n$ be given by the commutative diagram in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$:

$$\begin{tikzcd} F^{\mathcal{V}}_{n-1} \arrow[r, "\tilde{f}_n"] \arrow[d] & F^{\mathcal{V}}_n \arrow[r] \arrow[d] & \tau^{\kappa,*}_{\mathcal{V}} T^{\mathcal{V}} \arrow[d] \\ p^{1,\kappa,\vee,*} F_{n-1} \arrow[r] & p^{1,\kappa,\vee,*} F_n \arrow[r] & p^{1,\kappa,\vee,*}\tau^* T, \end{tikzcd}$$

where both small squares are Cartesian. By [Lur09, Lemma 4.4.2.1], the big square is Cartesian too. It follows from the pullback square 7.7 that $\tau^{\kappa,*}_{\mathcal{V}} \partial^{\mathrm{ext}} T^{\mathcal{V}} \to \tau^{\kappa,*}_{\mathcal{V}} T^{\mathcal{V}}$ is given by the composition of the maps $\tilde{f}_n$. Hence, to verify the claim, we only need to show that each map $\tilde{f}_n \colon F^{\mathcal{V}}_{n-1} \to F^{\mathcal{V}}_n$ lies in $\overline{S}_{\mathcal{F},1}$.

By Proposition 7.30, for each map f_n, there is an object $X \in \Delta^1_{\mathcal{F}}$ and a set s_n such that f_n is given by a pushout square in $\mathrm{P}(\Delta^1_{\mathcal{F}})$ of the form:

$$\begin{tikzcd} \coprod_{s_n} \Lambda^n_{n-1} X \arrow[r] \arrow[d] & F_{n-1} \arrow[d, "f_n"] \\ \coprod_{s_n} X \arrow[r] & F_n. \end{tikzcd} \tag{7.8}$$

If $\overline{X}^{\mathcal{V}} \to T^{\mathcal{V}}$ is the $p_{\Omega}^{\kappa,\vee}$-Cartesian lift of the map $\tau(X) \to T$, then there is a pullback square

$$\begin{tikzcd}
\overline{X}^{\mathcal{V}} \arrow[r] \arrow[d] & T^{\mathcal{V}} \arrow[d] \\
p_{\Omega}^{\kappa,\vee,*}\tau(X) \arrow[r] & p_{\Omega}^{\kappa,\vee,*}T
\end{tikzcd} \tag{7.9}$$

in $\mathrm{P}(\mathcal{V}_{\Omega}^{\kappa,\vee})$ by Lemma 5.8. Since the object $\overline{X}^{\mathcal{V}} \in \mathcal{V}_{\Omega}^{\kappa,\vee}$ lies over $\tau(X) \in \Omega$ and since $\mathcal{V}^{1,\kappa,\vee} \simeq \Delta^1_{\mathcal{F}} \times_{\Omega} \mathcal{V}_{\Omega}^{\kappa,\vee}$ by Remark 7.18, there exists an object $X^{\mathcal{V}} \in \mathcal{V}^{1,\kappa,\vee}$ such that $\tau^{\mathcal{V}}(X^{\mathcal{V}}) \simeq \overline{X}^{\mathcal{V}} \in \mathcal{V}_{\Omega}^{\kappa,\vee}$ and we then obtain the equivalence $\overline{X}^{\mathcal{V}} \simeq \tau_!^{\mathcal{V}}(X^{\mathcal{V}})$ of presheaves. The equivalence $\tau_{\mathcal{V}}^{\kappa,*}(\overline{X}^{\mathcal{V}}) \simeq \tau_{\mathcal{V}}^{\kappa,*}\tau_!^{\mathcal{V}}(X^{\mathcal{V}})$ in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ and $\tau_{\mathcal{V}}^{\kappa,*}p_{\Omega}^{\kappa,\vee,*} \simeq p^{1,\kappa,\vee,*}\tau^*$ induce the right hand square of the following diagram

$$\begin{tikzcd}
X^{\mathcal{V}} \arrow[r] \arrow[d] & \tau_{\mathcal{V}}^{\kappa,*}\tau_{\mathcal{V},!}^{\kappa}(X^{\mathcal{V}}) \arrow[r] \arrow[d] & \tau_{\mathcal{V}}^{\kappa,*}(T^{\mathcal{V}}) \arrow[d] \\
p^{1,\kappa,\vee,*}(X) \arrow[r] & p^{1,\kappa,\vee,*}\tau^*\tau_!(X)) \arrow[r] & p^{1,\kappa,\vee,*}\tau^*(T)
\end{tikzcd} \tag{7.10}$$

in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$; its left hand square is induced by the adjoint units $X \to \tau^*\tau_!(X)$ and $X^{\mathcal{V}} \to \tau_{\mathcal{V}}^{\kappa,*}\tau_{\mathcal{V},!}^{\kappa}(X^{\mathcal{V}})$. The right hand square of this diagram is a pullback square, because it is the image of the pullback diagram 7.9 under the right adjoint functor $\tau_{\mathcal{V}}^{\kappa,*}$. In order to see that the left hand square is also a pullback, it suffices to show that applying the functor $\mathrm{Map}_{\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})}(F,-)$ to this square gives a pullback square for every object $F \in \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$. Since every presheaf is given by a colimit of representable presheaves, we can assume that F is represented by an object $Y^{\mathcal{V}} \in \mathcal{V}^{1,\kappa,\vee}$. The adjoint pairs $(\tau_{\mathcal{V},!}^{\kappa}, \tau_{\mathcal{V}}^{\kappa,*})$ and $(\tau_!, \tau^*)$ imply that we only need to verify that the following diagram of mapping spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{V}^{1,\kappa,\vee}}(Y^{\mathcal{V}}, X^{\mathcal{V}}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{V}_{\Omega}^{\kappa,\vee}}(\tau_{\mathcal{V}}^{\kappa}(Y^{\mathcal{V}}), \tau_{\mathcal{V}}^{\kappa}(X^{\mathcal{V}})) \arrow[d] \\
\mathrm{Map}_{\Delta^1_{\mathcal{F}}}(Y, X) \arrow[r] & \mathrm{Map}_{\Omega}(\tau(Y), \tau(X))
\end{tikzcd}$$

is a pullback, which is true because of $\mathcal{V}^{1,\kappa,\vee} = \Delta^1_{\mathcal{F}} \times_{\Omega} \mathcal{V}_{\Omega}^{\kappa,\vee}$. Thus, by [Lur09, Lemma 4.4.2.1] the big square of the diagram 7.10

$$\begin{tikzcd}
X^{\mathcal{V}} \arrow[r] \arrow[d] & \tau_{\mathcal{V}}^{\kappa,*}(T^{\mathcal{V}}) \arrow[d] \\
p^{1,\kappa,\vee,*}(X) \arrow[r] & p^{1,\kappa,\vee,*}\tau^*(T).
\end{tikzcd} \tag{7.11}$$

is Cartesian in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$. Since the functor $p^{1,\kappa,\vee,*}$ is left adjoint to the functor $p_*^{1,\kappa,\vee}$ given by right Kan extension, it carries the pushout diagram 7.8 to a pushout which can be extended to a cube diagram in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$

$$\begin{tikzcd}
& G(s_n)^- \arrow[rr] \arrow[dl] \arrow[dd] & & F^{\mathcal{V}}_{n-1} \arrow[dl, "\tilde{f}_n"] \arrow[dd] \\
G(s_n)^+ \arrow[rr] \arrow[dd] & & F^{\mathcal{V}}_n \arrow[dd] & \\
& \coprod_{s_n} \Lambda^n_{n-1} p^{1,\kappa,\vee,*} X \arrow[rr] \arrow[dl] & & p^{1,\kappa,\vee,*} F_{n-1} \arrow[dl, "p^{1,\kappa,\vee,*}(f_n)"] \\
\coprod_{s_n} p^{1,\kappa,\vee,*} X \arrow[rr] & & p^{1,\kappa,\vee,*} F_n, &
\end{tikzcd}$$

where all the vertical squares are given by Cartesian. Since pullbacks preserve colimits in an ∞-topos, the definition of $F^{\mathcal{V}}_{n-1}, F^{\mathcal{V}}_n$ and the publlback square 7.11 imply that there are equivalences $G(s_n)^+ \simeq \coprod_{s_n} X^{\mathcal{V}}$ and $G(s_n)^- \simeq \coprod_{s_n} \Lambda^n_{n-1} X^{\mathcal{V}}$ in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$. Using the fact that pullbacks preserve colimits in an ∞-topos once more, we have that the bottom square of the cube above can be identified with a pushout diagram

$$\begin{tikzcd}
\coprod_{s_n} \Lambda^n_{n-1} X^{\mathcal{V}} \arrow[r] \arrow[d] & F^{\mathcal{V}}_{n-1} \arrow[d, "\tilde{f}_n"] \\
\coprod_{s_n} X^{\mathcal{V}} \arrow[r] & F^{\mathcal{V}}_n.
\end{tikzcd}$$

in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$. By Lemma 7.17 and Lemma 7.31, the map $\tilde{f}_n$ lies in $S^{\mathrm{Seg}}_{\mathcal{F},1}$, because the left vertical map of the pushout diagram above does. $\square$

Proposition 7.33. *The functor $\tau^{\kappa,*}_{\mathcal{V}} \colon \mathrm{P}(\mathcal{V}^{\kappa,\vee}_{\Omega}) \to \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ induced by $\tau^{\kappa}_{\mathcal{V}}$ carries the set S^{Seg} to the set $\overline{S}_{\mathcal{F},1}$.*

