THE BROKEN CIRCUIT COMPLEX AND THE ORLIK–TERAO ALGEBRA OF A HYPERPLANE ARRANGEMENT

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List of publications

This cumulative thesis is based on the following papers:

[LR13] D. V. Le and T. Römer, *Broken circuit complexes and hyperplane arrangements*. J. Algebraic Combin. **38** (2013), no. 4, 989–1016.

[Le14] D. V. Le, *On the Gorensteinness of broken circuit complexes and Orlik–Terao ideals*. J. Combin. Theory Ser. A **123** (2014), no. 1, 169–185.

[Le16] D. V. Le, *Broken circuit complexes of series–parallel networks.* European J. Combin. **51** (2016), 12–36.

[LM15] D. V. Le and F. Mohammadi, *On the Orlik–Terao ideal and the relation space of a hyperplane arrangement*. Adv. in Appl. Math. **71** (2015), 34–51.

CHAPTER 1

Introduction

In this thesis we are mostly concerned with algebraic and combinatorial aspects of the theory of hyperplane arrangements. More specifically, we study the Orlik–Terao algebra of a hyperplane arrangement and the broken circuit complex of a matroid. These objects are closely related to each other and they both encode interesting information.

A hyperplane arrangement (or simply an arrangement) is merely a finite collection of codimension one subspaces in a finite dimensional vector space. Despite this simple definition, the study of arrangements has led to profound and beautiful results. By now there has developed an extensive and rapidly growing theory of hyperplane arrangements, which combines methods and inspirations from many branches of mathematics, including algebra, algebraic geometry, analysis, combinatorics, topology, differential geometry, group theory and representation theory. Applications of the theory are abundant, ranging from random walks, ranking patterns, braid group representations to knot theory and conformal field theory. We refer to the classical book by Orlik and Terao [**OT92**] for a comprehensive account of the theory as well as a general reference for facts about hyperplane arrangements used in this thesis. The reader may also consult the monographs [**OT07, OW07**] and the ongoing project [**CDF$^+$**] for recent progress in the field.

Let $\mathscr{A} = \{H_1,\dots,H_n\}$ be an arrangement in a vector space V of dimension ℓ over some field $\mathbb{K}$. Throughout this thesis, $\mathscr{A}$ is always assumed to be central, i.e., each hyperplane H_i is a linear subspace of V. (The reason is that the Orlik–Terao algebra is only defined for central arrangements. Note, however, that the results of Zaslavsky and Orlik–Solomon mentioned below can be extended to non-central arrangements.) We choose a basis $\{z_1,\dots,z_\ell\}$ for the dual space V^* of V and identify the symmetric algebra of V^* with the polynomial algebra $R=\mathbb{K}[z_1,\dots,z_\ell]$. Then each hyperplane H_i is the kernel of a unique (up to a constant) linear polynomial α_i in R. Let $\mathsf{M}=\mathsf{M}(\mathscr{A}) := V-\bigcup_{i=1}^n H_i$ be the complement of the hyperplanes of $\mathscr{A}$. In its early stages, the study of arrangements mostly focused on real arrangements and one of the main problems was to count the number of regions (i.e., connected components) of M. Partial cases of this problems had been solved by many authors (see, e.g., [**Ste26, Schl52, Rob87, Bu43, Win66, Gr71**]) before Zaslavsky [**Za75**] gave a general solution for it. He found out that the solution depends only on the intersections of the hyperplanes, but not on their location. More precisely, let $L(\mathscr{A})$ be the intersection poset of $\mathscr{A}$ which consists of all the intersections of hyperplanes of $\mathscr{A}$ partially ordered by reverse inclusion. This poset is a lattice that encodes the combinatorics of $\mathscr{A}$. Define the characteristic polynomial of $\mathscr{A}$ as follows

$$\chi(\mathscr{A},t) = \sum_{X\in L(\mathscr{A})} \mu(X)t^{r(\mathscr{A})-r(X)},$$

where μ and r are the Möbius function and rank function of $L(\mathscr{A})$. Then Zaslavsky proved that the number of regions of M is the evaluation $(-1)^{r(\mathscr{A})}\chi(\mathscr{A},-1)$.

Another significant advance in arrangement theory was made by Orlik and Solomon [**OS80**] who studied the cohomology algebra of the complement M of a complex arrangement. Based on works of Arnol'd [**Ar69**] and Brieskorn [**Bri73**], they gave a presentation of $H^*(\mathsf{M})$ in terms of generators and relations. This algebra, which is now called the Orlik–Solomon algebra of $\mathscr{A}$ and has been much studied, is a quotient of an exterior algebra and depends only on the intersection lattice $L(\mathscr{A})$.

The aforementioned results of Zaslavsky and Orlik–Solomon gave rise to several open questions and conjectures, which ask whether certain properties of the arrangement $\mathscr{A}$ are *combinatorial*, i.e., depends only on the intersection lattice $L(\mathscr{A})$. We collect here some of them which are of great interest.

Over the past decades much of the work in the topology of arrangements has centered around the $K(\pi,1)$ problem, the problem of determining when M is a $K(\pi,1)$ space, i.e., when the higher homotopy groups $\pi_i(\mathsf{M})$ vanish for all $i>1$. The first result in this direction is due to Fadell and Neuwirth [**FN62**], who showed that the braid arrangement is $K(\pi,1)$. Here the braid arrangement $\mathscr{A}_{\ell-1}$ is the complex arrangement in $\mathbb{C}^\ell$ consisting of the hyperplanes $H_{i,j}=\ker(z_i-z_j)$ for $1\leq i<j\leq \ell$. This is the reflection arrangement of the symmetric group $\mathrm{Sym}(\ell)$ in the sense that it consists of reflecting hyperplanes of the reflections in $\mathrm{Sym}(\ell)$. Subsequently, the result of Fadell and Neuwirth was generalized in two directions.

The first generalization was to replace the symmetric group and the braid arrangement by a reflection group and the corresponding reflection arrangement. The pioneer in this direction was Brieskorn [**Bri73**], who proved the $K(\pi,1)$ property for some of the Coxeter arrangements – the reflection arrangements of finite Coxeter groups (which are also known as complexified real reflection groups). He conjectured that all Coxeter arrangements are $K(\pi,1)$. Coxeter arrangements have the property that their real parts are simplicial arrangements. Recall that a real arrangement is simplicial if every region of its complement is an open simplicial cone. Brieskorn's conjecture was settled by Deligne [**De72**], who actually proved the much stronger result that the complexification of a simplicial arrangement is $K(\pi,1)$. From this result it is natural to expect that all complex reflection arrangements are $K(\pi,1)$. Nakamura [**Na83**] and Orlik and Solomon [**OS88**] had confirmed that this is true for several classes of complex reflection arrangements quite a long time before a proof for the remaining cases was recently found by Bessis [**Be15**].

The second generalization of Fadell and Neuwirth's result comes from the fact that the braid arrangement is supersolvable. Supersovability was introduced by Stanley [**Sta72**] as a combinatorial sufficient condition for the factorization of the characteristic