Proof. Let $T^{\mathcal{V}}_{\mathrm{Seg}} \to T^{\mathcal{V}}$ be an element in S^{Seg} (see Notation 7.11), we wish to show that its image $\tau^{\kappa,*}_{\mathcal{V}}(\alpha)$ in $\mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ lies in $\overline{S}_{\mathcal{F},1}$. We will prove the proposition by an induction of the cardinality of the set of vertices in $T \in \Omega$. If T has only one vertex, i.e. it is a corolla, then $T^{\mathcal{V}}_{\mathrm{Seg}} \simeq T^{\mathcal{V}}$ obviously lies in $\overline{S}_{\mathcal{F},1}$. If the tree T has at least two vertices, then Lemma 7.28 implies that the left vertical map of the commutative square

$$\begin{tikzcd}
(\partial^{\mathrm{ext}} T^{\mathcal{V}})_{\mathrm{Seg}} \arrow[r] \arrow[d] & \partial^{\mathrm{ext}} T^{\mathcal{V}} \arrow[d] \\
T^{\mathcal{V}}_{\mathrm{Seg}} \arrow[r] & T^{\mathcal{V}}
\end{tikzcd}$$

in $\mathrm{P}(\mathcal{V}^{\kappa,\vee}_{\Omega})$ is an equivalence. Since $\tau^{\kappa,*}_{\mathcal{V}}$ is left adjoint to $\tau_{\mathcal{V},*}$, it preserves colimits and since, for every object $\alpha \colon S \to T$ of $\mathrm{Sub}(T^{\mathcal{V}})$, the tree S has fewer vertices than T the induction hypothesis implies that $\tau^{\kappa,*}_{\mathcal{V}}$ carries the upper horizontal map to an element in $\overline{S}_{\mathcal{F},1}$. Hence, by the 2-of-3 property, we only need to show that $\tau_{\mathcal{V}}^{\kappa,*}$ carries the right vertical map $f\colon \partial^{\mathrm{ext}}T^{\mathcal{V}} \to T^{\mathcal{V}}$ to an element in $\overline{S}_{\mathcal{F},1}$ and this is true by the previous lemma. □

Proposition 7.34. *The functor $\tau_{\mathcal{V}}^{\kappa,*}\colon \mathrm{P}(\mathcal{V}_{\Omega}^{\vee,\kappa}) \to \mathrm{P}(\mathcal{V}^{1,\kappa,\vee})$ induced by $\tau_{\mathcal{V}}^{\kappa}$ carries the set $\overline{S}$ to the set $\overline{S}_{\mathcal{F},1}$. Hence, it induces a functor $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega}^{\vee}) \to \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$.*

Proof. The claim of the proposition follows from Proposition 7.33 and Lemma 7.21. □

This result clearly implies the following proposition.

Proposition 7.35. *The adjunction $\tau_{\mathcal{V}}^{*}\colon \mathrm{P}(\mathcal{V}_{\Omega}^{\vee}) \rightleftarrows \mathrm{P}(\mathcal{V}^{1,\vee}) :\tau_{\mathcal{V},*}$, where the right adjoint is given by the right Kan extension induces an adjunction*

$$\tau_{\mathcal{V}}^{*}\colon \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega}^{\vee}) \rightleftarrows \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) :\tau_{\mathcal{V},*}.$$

Before we can prove the main theorem of this section, we need one last result.

Lemma 7.36. *The canonical map $\tau_{\mathcal{V},\mathrm{in}}^{*}\circ\iota_{\mathrm{in}}^{*}\circ\tau_{\mathcal{V},*} \simeq \iota_{\mathrm{in}}^{1,*}\circ\tau_{\mathcal{V}}^{*}\circ\tau_{\mathcal{V},*} \to \iota_{\mathrm{in}}^{1,*}$ is a natural equivalence.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega}^{\vee}) \arrow[r, "\tau_{\mathcal{V}}^{*}"] \arrow[d, "\iota_{\mathrm{in}}^{*}"'] & \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) \arrow[d, "\iota_{\mathrm{in}}^{1,*}"] \\
\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega,\mathrm{in}}^{\vee}) \arrow[r, "\tau_{\mathcal{V},\mathrm{in}}^{*}"'] \arrow[d] & \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\mathrm{in}}^{1,\vee}) \arrow[d] \\
\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega,\mathrm{el}}^{\vee}) \arrow[r] & \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\mathrm{el}}^{1,\vee}),
\end{tikzcd}$$

where all the maps in the lower square are equivalences by the proof of Lemma 7.23. It follows that in order to show that $\iota_{\mathrm{in}}^{1,*}\circ\tau_{\mathcal{V}}^{*}\circ\tau_{\mathcal{V},*} \to \iota_{\mathrm{in}}^{1,*}$ is an equivalence, we only need to verify that for every presheaf $F \in \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}_{\Omega,\mathrm{in}}^{\vee})$ and every object $x \in \mathcal{V}_{\mathrm{el}}^{1,\vee}$ the map $\tau_{\mathcal{V}}^{*}\circ\tau_{\mathcal{V},*}F(x) \to F(x)$ induced by the adjunction counit is an equivalence. Since $\tau_{\mathcal{V},*}F$ is given by the right Kan extension, we have

$$\tau_{\mathcal{V}}^{*}\circ\tau_{\mathcal{V},*}F(x) \simeq \tau_{\mathcal{V}}^{*}F(\tau_{\mathcal{V}}(x)) \simeq \lim\nolimits_{a\in(\mathcal{V}^{1,\vee,\mathrm{op}})_{\tau_{\mathcal{V}}(x)/}} F(a),$$

where $(\mathcal{V}^{1,\vee,\mathrm{op}})_{\tau_{\mathcal{V}}(x)/} \simeq ((\mathcal{V}^{1,\vee})_{/\tau_{\mathcal{V}}(x)})^{\mathrm{op}}$ and $(\mathcal{V}^{1,\vee})_{/\tau_{\mathcal{V}}(x)}$ denotes $\mathcal{V}^{1,\vee}\times_{\mathcal{V}_{\Omega}^{\vee}}(\mathcal{V}_{\Omega}^{\vee})_{/\tau_{\mathcal{V}}(x)}$.

In the following we show that the object $(x, \mathrm{id}_{\tau_{\mathcal{V}}(x)})$ in the ∞-category $(\mathcal{V}^{1,\vee})_{/\tau_{\mathcal{V}}(x)}$ is initial. This then immediately implies that

$$\lim\nolimits_{a\in(\mathcal{V}^{1,\vee,\mathrm{op}})_{\tau_{\mathcal{V}}(x)/}} F(a) \simeq F(x).$$

The commutative diagram

$$\begin{tikzcd}
\mathcal{V}_{/x}^{1,\vee} \arrow[r] \arrow[d] & (\mathcal{V}_{\Omega}^{\vee})_{/\tau_{\mathcal{V}}(x)} \arrow[d] \\
\mathcal{V}^{1,\vee} \arrow[r, "\tau_{\mathcal{V}}"'] & \mathcal{V}_{\Omega}^{\vee}
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathcal{V}^{1,\vee}_{/x} \arrow[rr, "h"] \arrow[dr] & & (\mathcal{V}^{1,\vee})_{/\tau_{\mathcal{V}}(x)} \arrow[dl] \\
& \mathcal{V}^{1,\vee}, &
\end{tikzcd}$$

where the diagonal maps are right fibrations by [Lur09, Proposition 2.1.2.1]. Hence, the horizontal map h above is an equivalence if and only if it is one on each fibre. For an object $y \in \mathcal{V}^{1,\vee}$, the map h_y can be identified with the horizontal map of the commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{V}^{1,\vee}}(y, x) \arrow[r, "h_y"] \arrow[d] & \mathrm{Map}_{\mathcal{V}^{\vee}_{\Omega}}(\tau_{\mathcal{V}}(y), \tau_{\mathcal{V}}(x)) \arrow[d] \\
\mathrm{Hom}_{\Delta^1_{\mathcal{F}}}(p^{1,\vee}(y), p^{1,\vee}(x)) \arrow[r] & \mathrm{Hom}_{\Omega}(p^{\vee}_{\Omega}\tau(y), p^{\vee}_{\Omega}\tau_{\mathcal{V}}(x)),
\end{tikzcd}$$

which is a pullback diagram by the equivalence $\mathcal{V}^{1,\vee} \simeq \mathcal{V}^{\vee}_{\Omega} \times_{\Omega} \Delta^1_{\mathcal{F}}$ of Remark 7.18. Since the lower horizontal map of this diagram is an equivalence by [CHH16, Lemma 4.6], we see that h_y is an equivalence. In particular, h is an equivalence and it carries the initial object $(x, \mathrm{id}_x) \in \mathcal{V}^{1,\vee}_{/x}$ to the initial object $(x, \mathrm{id}_{\mathcal{V}(x)}) \in (\mathcal{V}^{\vee}_{\Omega})_{/\tau_{\mathcal{V}}(x)}$. □

proof of Theorem 7.1. The equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee})$ provided by Lemma 7.17 implies that it suffices to show that there is an equivalence $\mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{1,\vee}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^{\vee}_{\Omega})$. By [Lur, Definition 4.7.4.16], we only need to show that the canonical natural transformation $F_{\mathcal{F},1}\tau^*_{\mathcal{V},\mathrm{in}} \to \tau_{\mathcal{V}} F_{\Omega}$ is an equivalence. Since both functors are left adjoints by Proposition 7.35, the natural transformation them is an equivalence if and only if there is an natural transformation $\iota^*\tau_{\mathcal{V},*} \to (\tau^*_{\mathcal{V},\in})^{-1}\iota^*_{\mathcal{F},1}$ between the corresponding right adjoints is an equivalence. This is follows from the lemma above. □

Chapter 8

Enrichment via Operad OOp

In this chapter we introduce another model for enriched ∞-operads. The motivation comes from the observation that every ordinary coloured operad with a fixed set X of colours can be regarded as an algebra over a particular coloured operad OOp_X in the symmetric monoidal category $\mathrm{Set}^{\times}$ of sets with the Cartesian product. If we replace $\mathrm{Set}^{\times}$ by any other symmetric monoidal category $\mathcal{V}$, then we obtain the notion of $\mathcal{V}$-enriched operads with fixed set of colours X. This definition of ordinary enriched operads can be generalized in the setting of ∞-operads as follows: If we write OOp_X also for the ∞-operad associated to OOp_X, then we can define ∞-operads enriched in a symmetric monoidal ∞-category $\mathcal{V}$ as OOp_X-algebras in $\mathcal{V}$. By varying the set X, we obtain the collection of all $\mathcal{V}$-enriched ∞-operads which can be organized in one ∞-category.

In the first section we define a simplicial category $\Omega(\mathcal{X})$, where $\mathcal{X}$ is a small simplicial groupoid. Its associated ∞-category turns out to be equivalent to $\Omega^{\mathrm{op}}_{\mathrm{N}\mathcal{X}}$ by Proposition 8.3.

The most important ∞-operad OOp_X is then introduced in Definition 8.10 of the second section. We will see that the operad whose algebras are monochromatic operads as well as the operad whose algebras are categories with a fixed set of objects are special cases of OOp_X. We will then provide a functor Θ_X from $\Omega(X)^{\mathrm{op}}$ to the category associated to the operad OOp_X in Definition 8.19.

In the last section we show that the functor Θ_X exhibits the ∞-operad OOp_X as a fibrant replacement of $\Omega(X)^{\mathrm{op}}$ in the model category $\mathrm{sSet}^{+}_{/\mathcal{F}^{\natural,0}_{*}}$ of ∞-operads in Corollary 8.25. Using this we prove our first important result of this chapter in Theorem 8.28 which asserts that the ∞-category of all OOp_X-algebras in $\mathcal{V}$ is equivalent to that of all Ω-∞-operads. This observation finally allows us to prove the second important result in Theorem 8.32 which provides an equivalence of ∞-categories $\mathrm{Alg}_{\mathcal{F}}(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V})[\mathrm{FFES}^{-1}]$ for every symmetric monoidal ∞-category $\mathcal{V}$. This shows that the correct ∞-category of $\mathcal{V}$-enriched $\mathcal{F}$-∞-operads is equivalent to the correct ∞-category of all OOp_X-algebras in $\mathcal{V}$. In other words, our first model and our this model for enriched ∞-operads are equivalent for every $\mathcal{V}$.

8.1 Category $\Omega(\mathcal{X})$

Definition 8.1. Let $\mathcal{X}$ be a small simplicial groupoid. Let $\Omega(\mathcal{X})$ denote the simplicial category which is defined by the following data:

1. The objects in $\Omega(\mathcal{X})$ are objects in Ω whose edges are labelled by objects in $\mathcal{X}$. Hence, we will use the notation $(T, \{y_i\}_{i\in T_0})$ also for objects in $\Omega(\mathcal{X})$.
2. For two objects $(T, \{y_i\}_{i\in T_0})$ and $(S, \{x_i\}_{i\in S_0})$, let the corresponding mapping space be defined by
$$\mathrm{Map}_{\Omega(\mathcal{X})}((T, \{y_i\}_{i\in T_0}), (S, \{x_i\}_{i\in S_0})) = \coprod_{\alpha\colon T\to S\in\Omega} \prod_{i\in T_0} \mathrm{Map}_{\mathcal{X}}(x_{\alpha(i)}, y_i).$$
3. The compositions are induced by those of $\mathcal{X}$.

There is a canonical projection $\Omega(\mathcal{X}) \to \Omega$ which carries an object $(S, \{x_i\}_{i\in S_0})$ to S and a mapping space $\mathrm{Map}_{\Omega(\mathcal{X})}((T, \{y_i\}_{i\in T_0}), (S, \{x_i\}_{i\in S_0}))$ to $\mathrm{Hom}_\Omega(T, S)$. We write $p_{\mathcal{X}}\colon \Omega(\mathcal{X})^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ for the opposite functor of the canonical projection.

Remark 8.2. *Definition 8.1 also makes sense, if $\mathcal{X}$ is a small simplical category and Proposition 8.3 still holds for this general case. The statements are not written in the most general form, because we only consider the case where $\mathcal{X}$ is a small simplical groupoid in this thesis.*

The definition above is an operadic version of [GH15, Definition 4.2.2] and the following proposition is a generalization of [GH15, Proposition 4.2.3]. By unwinding the definition of $\Omega(\mathcal{X})$, we realize that the simplicial functor $p_{\mathcal{X}}\colon \Omega(\mathcal{X})^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ is a Grothendieck opfibration in the simplicial setting. In the Proposition below we first verify the expected fact that its coherent nerve is a coCartesian fibration, then we show that this coCartesian fibration is equivalent to $p_X\colon \Omega^{\mathrm{op}}_{\mathrm{N}\,\mathcal{X}} \to \Omega^{\mathrm{op}}$. In other words the nerve of $p_{\mathcal{X}}$ and $p_{\mathrm{N}\,\mathcal{X}}$ corresponds to the same functor $\tilde{i}_{\mathrm{N}\,\mathcal{X}}$ which is given by a right Kan extension.

Proposition 8.3. *The following hold:*

1. *The nerve of the canonical projection map $p_{\mathcal{X}}$ induces a coCartesian fibration of ∞-categories $\Omega(\mathcal{X})^{\mathrm{op}} \to \Omega^{\mathrm{op}}$.*
2. *The homotopy fibre of $\Omega(\mathcal{X})^{\mathrm{op}}$ over the trivial tree $\eta \in \Omega^{\mathrm{op}}$ is equivalent to $\mathrm{N}\,\mathcal{X}$.*
3. *There is a canonical map $\theta\colon \Omega(\mathcal{X})^{\mathrm{op}} \to \Omega^{\mathrm{op}}_{\mathrm{N}\,\mathcal{X}}$ which preserves coCartesian morphisms.*
4. *This canonical map $\theta\colon \Omega(\mathcal{X})^{\mathrm{op}} \to \Omega^{\mathrm{op}}_{\mathrm{N}\,\mathcal{X}}$ is an equivalence of ∞-categories.*

Proof. 1. The nerve of the projection map $p_{\mathcal{X}}$ is an inner fibration in sSet, because its domain is an ∞-category and its codomain is the nerve of an ordinary category. Therefore, we only have to show that every morphism in Ω^{op} has a $p_{\mathcal{X}}$-coCartesian lift. Let $(S, \{x_i\}_{i\in S_0})$ be an object in $\Omega(\mathcal{X})^{\mathrm{op}}$ and $\alpha\colon T \to p_{\mathcal{X}}(S, \{x_i\}_{i\in S_0}) = S$ be a morphism in Ω. It follows that the object $(T, \{x_{\alpha(i)}\}_{i\in T_0}) \in \Omega(\mathcal{X})^{\mathrm{op}}$ lies over T and we claim that the map $\tilde{\alpha} \in \mathrm{Map}_{\Omega(\mathcal{X})^{\mathrm{op}}}((S, \{x_i\}_{i\in S_0}), (T, \{x_{\alpha(i)}\}_{i\in T_0}))$ induced by α and $\mathrm{id}_{x_{\alpha(i)}} \in \mathrm{Map}_{\mathcal{X}}(x_{\alpha(j)}, x_{\alpha(j)})$, for $i \in T_0$, is a coCartesian lift of $\alpha^{\mathrm{op}} \in \mathrm{Hom}_\Omega(S, T)$. It suffices to verify that for every object $U \in \Omega$ and $(U, \{z_i\}_{i\in U_1}) \in \Omega(\mathcal{X})$ the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\Omega(\mathcal{X})^{\mathrm{op}}}((T, \{x_{\alpha(i)}\}_{i\in T_0}), (U, \{z_i\}_{i\in U_1})) \arrow[r, "\tilde{\alpha}^*"] \arrow[d] & \mathrm{Map}_{\Omega(\mathcal{X})^{\mathrm{op}}}((S, \{x_i\}_{i\in S_0}), (U, \{z_i\}_{i\in U_1})) \arrow[d] \\
\mathrm{Hom}_{\Omega^{\mathrm{op}}}(T, U) \arrow[r, "\alpha^*"'] & \mathrm{Hom}_{\Omega^{\mathrm{op}}}(S, U),
\end{tikzcd}$$

induced by $\tilde{\alpha}$ is a homotopy pullback square.

The vertical maps in the square are Kan fibrations between Kan complexes, hence it suffices to show that the fibres are equivalent, which follows from the definition of $\Omega(\mathcal{X})^{\mathrm{op}}$ and the choice of $(T, \{x_{\alpha(i)}\}_{i\in T_0})$.

2. If we regard $p_{\mathcal{X}}$ as a map of simplicical categories, then the fibre of $p_{\mathcal{X}}\colon \Omega(\mathcal{X})^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ over the trivial tree η is the full simplicial category of $\Omega(\mathcal{X})^{\mathrm{op}}$ spanned by the objects of the form $(\eta, \{x\}), x \in \mathcal{X}$. Since $\mathrm{Map}_{\Omega(\mathcal{X})^{\mathrm{op}}}((\eta,\{x\}),(\eta,\{y\})) = \mathrm{Map}_{\mathcal{X}}(x,y)$ by definition, the fibre over η is isomorphic to $\mathcal{X}$. It follows that the commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \Omega(\mathcal{X})^{\mathrm{op}} \arrow[d, "p_{\mathcal{X}}"] \\ \{\eta\} \arrow[r] & \Omega^{\mathrm{op}} \end{tikzcd}$$

is a pullback in the category of simplicial categories. Being a right adjoint the coherent nerve functor N carries this square to a pullback square in sSet such that the right vertical map is a coCartesian fibration by the observation 1. above. [Lur09, Definition 3.3.1.4] implies that this square is a homotopy Cartesian square and that $\mathrm{N}\,\mathcal{X}$ can be identified with the fibre of the ∞-category $\Omega(\mathcal{X})$ over η.

3. The inclusion $i\colon \{\eta\} \hookrightarrow \Omega^{\mathrm{op}}$ induces an adjoint pair

$$i^*\colon \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathrm{Cat}_\infty) \rightleftarrows \mathrm{Fun}(\{\eta\}, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty : i_*,$$

where i^* and i_* are given by evaluation at $\{\eta\}$ and right Kan extension, respectively. If we identify $\mathrm{Fun}(\Omega^{\mathrm{op}}, \mathrm{Cat}_\infty)$ with the category of coCartesian fibrations over Ω^{op} together with maps which preserve coCartesian morphisms, then it follows from above that $i^*\Omega(\mathcal{X})^{\mathrm{op}} \simeq \mathrm{N}\,\mathcal{X}$ and that the unit of the adjunction induces a map $\theta\colon \Omega(\mathcal{X})^{\mathrm{op}} \to i_*i^*\Omega(\mathcal{X})^{\mathrm{op}} \simeq i_*\,\mathrm{N}\,\mathcal{X} = \Omega^{\mathrm{op}}_{\mathrm{N}\,\mathcal{X}}$ between coCartesian fibrations preserving coCartesian morphisms.

4. We have a commutative diagram

$$\begin{tikzcd} \Omega(\mathcal{X})^{\mathrm{op}} \arrow[rr, "\theta"] \arrow[dr, "p_{\mathcal{X}}"'] & & \Omega^{\mathrm{op}}_{\mathrm{N}\,\mathcal{X}} \arrow[dl] \\ & \Omega^{\mathrm{op}} & \end{tikzcd}$$

from the previous part of the proof such that the diagonal maps are coCartesian fibrations and θ preserves coCartesian morphisms. By [Lur09, Definition 2.4.4.4], the map θ is a categorical equivalence if and only if, for every object $T \in \Omega^{\mathrm{op}}$, the induced map of fibres $\theta_T\colon \Omega(\mathcal{X})_T \to (\Omega^{\mathrm{op}}_X)_T$ is a categorical equivalence.

The same arguments as before show that the diagram

$$\begin{tikzcd}
\prod_{i\in T_0}\mathcal{X} \arrow[r] \arrow[d] & \Omega(\mathcal{X})^{\mathrm{op}} \arrow[d, "p_{\mathcal{X}}"] \\
\{T\} \arrow[r] & \Omega^{\mathrm{op}}
\end{tikzcd}$$

is a pullback in the category of simplicial categories. After applying the coherent nerve functor to this commutative square, we have that $\prod_{i\in T_0}\mathrm{N}\,\mathcal{X}$ can be identified with the ∞-category $\Omega(\mathcal{X})_T$. It follows from the definition of Ω_X^{op} that $(\Omega_X^{\mathrm{op}})_T$ is equivalent to $\prod_{i\in T_0}\mathrm{N}\,\mathcal{X}$. Since the map θ_T is equivalent to the identity map of $\prod_{i\in T_0}\mathrm{N}\,\mathcal{X}$, the map θ is a categorical equivalence.

□

8.2 The ∞-Operad OOp_X

We will first define a symmetric coloured operad OOp_X (see Definition 8.10) whose algebras in a symmetric monoidal category $\mathcal{V}$ are the $\mathcal{V}$-enriched operads with set X of colours. We will see that the operad whose algebras are $\mathcal{V}$-enriched ∞-categoies with fixed set of objects as well as the operad whose algebras are monochromatic operads in sets with are variants of the operad OOp_X. Then we will define a simplicial functor Θ_X in Definition 8.19 which induces a functor from the ∞-category $\Omega(X)^{\mathrm{op}}$ (where we view X as a discrete simplicial groupoid) to the ∞-operad associated to OOp_X. This functor will be then used in the proof of Proposition 8.24 in the next section.

Notation 8.4. *For the whole section, we will assume that X is a set.*

Roughly speaking, in the following we are going to define $\Omega(C_{n_1},\ldots,C_{n_m};C_n)$ to be the set of all possible trees $T\in\Omega$ which can be built out of the m-many corollas $C_{n_1},\ldots,C_{n_m}\in\Omega$ under the constraint that the leaves and the root of T coincide with those of C_n. This idea can be made precise as follows:

Definition 8.5. Let $C_{n_1},\ldots,C_{n_m},C_n$ be corollas in Ω. Define $\tilde{\Omega}(C_{n_1},\ldots,C_{n_m};C_n)$ to be the category whose objects are tuples $(T,\sigma,\{f_i\}_{i\in\mathbf{m}},f)$ such that:

- T is an object in Ω.
- $\sigma\colon \mathbf{m}\to T_1$ is a bijection of sets.
- For $i\in\mathbf{m}$, $f_i\colon C_{n_i}\to T$ is an inert map carrying $(C_{n_i})_1=\{0\}$ to $i\in T_1$.
- $f\colon C_n\to T$ is an active map.

The morphisms in $\tilde{\Omega}(C_{n_1},\ldots,C_{n_m};C_n)$ defined to be isomorphisms in Ω which are compatible with the aditional data.

Notation 8.6. *Since every morphism in Ω determines and is determined by the values of edges or corollas, between any two objects in the category $\tilde{\Omega}(C_{n_1},\ldots,C_{n_m};C_n)$ there is atmost one* *isomorphism connecting them. Hence, the category $\tilde{\Omega}(C_{n_1}, \ldots, C_{n_m}; C_n)$ is equivalent to a set which we denote by $\Omega(C_{n_1}, \ldots, C_{n_m}; C_n)$. We will often write $T \in \Omega(C_{n_1}, \ldots, C_{n_m}; C_n)$ instead of $(T, \sigma, \{f_i\}_{\mathbf{m}}, f) \in \Omega(C_{n_1}, \ldots, C_{n_m}; C_n)$ and leave the additional data implicit.*

The definition above implies that each inner edge $e \in T_0$ lies in the image of exactly two inclusions of corollas. If the edge e is a leaf or the root of T, then it lies in the image of an inclusion of a unique corolla as well as in the image of the active morphism f. In particular, the set $\coprod_{1 \leq i \leq m} f_i^{-1}(e) \coprod f^{-1}(e)$ always consists of two elements.

Lemma 8.7. *There is an ordinary symmetric operad $\mathcal{O}_\Omega$ whose colours are corollas in Ω and whose multihoms are of the form $\Omega(C_{n_1}, \ldots, C_{n_m}; C_n)$.*

Proof. For corollas $C_{n_1^1}, \ldots, C_{n_m^1}, \ldots, C_{n_1^k}, \ldots, C_{n_m^k}, C_{n^1}, \ldots, C_{n^k}$ and C_n, we have to define a composition map μ:

$$(\Omega(C_{n_1^1}, \ldots, C_{n_m^1}; C_{n^1}) \times \ldots \times \Omega(C_{n_1^k}, \ldots, C_{n_m^k}; C_{n^k})) \times \Omega(C_{n^1}, \ldots, C_{n^k}; C_n) \xrightarrow{\mu} \Omega(C_{n_1^1}, \ldots, C_{n_m^k}; C_n)$$

which satisfies the associativity and unitality conditions. If $T^j \in \Omega(C_{n_1^j}, \ldots, C_{n_m^j}; C_{n^j})$, for $1 \leq j \leq k$, and $T \in \Omega(C_{n^1}, \ldots, C_{n^k}; C_n)$, then each inner edge of T corresponds to a leaf and the root of exactly two corollas. Hence, we can define $\mu(T^1, \ldots, T^k, T)$ to be the tree in Ω given by grafting the trees T^k along the inner edges of T. We can think of $\mu(T^1, \ldots, T^k, T)$ as the tree given by replacing each corolla in T by the corresponding tree T^j. It is easy to check that the μ is associative and respects the unit. $\square$

It is not difficult to see that an $\mathcal{O}_\Omega$-algebra in the category of sets is the same as a symmetric monochromatic operad. In the following, we will extend $\mathcal{O}_\Omega$ to a simplicial coloured operad OOp_X. For this reason we first need the following definition.

Definition 8.8. Suppose $(C_{n_1}, \{x_{1_i}\}_{1_i \in \mathbf{n_1+1}}), \ldots, (C_{n_m}, \{x_{m_i}\}_{m_i \in \mathbf{n_m+1}})$ and $(C_n, \{x_i\}_{i \in \mathbf{n+1}})$ are corollas whose edges are labelled by elements in X and let $(T, \sigma, \{f_i\}_{i \in \mathbf{m}}, f)$ be an element in $\Omega(C_{n_1}, \ldots, C_{n_m}; C_n)$. For an edge $e \in T_0$, let x' and x'' be two objects in $\coprod_{k \in \mathbf{m}} \{x_{k_i}\}_{k_i \in \mathbf{n_k+1}} \coprod \{x_i\}_{i \in \mathbf{n+1}}$ corresponding to the two elements in $\coprod_{1 \leq i \leq m} f_i^{-1}(e) \coprod f^{-1}(e)$. We define

$$K^T(e) = \mathrm{Hom}_X(x', x'') = \begin{cases} * & \text{if } x' = x'', \\ \emptyset & \text{otherwise.} \end{cases}$$

Remark 8.9. *Let $p_X \colon \Omega_X^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ be as in Definition 2.66. Suppose there exist corollas $(C_{n_1}, \{x_{1_i}\}_{1_i \in \mathbf{n_1+1}}), \ldots, (C_{n_m}, \{x_{m_i}\}_{m_i \in \mathbf{n_m+1}}), (C_n, \{x_i\}_{i \in \mathbf{n+1}})$ and an object $\overline{T} \in \Omega_X^{\mathrm{op}}$ such that:*

- *We have $p_X(\overline{T}) = T$.*
- *The corollas in $\overline{T}$ are given by $(C_{n_1}, \{x_{1_i}\}_{1_i \in \mathbf{n_1+1}}), \ldots, (C_{n_m}, \{x_{m_i}\}_{m_i \in \mathbf{n_m+1}})$.*
- *The labelling of leaves and root of $\overline{T}$ and $(C_n, \{x_i\}_{i \in \mathbf{n+1}})$ coincide.*

Then there an object $(T, \sigma, \{f_i\}_{i \in \mathbf{m}}, f) \in \Omega(C_{n_1}, \ldots, C_{n_m}; C_n)$ such that that $K^T(e) = *$ for every edge $e \in T_0$.

Definition 8.10. Define OOp_X to be the coloured symmetric operad determined by the following properties:

1. The collection of colours of OOp_X is the collection of corollas whose edges are labelled by elements in X, i.e. we can identify the colours with objects in $\Omega(X)$.

2. For colours $(C_{n_1}, \{x_{1_i}\}_{1_i \in \mathbf{n_1}+\mathbf{1}}), \dots, (C_{n_m}, \{x_{m_i}\}_{m_i \in \mathbf{n_m}+\mathbf{1}})$ and $(C_n, \{x_i\}_{i \in \mathbf{n}+\mathbf{1}})$, the multi-mapping space

$$\mathrm{Mul}_{\mathrm{OOp}_X}((C_{n_1}, \{x_{1_i}\}_{1_i \in \mathbf{n_1}+\mathbf{1}}), \dots, (C_{n_m}, \{x_{m_i}\}_{m_i \in \mathbf{n_m}+\mathbf{1}}); (C_n, \{x_i\}_{i \in \mathbf{n}+\mathbf{1}})) \tag{8.1}$$

   is defined to be the set

$$\coprod_{T \in \Omega(C_{n_1}, \dots, C_{n_m}; C_n)} \prod_{e \in T_0} K^T(e).$$

3. The composition is induced by the composition of the symmetric operad $\mathcal{O}_\Omega$ (see Lemma 8.7) and that of the set X.

In order to ease the notation, we will write OOp_X also for the ∞-operad associated to the simplicial operad OOp_X and we write $\overline{C}_n$ instead of $(C_n, \{x_i\}_{i \in \mathbf{n}+\mathbf{1}})$. In particular, an object in the ∞-operad OOp_X lying over $\langle m \rangle \in \mathcal{F}_*$ is of the form $(\overline{C}_{n_1}, \dots, \overline{C}_{n_m})$.

Remark 8.11. *By the definitions of $\Omega(C_{n_1}, \dots, C_{n_m}; C_n)$ and $K^T(e)$, we see that the multi-hom set 8.1 from above is given by the collection of all possible trees $\overline{T}$ whose edges are labelled by elements in X such that its corollas are given by $\overline{C}_{n_1}, \dots, \overline{C}_{n_m}$ and its labelled leaves and labelled root coincide with those of C_n. In particular, the underlying tree T by forgetting the labelling is an element of $\Omega(C_{n_1}, \dots, C_{n_m}; C_n)$.*

Remark 8.12. *Let us write $\overline{C_n}$ for an object $(C_n, \{x_i\}_{i \in \mathbf{n}+\mathbf{1}})$ in OOp_X. It follows from Definition 8.10 that, for every object $\overline{C_n}$, we have $\mathrm{Mul}_{\mathrm{OOp}_X}(;\overline{C_n}) = \emptyset$. This implies that, for two objects $(\overline{C}_{i_1}, \dots, \overline{C}_{i_m})$ and $(\overline{C}_{j_1}, \dots, \overline{C}_{j_n})$ of the ∞-operad OOp_X, every active morphism in $\mathrm{Map}_{\mathrm{OOp}_X}((\overline{C}_{i_1}, \dots, \overline{C}_{i_m}), (\overline{C}_{j_1}, \dots, \overline{C}_{j_n}))$ has to lie over a surjective active morphism $\langle m \rangle \to \langle n \rangle$ in $\mathcal{F}_*$.*

In order to have a better understanding of the definition above, we want to look at some of its variations.

Example 8.13. *If $X = \{*\}$, then $K^T(e) = *$ for every T and $e \in T_0$ and OOp_X coincides with the symmetric operad $\mathcal{O}_\Omega$ and for every symmetric monoidal category $\mathcal{V}$, the $\mathcal{O}_\Omega$-algebras are $\mathcal{V}$-enriched symmetric monochromatic operads.*

Example 8.14. *Let OOp^1_X denote the full simplicial suboperad of OOp_X spanned by colours which are labelled 1-corollas. The colours are of the form $(C_1, \{x, y\})$ for which we write (x, y), if x corresponds to the leaf of C_1. For colours $(x_1, y_1), \dots, (x_n, y_n), (x, y)$, an object $(T, \sigma, \{f_i\}_{i \in \mathbf{m}}, f) \in \Omega(C_1, \dots, C_1; C_1)$ is determined by a linear tree Δ^n and a permutation $\sigma \colon \mathbf{n} \to \mathbf{n}$. Hence, it follows from the definition that the multi-hom sets of the operad OOp^1_X can be described as follows:*

$$\mathrm{Mul}_{\mathrm{OOp}^1_X}((x_1, y_1), \dots, (x_n, y_n); (x, y)) = \begin{cases} * & \text{if } x = x_1, y_1 = x_2, \dots, y_n = y \\ \emptyset & \text{otherwise.} \end{cases}$$

We observe that an OOp^1_X-algebra in a symmetric symmetric monoidal category $\mathcal{V}$ is a $\mathcal{V}$-enriched category with X as its collection of objects.

Definition 8.15. Let X be a set and let $\mathcal{V}$ denote a symmetric monoidal ∞-category. We write $\mathrm{Alg}_{\mathrm{OOp}_X}(\mathcal{V})$ for the ∞-category of OOp_X-algebras in $\mathcal{V}$ and we call an object in $\mathrm{Alg}_{\mathrm{OOp}_X}(\mathcal{V})$ a *$\mathcal{V}$-enriched Ω-∞-operad with the set of colours X.*

Remark 8.16. *In Theorem 8.28 we will show that the ∞-category $\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V})$ is equivalent to $\mathrm{Alg}^{\mathrm{Set}}_{\mathcal{F}}(\mathcal{V})$ which implies that $\mathcal{V}$-enriched Ω-∞-operad are the $\mathcal{V}$-enriched $\mathcal{F}$-∞-operads introduced in the first chapter.*

Remark 8.17. *If we use the planar variant of Ω in Definition 8.10, then we obtain a simplicial operad OOp'_X whose algebras are ordinary planar/non-symmetric operads with X as its set of colours. Similar arguments as in the proof of Theorem 8.28 shows that the ∞-categories $\mathrm{Alg}_{\mathrm{OOp}'_X}(\mathcal{V})$ and $\mathrm{Alg}_{\mathrm{Ord}}(\mathcal{V})$ are equivalent, where* Ord *denotes the operad category of ordered finite sets.*

In the following we first recall the definition of the category associated to a symmetric monoidal category. Then we define a functor Θ_X from the category $\Omega(X)^{\mathrm{op}}$ to the category associated to the operad OOp_X. In the next section (see Proposition 8.3) we will realize that this functor allows us to regard OOp_X as a fibrant replacement of Ω_X^{op} in the model category of ∞-operads.

For a symmetric coloured operad $\mathcal{O}$, let us recall the construction of the category $\widetilde{\mathcal{O}}$ associated to $\mathcal{O}$ as defined in [Lur, Construction 2.1.1.7].

Definition 8.18. Let $\mathcal{O}$ be a symmetric coloured operad. The category $\widetilde{\mathcal{O}}$ associated to $\mathcal{O}$ is determined by the following properties:

1. The objects in $\widetilde{\mathcal{O}}$ are pairs $(\langle n\rangle, (c_1, \dots, c_n))$, where $\langle n\rangle \in \mathcal{F}_*$ and c_i are colours of $\mathcal{O}$ for $1 \le i \le n$.
2. For two objects $c = (\langle m\rangle, (c_1, \dots, c_n))$ and $c' = (\langle n\rangle, (c'_1, \dots, c'_n))$, the set $\mathrm{Hom}_{\widetilde{\mathcal{O}}}(c, c')$ is given by
$$\mathrm{Hom}_{\widetilde{\mathcal{O}}}(c, c') = \coprod_{\alpha\colon \langle m\rangle \to \langle n\rangle} \prod_{1\le j\le n} \mathrm{Mul}_{\mathcal{O}}(\{c_i\}_{i\in\alpha^{-1}(j)}, c'_j).$$
3. The composition in $\widetilde{\mathcal{O}}$ is induced by the compositions in $\mathcal{F}_*$ and $\mathcal{O}$.

Definition 8.19. Let X be a set. The functor $\Theta_X\colon \Omega(X)^{\mathrm{op}} \to \widetilde{\mathrm{OOp}_X}$ of categories is defined by the following data:

1. If $\overline{S} \in \Omega(X)^{\mathrm{op}}$, $\mathrm{Cr}(S) = \langle m\rangle \in \mathcal{F}_*$ and $\overline{C}_{n_1}, \dots, \overline{C}_{n_m}$ denote the corollas of $\overline{S}$, then we define
$$\Theta_X(\overline{S}) = (\langle m\rangle, (\overline{C}_{n_1}, \dots, \overline{C}_{n_m}) \in \widetilde{\mathrm{OOp}_X}.$$
2. Let $\overline{S} = (S, \{x_i\}_{i\in S_0})$ and $\overline{T} = (T, \{y_i\}_{i\in T_0})$ be objects in $\Omega(X)^{\mathrm{op}}$ and let $\mathrm{Cr}(S) = \langle m\rangle, \mathrm{Cr}(T) = \langle n\rangle$. If $\overline{C}_{i_1}, \dots, \overline{C}_{i_m}$ and $\overline{C}_{j_1}, \dots, \overline{C}_{j_n}$ are the corollas of $\overline{S}$ and $\overline{T}$, respectively, then, by definition we have
$$\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{S}, \overline{T}) = \coprod_{\alpha\colon T\to S\in\Omega} \prod_{e\in T_0} \mathrm{Hom}_X(x_{\alpha(e)}, y_e) \text{ and}$$

$$\mathrm{Hom}_{\widetilde{\mathrm{OOp}_X}}(\Theta_X(\overline{S}),\Theta_X(\overline{T})) = \coprod_{\mathrm{Cr}(\alpha)\colon \langle m\rangle\to\langle n\rangle}\prod_{k\in\mathbf{n}} \mathrm{Mul}_{\mathrm{OOp}_X}(\{\overline{C}_{i_l}\}_{l\in\mathrm{Cr}(\alpha)^{-1}(k)},\overline{C}_{j_k}).$$

We see that the set $\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{S},\overline{T})$ is given by $\{\alpha\in\mathrm{Hom}_\Omega(T,S)\colon x_{\alpha(e)}=y_e,\forall e\in T_0\}$. Since a map $\alpha\colon T\to S$ in Ω is determined by the images of each corolla in T by Corollary 2.62, if we write $\alpha_{|C_{j_k}}: C_{j_k}\to S$ for the restrictions, then we can identify α with the tuple $(\alpha_{|C_{j_1}},\dots,\alpha_{|C_{j_n}})$. Moreover, Corollary 2.62 implies that, every map in Ω^{op} is determined by its image under Cr, which means the functor $\mathrm{Cr}\colon\Omega^{\mathrm{op}}\to\mathcal{F}_*$ induces an inclusion of sets $\mathrm{Hom}_{\Omega^{\mathrm{op}}}(S,T)\to\mathrm{Hom}_{\mathcal{F}_*}(\langle m\rangle,\langle n\rangle)$. Therefore, this inclusion of Hom-sets defines an assignment $\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{S},\overline{T})\to\mathrm{Hom}_{\widetilde{\mathrm{OOp}_X}}(\Theta_X(\overline{S}),\Theta_X(\overline{T}))$, if each tuple $(\alpha_{|C_{j_1}},\dots,\alpha_{|C_{j_n}})$ can be identified with an element in

$$\prod_{k\in\mathbf{n}}\mathrm{Mul}_{\mathrm{OOp}_X}(\{\overline{C}_{i_l}\}_{l\in\mathrm{Cr}(\alpha)^{-1}(k)},\overline{C}_{j_k}) = \prod_{k\in\mathbf{n}}\Big(\coprod_{R\in\Omega(\{C_{i_l}\}_{l\in\mathrm{Cr}(\alpha)^{-1}(k)};C_{j_k})}\prod_{e\in R_0}K^R(e)\Big).$$

This can be done by the following construction: For every corolla C_{j_k} of T, the restriction $\alpha_{|C_{j_k}}: C_{j_k}\to S$ determines and is determined by a unique subtree R_{j_k} of S containing all the edges which lies on a path from the image of a leaf in C_{j_k} to the root of C_{j_k} under $\alpha_{|C_{j_k}}$. Since there exists a unique labelled subtree $\overline{R}_{j_k}\subseteq\overline{S}$ lying over R_{j_k}, Remark 8.9 and the condition $x_{\alpha(e)}=y_e,\forall e\in T_0$, implies that $K^{R_{j_k}}(e))=*$ for every edge $e\in R_{j_k}$. Hence, by identifying $(\alpha_{|C_{j_1}},\dots,\alpha_{|C_{j_n}})$ with $(R_{j_1},\dots,R_{j_n})$, we see that it is an element in

$$\prod_{k\in\mathbf{n}}\Big(\coprod_{R\in\Omega(\{C_{i_l}\}_{l\in\mathrm{Cr}(\alpha)^{-1}(k)};C_{j_k})}\prod_{e\in R_0}K^R(e)\Big).$$

Notation 8.20. *The assignment* $\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{S},\overline{T})\mapsto\mathrm{Hom}_{\widetilde{\mathrm{OOp}_X}}(\Theta_X(\overline{S}),\Theta_X(\overline{T}))$ *is compatible with compositions, hence, it defines a functor* $\Theta_X\colon\Omega(X)^{\mathrm{op}}\to\widetilde{\mathrm{OOp}_X}$. *By abusing the notation, we will write*

$$\Theta_X\colon\Omega(X)^{\mathrm{op}}\to\mathrm{OOp}_X$$

also for its coherent nerve.

Remark 8.21. *Since the assignment* $\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{S},\overline{T})\mapsto\mathrm{Hom}_{\widetilde{\mathrm{OOp}_X}}(\Theta_X(\overline{S}),\Theta_X(\overline{T}))$ *is a composite of inclusions, the functor* Θ_X *is fully faithful.*

8.3 Approximation

Lemma 8.22. *Suppose* $f\colon(\langle m\rangle,\overline{C}_{i_1},\dots,\overline{C}_{i_m})\to(\langle n\rangle,\overline{C}_{j_1},\dots,\overline{C}_{j_n})$ *is a morphism in* OOp_X *lying over an active map* $\alpha\colon\langle m\rangle\to\langle n\rangle$ *and suppose there exists an object* $\overline{T}\in\Omega(X)^{\mathrm{op}}$ *such that* $\Theta_X(\overline{T})=(\langle n\rangle,\overline{C}_{j_1},\dots,\overline{C}_{j_n})$. *Then we can lift* f *to a morphism* $\overline{f}\colon\overline{S}\to\overline{T}$ *in* $\Omega(X)$.

Proof. We will first show that the map f allows us to define a tree S and a map $\tilde{f}\colon S\to T$ in Ω^{op} and then we see that the labelling of edges of corollas $\overline{C}_{i_l}$, for $1\le l\le m$, defines a map $\overline{f}\colon\overline{S}\to\overline{T}$ in $\Omega(X)^{\mathrm{op}}$ lying over $\tilde{f}$.

By definition, the map f is of the form $\prod_{k\in\mathbf{n}}f_k$ such that $f_k\in\mathrm{Mul}_{\mathrm{OOp}_X}(\{\overline{C}_{i_l}\}_{l\in\alpha^{-1}(k)},\overline{C}_{j_k})$, for $k\in\mathbf{n}$. By Definition 8.10, each map f_k determines an object $S^k\in\Omega(\{C_{i_l}\}_{l\in\alpha^{-1}(k)};C_{j_k})$

such that there is an active map $C_{j_k} \to S^k$ in Ω and the corollas in S^k are elements of the set $\{C_{i_l}\}_{l\in\alpha^{-1}(k)}$. Since f lies over an active morphism $\alpha\colon \langle m\rangle \to \langle n\rangle$, Remark 8.12 guarantees that α is surjective. Therefore, the collection of corollas in all the trees S^k, $k \in \mathbf{n}$, coincides with the set $\coprod_{1\leq l\leq m}\{C_{i_l}\}$ and we want to glue these trees S^k to one tree S. Let each tree S^k be given by the polynomial endofunctor

$$S_0^k \leftarrow S_2^k \to S_1^k \to S_0^k$$

and let R_T denote the root of T. We define $S \in \Omega$ to be the object given by the polynomial endofunctor

$$S_0 \leftarrow S_2 \to S_1 \to S_0,$$

where $S_0 = \coprod_{k\in\mathbf{n}} S_2^k \coprod\{R_T\}$, $S_1 = \coprod_{k\in\mathbf{n}} S_1^k$ and $S_2 = \coprod_{k\in\mathbf{n}} S_2^k$. [Koc11, Proposition 1.1.19] implies that the tree S is given by successive grafting of the trees S_k indexed by the set of inner edges of T. Roughly speaking, we can think of S as be given by replacing each corolla C_{j_k} in T by the tree S^k. This is possible by the existence of an active morphisms $C_{j_k} \to S^k$ determined by f_k.

By Definition 2.59, defining a map $\tilde{f}\colon S \to T$ in Ω^{op} is equivalent to providing a commutative diagram diagram of the form

$$\begin{tikzcd} T_0 \arrow[d, "\psi_0"'] & \mathrm{sub}'(T) \arrow[l] \arrow[r] \arrow[d, "\psi_2"'] & \mathrm{sub}(T) \arrow[r] \arrow[d, "\psi_1"'] & T_0 \arrow[d, "\psi_0"'] \\ S_0 & \mathrm{sub}'(S) \arrow[l] \arrow[r] & \mathrm{sub}(S) \arrow[r] & S_0 \end{tikzcd} \tag{8.2}$$

such that the square in the middle is a Cartesian square. The set of inner edges of the tree S^k is given by $S_1^k \times_{S_0^k} S_2^k$. Since every active map $C_{j_k} \to S_j$ is boundary preserving, the set S_2^k which is the set of all inner edges and leaves of S^k coincides with the set $S_1^k \times_{S_0^k} S_2^k \coprod \mathbf{j_k}$. Since $\coprod_{1\leq k\leq n} \mathbf{j_k} \coprod\{R_T\} = T_0$, we have $S_0 = \coprod_{1\leq k\leq n} S_2^k \coprod\{R_T\} = \coprod_{1\leq k\leq n} S_1^k \times_{S_0^k} S_2^k \coprod T_0$. We define $\psi_0\colon T_0 \to S_0$ to be given by the canonical inclusion.

If $T' \in \mathrm{sub}(T)$, then by [Koc11, Definition 1.1.24], it is given by a grafting of a set corollas $\{C_{j_k}\}_{k\in K}$ of T. Since the maps f_j determine active maps $C_{j_k} \to S^k$, we have these maps also determine a map $S' \to T'$, where S' is given by grafting of trees S^k for $k \in K$. In particular, we obtain a map $\psi_1\colon \mathrm{sub}(T) \to \mathrm{sub}(S)$. Now, by defining $\psi_2\colon \mathrm{sub}'(T) \to \mathrm{sub}'(S)$ to be given by the pullback of ψ_1, we obtain a map $\tilde{f}\colon S \to T$ in Ω^{op}.

It follows from the construction that the corollas in S are given by $C_{i_1}, \dots, C_{i_m}$. Therefore, the object $(\langle m\rangle, \overline{C}_{i_1}, \dots, \overline{C}_{i_m}) \in \mathrm{OOp}_X$ defines a labelling of the edges of S. Since the map $\tilde{f}$ is induced by f, it respects the labelling and there is a map $\overline{f}\colon \overline{S} \to \overline{T}$ lying over $\tilde{f}$ and it is clear that $\Theta_X(\overline{f}) = f$. □

Let us recall the definition of an approximation in the sense of [Lur, Definition 2.3.3.6].

Definition 8.23. Suppose $p\colon \mathcal{O} \to \mathcal{F}_*$ is an ∞-operad and $\mathcal{C}$ an ∞-category. We call a categorical fibration of simplicial sets $f\colon \mathcal{C} \to \mathcal{O}$ an *approximation to* $\mathcal{O}$, if the following conditions are satisfied:

1. Let $p' = p \circ f$, let $c \in \mathcal{C}$ be an object and let $p'(c) = \langle n\rangle$. For every $1 \leq i \leq n$, there is a locally p'-coCartesian morphism $\alpha_i\colon c \to c_i$ in $\mathcal{C}$ lying over the inert map $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ given by the projection at the i-th element and the morphism $f(\alpha_i)$ in $\mathcal{O}$ is inert.

2. Let $c \in \mathcal{C}$ and let $\alpha\colon u \to f(c)$ be an active morphism in $\mathcal{O}$. There exists an f-Cartesian morphism $\overline{\alpha}\colon \overline{u} \to c$ lifting α.

Proposition 8.24. *There exists a commutative diagram*

$$\begin{tikzcd} \Omega(X)^{\mathrm{op}} \arrow[rr, "\simeq"] \arrow[dr, "\Theta_X"'] & & \Omega'(X)^{\mathrm{op}} \arrow[dl, "\Theta'_X"] \\ & \mathrm{OOp}_X, & \end{tikzcd}$$

such that the upper horizontal map is an equivalence of ∞*-categories and* Θ'_X *is an approximation to* OOp_X.

Proof. We can factorize Θ_X into an equivalence followed by a categorical fibration Θ'_X. Since Cartesian and coCartesian morphisms are preserved by equivalences, the categorical fibration Θ'_X is an approximation to OOp_X, if the functor Θ_X satisfies the two conditions of the previous definition and this is what we want to check in the following.

1. Let $p\colon \mathrm{OOp}_X \to \mathcal{F}_*$ denote the structure map of the ∞-operad OOp_X and let p' denote the composite $p' = p \circ \Theta_X$. It follows from the definition of Θ_X that the following diagram commutes

$$\begin{tikzcd} \Omega(X)^{\mathrm{op}} \arrow[r, "\Theta_X"] \arrow[d, "p_X"'] & \mathrm{OOp}_X \arrow[d, "p"] \\ \Omega^{\mathrm{op}} \arrow[r, "\mathrm{Cr}"'] & \mathcal{F}_*, \end{tikzcd}$$

where $p_X\colon \Omega(X)^{\mathrm{op}} \to \Omega$ is as defined in Definition 2.66 For an object $\overline{S} = (S, \{x_i\}_{i \in S_0}) \in \Omega(X)$ and $1 \le i \le m$, let $\rho^i\colon p'(\overline{S}) = \langle m \rangle \to \langle 1 \rangle$ denote the inert morphism in $\mathcal{F}_*$ given by the i-th projection. Since OOp_X is an ∞-operad, we can lift ρ^i to a p-coCartesian morphism $\tilde{\rho}^i\colon \overline{S} \to \overline{C}_n$ in OOp_X, where $\overline{C}_n$ is a corolla in $\overline{S}$. In particular, C_n is a corolla in S and there exists a map $\alpha\colon S \to C_n$ in Ω^{op} which can be lifted to a p_X-coCartesian morphism $\overline{\alpha}\colon \overline{S} \to \overline{C}_n$ in $\Omega(X)^{\mathrm{op}}$, because $p_X\colon \Omega(X)^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ is a coCartesian fibration by Proposition 8.3. Clearly we have $\Theta_X(\overline{\alpha}) = \tilde{\rho}^i$ which is inert in OOp_X by construction.

Therefore, we only need to verify that $\overline{\alpha}$ is locally p'-coCartesian. Let $\Omega(X)_{\rho^i}$ be given by the pullback

$$\begin{tikzcd} \Omega(X)^{\mathrm{op}}_{\rho^i} \arrow[r] \arrow[d] & \Omega(X)^{\mathrm{op}} \arrow[d, "p'"] \\ \Delta^1 \arrow[r, "\rho^i"'] & \mathcal{F}_*, \end{tikzcd}$$

where the bottom horizontal map is induced by ρ^i. For an object $\overline{C}_m \in \Omega(X)^{\mathrm{op}}$, we have to show that the following commutative diagram induced by $\overline{\alpha}$ is a homotopy pullback square.

$$\begin{tikzcd}
\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}_{\rho^i}}(\overline{C}_n, \overline{C}_m) \arrow[r] \arrow[d] & \mathrm{Hom}_{\Omega(X)^{\mathrm{op}}_{\rho^i}}(\overline{S}, \overline{C}_m) \arrow[d] \\
\mathrm{Map}_{\Delta^1}([1],[1]) \arrow[r] & \mathrm{Map}_{\Delta^1}([0],[1])
\end{tikzcd}$$

Since $\mathrm{Map}_{\Delta^1}([1],[1]) \simeq \{*\} \simeq \mathrm{Map}_{\Delta^1}([0],[1])$, the bottom map is an equivalence and the vertical maps in the diagram are Kan fibrations and it suffices to show that upper horizontal map is also an equivalence. This follows from Definition 8.1 and the fact that $\overline{\alpha}$ is an inclusion of a corolla.

2. Let $f\colon (\langle m\rangle, (\overline{C}_{i_1}, \ldots, \overline{C}_{i_m}) \to (\langle n\rangle, (\overline{C}_{j_1}, \ldots, \overline{C}_{j_n})$ be an active morphism in OOp_X lying over $\alpha\colon \langle m\rangle \to \langle n\rangle$ and let $\overline{T} \in \Omega(X)^{\mathrm{op}}$ such that $\Theta_X(\overline{T}) = (\langle n\rangle, (\overline{C}_{j_1}, \ldots, \overline{C}_{j_n})$. According to Lemma 8.22, there exists an object $\overline{S}$ and a morphism $\overline{f}\colon \overline{S} \to \overline{T}$ in $\Omega(X)^{\mathrm{op}}$.

   We want to show that $\overline{f}$ is an Θ_X-Cartesian morphism, i.e. we have to show that, for every object $\overline{U} \in \Omega(X)^{\mathrm{op}}$, the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{U}, \overline{S}) \arrow[r, "\overline{f}_*"] \arrow[d] & \mathrm{Hom}_{\Omega(X)^{\mathrm{op}}}(\overline{U}, \overline{T}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{OOp}_X}(\Theta_X(\overline{U}), \Theta_X(\overline{S})) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathrm{OOp}_X}(\Theta_X(\overline{U}), \Theta_X(\overline{T}))
\end{tikzcd}$$

   is a homotopy pullback square. The vertical maps are Kan fibrations, because they are maps between sets. Since the functor Θ_X is fully faithful, the vertical maps are inclusions and the fibres are equivalent. Hence, the commutative diagram above is a homotopy pullback square.

□

Corollary 8.25. *For a set X, the ∞-operad $p\colon \mathrm{OOp}_X \to \mathcal{F}_*$ is a fibrant replacement of $p \circ \Theta_X\colon \Omega(X)^{\mathrm{op}} \to \mathcal{F}_*$ in the operadic model structure* $\mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}}$.

Proof. By Proposition 8.24 and [Lur, Lemma 2.3.3.10], there exists a weak approximation to $\Theta'_X\colon \Omega'(X)^{\mathrm{op}} \to \mathrm{OOp}_X$. Since $\Omega(X)^{\mathrm{op}}$ is equivalent to $\Omega'(X)^{\mathrm{op}}$ and the fibres over $\langle 1\rangle$ of Ω_X^{op} and OOp_X are equivalent, [Lur, Theorem 2.3.3.23.(1)] implies that Θ'_X is a weak equivalence in $\mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}}$. The equivalence $\Omega(X)^{\mathrm{op}} \simeq \Omega'(X)^{\mathrm{op}}$ implies that Θ_X is also a weak equivalence and OOp_X is a fibrant replace of $\Omega(X)^{\mathrm{op}}$. □

Definition 8.26. Let $*^\natural$ denote the categorical pattern on Δ^0 as defined in Remark 2.44. Let

$$\mathrm{sSet}^+_{/*^\natural} \times \mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}} \to \mathrm{sSet}^+_{/\mathcal{F}_*^{\natural,0}} \tag{8.3}$$

denote the left Quillen bifunctor adjunction provided by [Lur, Remark B.2.5]. We write

$$\mathrm{Cat}_\infty \times \mathrm{Op}_\infty \to \mathrm{Op}_\infty$$

for the induced functor of ∞-categories and

$$\mathrm{Alg}_{(-)/\mathcal{F}_*}(-)\colon \mathrm{Op}_\infty^{\mathrm{op}} \times \mathrm{Op}_\infty \to \mathrm{Cat}_\infty$$

for the the functor induced by adjunction.

Definition 8.27. • Let $\mathrm{OOp}_{(-)}\colon \mathrm{Set} \to \mathrm{Op}_\infty$ denote the functor which assigns to each set X to the ∞-operad OOp_X For a symmetric monoidal ∞-category $\mathcal{V}$, we write $\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V})$ for the ∞-category given by the pullback square

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V}) \arrow[r] \arrow[d] & \mathrm{Alg}(\mathcal{V}) \arrow[d] \\
\mathrm{Set} \arrow[r, "\mathrm{OOp}_{(-)}"'] & \mathrm{Op}_\infty,
\end{tikzcd} \tag{8.4}$$

where the right vertical map is the Cartesian fibration associated to the functor

$$\mathrm{Alg}_{(-)/\mathcal{F}_*}(\mathcal{V})\colon \mathrm{Op}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty .$$

• Let $\Omega^{\mathrm{op}}_{(-)}\colon \mathrm{Set} \to \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$ denote the functor which assigns each set X to Ω^{op}_X (see Definition 2.66). We write $\mathrm{Alg}^{\mathrm{Set}}_{\Omega}(\mathcal{V})$ for the ∞-category given by the pullback

$$\begin{tikzcd}
\mathrm{Alg}^{\mathrm{Set}}_{\Omega}(\mathcal{V}) \arrow[r] \arrow[d] & \Omega\text{-}\mathrm{Alg}(\mathcal{V}) \arrow[d] \\
\mathrm{Set} \arrow[r, "\Omega^{\mathrm{op}}_{(-)}"'] & \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}},
\end{tikzcd} \tag{8.5}$$

where the right vertical map is the Cartesian fibration associated to the functor

$$\mathrm{Alg}^{\mathrm{gen}}_{(-)/\Omega^{\mathrm{op}}}(\mathrm{Cr}^* \mathcal{V})\colon \mathrm{coCart}^{\Omega,\mathrm{gen},\mathrm{op}}_{\mathrm{Seg}} \to \mathrm{Cat}_\infty$$

introduced in Definition 2.74.

• We write $\mathrm{Alg}_\Omega(\mathcal{V})$ for the ∞-category given by the pullback

$$\begin{tikzcd}
\mathrm{Alg}_{\Omega}(\mathcal{V}) \arrow[r] \arrow[d] & \Omega\text{-}\mathrm{Alg}(\mathcal{V}) \arrow[d] \\
\mathcal{S} \arrow[r] & \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}},
\end{tikzcd}$$

where the bottom horizontal map $\mathcal{S} \to \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$ carries every ∞-groupoid X to the object $p_X\colon \Omega^{\mathrm{op}}_X \to \Omega^{\mathrm{op}}$ in $\mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$. The objects in $\mathrm{Alg}_\Omega(\mathcal{V})$ are called *$\mathcal{V}$-enriched Ω-operads.*

Theorem 8.28. *For every symmetric monoidal ∞-category $\mathcal{V}$, there is an equivalence of ∞-categories*

$$\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V}) \simeq \mathrm{Alg}^{\mathrm{Set}}_{\Omega}(\mathcal{V}).$$

Proof. By Lemma 2.75 and Lemma 2.76, there exists an adjunction of ∞-categories

$$\mathrm{Cr}_!\colon \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}} \rightleftarrows \mathrm{Op}_\infty : \mathrm{Cr}^* .$$

The commutative diagram

$$\begin{tikzcd} \Omega(X)^{\mathrm{op}} \arrow[r, "\Theta_X"] \arrow[d, "p_X"'] & \mathrm{OOp}_X \arrow[d, "p"] \\ \Omega^{\mathrm{op}} \arrow[r, "\mathrm{Cr}"'] & \mathcal{F}_*. \end{tikzcd}$$

and Corollary 8.25 show that the map Θ_X induces an equivalence $\mathrm{Cr}_!\,\Omega(X)^{\mathrm{op}} \to \mathrm{OOp}_X$ in the ∞-category Op_∞. Together with equivalence $\theta\colon \Omega(X)^{\mathrm{op}} \to \Omega_X^{\mathrm{op}}$ of Proposition 8.3 we obtain an equivalence $\mathrm{Cr}_!\,\Omega_X^{\mathrm{op}} \to \mathrm{OOp}_X$. Since θ is natural in X, we can extend the commutative triangle

$$\begin{tikzcd} & \mathrm{Set} \arrow[dl, "\Omega^{\mathrm{op}}_{(-)}"'] \arrow[dr, "\mathrm{OOp}_{(-)}"] & \\ \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}} \arrow[rr, "\mathrm{Cr}_!"] & & \mathrm{Op}_\infty \end{tikzcd}$$

to the following commutative diagram

$$\begin{tikzcd} & \mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V}) \arrow[rr] \arrow[dd] & & \mathrm{Alg}(\mathcal{V}) \arrow[dd] \\ \mathrm{Alg}^{\mathrm{Set}}_{\Omega}(\mathcal{V}) \arrow[ur] \arrow[rr] \arrow[dd] & & \Omega\text{-}\mathrm{Alg}(\mathcal{V}) \arrow[ur] \arrow[dd] & \\ & \mathrm{Set} \arrow[rr, "\mathrm{OOp}_{(-)}"'] & & \mathrm{Op}_\infty \\ \mathrm{Set} \arrow[ur, "="] \arrow[rr, "\Omega^{\mathrm{op}}_{(-)}"'] & & \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}, \arrow[ur, "\mathrm{Cr}_!"'] & \end{tikzcd}$$

where the front and back squares are the pullbacks squares 8.4 and 8.5 of the previous definition. Therefore, if the commutative square on the right hand side of the cube is a pullback, then, by [Lur09, Lemma 4.4.2.1], the left square is a pullback and the ∞-categories $\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V}) \simeq \mathrm{Alg}^{\mathrm{Set}}_{\Omega}(\mathcal{V})$ are equivalent.

The square on right hand side is a pullback if and only if the functor $\mathrm{Alg}^{\mathrm{gen}}_{(-)/\Omega^{\mathrm{op}}}(\mathrm{Cr}^*\,\mathcal{V})$ associated to the Cartesian fibration $\Omega\text{-}\mathrm{Alg}(\mathcal{V}) \to \mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}$ is is equivalent to the composite $\mathrm{Alg}_{(-)/\mathcal{F}_*}(\mathcal{V}) \circ \mathrm{Cr}_!\colon \mathrm{coCart}^{\Omega,\mathrm{gen},\mathrm{op}}_{\mathrm{Seg}} \to \mathrm{Cat}_\infty$. In other words, for every object $\mathcal{O}_\Omega \in \mathrm{coCart}^{\Omega,\mathrm{gen},\mathrm{op}}_{\mathrm{Seg}}$, we have to provide an equivalence $\mathrm{Alg}_{\mathrm{Cr}_!\,\mathcal{O}_\Omega/\mathcal{F}_*}(\mathcal{V}) \simeq \mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}_\Omega/\Omega^{\mathrm{op}}}(\mathrm{Cr}^*\,\mathcal{V})$ of ∞-

categories. By Yoneda Lemma, we only have to show that there is an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}_{\mathrm{Cr}_!\,\mathcal{O}_\Omega/\mathcal{F}_*}(\mathcal{V})) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}_\Omega/\Omega^{\mathrm{op}}}(\mathrm{Cr}^*\,\mathcal{V}))$$

for every ∞-category $\mathcal{C}$. By adjunctions, we have

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}_{\mathrm{Cr}_!\,\mathcal{O}_\Omega/\mathcal{F}_*}(\mathcal{V})) \simeq \mathrm{Map}_{\mathrm{Op}_\infty}(\mathcal{C} \times \mathrm{Cr}_!\,\mathcal{O}_\Omega, \mathcal{V})$$

and

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}^{\mathrm{gen}}_{\mathcal{O}_\Omega/\Omega^{\mathrm{op}}}(\mathrm{Cr}^*\,\mathcal{V})) \simeq \mathrm{Map}_{\mathrm{coCart}^{\Omega,\mathrm{gen}}_{\mathrm{Seg}}}(\mathcal{C} \times \mathcal{O}_\Omega, \mathrm{Cr}^*\,\mathcal{V}) \simeq \mathrm{Map}_{\mathrm{Op}_\infty}(\mathrm{Cr}_!(\mathcal{C} \times \mathcal{O}_\Omega), \mathcal{V}).$$

The claim follows from the construction of the Quillen bifunctor 8.3 in Definition 8.26 which implies that $\mathcal{C} \times \mathrm{Cr}_!\,\mathcal{O}_\Omega \simeq \mathrm{Cr}_!(\mathcal{C} \times \mathcal{O}_\Omega)$. □

By replacing Δ_Φ by Ω, the proof of Theorem 4.22 implies the following result:

Proposition 8.29. *Let $\mathcal{V}$ be a presentable symmetric monoidal ∞-category. There is an equivalence of ∞-categories* $\mathrm{Alg}_\Omega(\mathcal{V}) \simeq \mathrm{P}_{\mathrm{Seg}}(\mathcal{V}^\vee_\Omega)$.

By an examination of the proof of [GH15, Theorem 5.3.17], we realize that the following theorem holds:

Theorem 8.30. *There is an equivalence of ∞-categories:*

$$\mathrm{Alg}_\Omega(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Alg}^{\mathrm{Set}}_\Omega(\mathcal{V})[\mathrm{FFES}^{-1}].$$

Definition 8.31. A morphism $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_\Omega(\mathcal{V})$ is called *fully faithful* or *essentially surjective*, if it one in $\mathrm{Alg}_\Phi(\mathcal{V})$ under the equivalence $\mathrm{Alg}_\Omega(\mathcal{V}) \simeq \mathrm{Alg}_\Phi(\mathcal{V})$ provided by Theorem 7.1. As in Notation 6.11, we write FFES for the class of morphisms in $\mathrm{Alg}_\Omega(\mathcal{V})$ which are fully faithful and essentially surjective morphisms.

Similarly, we write FFES for the class of morphisms in $\mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V})$ which are fully faithful and essentially surjective in $\mathrm{Alg}^{\mathrm{Set}}_\Omega(\mathcal{V})$ under the equivalence provided by Theorem 8.28.

Let $\mathcal{O}$ be an object in $\mathrm{Alg}_\Omega(\mathcal{V})$. By unwinding the definition above, we see that a morphism $f\colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Alg}_\Omega(\mathcal{V})$ is fully faithful if the morphism $\mathcal{O}(C_n(x_1, \ldots, x_n; x)) \to \mathrm{P}(C_n(f(x_1), \ldots, f(x_n); f(x)))$ is an equivalence in $\mathcal{V}_{\langle 1\rangle}$ for every object $C_n(x_1, \ldots, x_n; x) \in \Omega^{\mathrm{op}}_X$.

Theorem 8.32. *For every symmetric monoidal ∞-category which is possibly large, there is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathcal{F}}(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Alg}^{\mathrm{Set}}_{\mathrm{OOp}}(\mathcal{V})[\mathrm{FFES}^{-1}].$$

Proof. If $\widehat{\mathrm{P}}(\mathcal{V})$ denotes the ∞-category of presheaves of large spaces on $\mathcal{V}$, then by [Lur, Corollary 4.8.1.12], there exists a symmetric monoidal structure on $\widehat{\mathrm{P}}(\mathcal{V})$ such that the Yoneda embedding $\mathcal{V} \to \widehat{\mathrm{P}}(\mathcal{V})$ is symmetric monoidal. Let $\widehat{\mathrm{Alg}}_{\mathcal{F}}(\widehat{\mathrm{P}}(\mathcal{V}))$ and $\widehat{\mathrm{Alg}}_\Omega(\widehat{\mathrm{P}}(\mathcal{V}))$ denote the very large ∞-category of $\widehat{\mathrm{P}}(\mathcal{V})$-enriched $\mathcal{F}$-∞-operads and $\mathcal{V}$-enriched Ω-operads, respectively. Then Theorem 7.1 provides an equivalence of ∞-categories

$$\mathrm{P}_{\mathrm{Seg}}(\widehat{\mathrm{P}}(\mathcal{V})^\vee) \simeq \mathrm{P}_{\mathrm{Seg}}(\widehat{\mathrm{P}}(\mathcal{V})^\vee_\Omega)$$

which induces the equivalence

$$\widehat{\mathrm{Alg}}_{\mathcal{F}}(\widehat{\mathrm{P}}(\mathcal{V})) \simeq \widehat{\mathrm{Alg}}_{\Omega}(\widehat{\mathrm{P}}(\mathcal{V}))$$

by the previous proposition. Since $\widehat{\mathrm{Alg}}_{\Omega}^{\mathrm{Set}}(\widehat{\mathrm{P}}(\mathcal{V}))$ is equivalent to $\widehat{\mathrm{Alg}}_{\mathrm{OOp}}^{\mathrm{Set}}(\widehat{\mathrm{P}}(\mathcal{V}))$ by Theorem 8.28, we want to see that the ∞-category $\mathrm{Alg}_{\mathcal{F}}(\mathcal{V}) \subseteq \widehat{\mathrm{Alg}}_{\mathcal{F}}(\widehat{\mathrm{P}}(\mathcal{V}))$ is equivalent to the ∞-category $\mathrm{Alg}_{\Omega}(\mathcal{V}) \subseteq \widehat{\mathrm{Alg}}_{\Omega}(\widehat{\mathrm{P}}(\mathcal{V}))$ under $f\colon \widehat{\mathrm{Alg}}_{\mathcal{F}}(\widehat{\mathrm{P}}(\mathcal{V})) \simeq \widehat{\mathrm{Alg}}_{\Omega}(\widehat{\mathrm{P}}(\mathcal{V}))$ given by the composite of equivalences above. But the equivalence $\Delta_{\mathcal{F},\mathrm{el}} \simeq \Omega_{\mathrm{el}}$ implies that, for every object $\mathfrak{c}(x_1, \ldots, x_n; x) \in \Delta^{\mathrm{op}}_{\mathcal{F},X}$, $\mathcal{O} \in \mathrm{Alg}_{\mathcal{F}}(\mathcal{V})$ and $f(\mathcal{O}) \in \mathrm{Alg}_{\Omega}(\mathcal{V})$, there is an equivalence $\mathcal{O}(x_1, \ldots, x_n; x) \simeq f(\mathcal{O})(x_1, \ldots, x_n; x) \in \mathcal{V}$. Hence, the equivalence f restricts to an equivalence $\mathrm{Alg}_{\mathcal{F}}(\mathcal{V}) \simeq \mathrm{Alg}_{\Omega}(\mathcal{V})$.

By Theorem 8.30, there is an equivalence between the following ∞-categories:

$$\mathrm{Alg}_{\Omega}(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Alg}_{\Omega}^{\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}].$$

Together with the equivalence $\mathrm{Alg}_{\Omega}^{\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Alg}_{\mathrm{OOp}}^{\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}]$ provided by Theorem 8.28 we then obtain an equivalence $\mathrm{Alg}_{\mathcal{F}}(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Alg}_{\mathrm{OOp}}^{\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}]$. $\square$

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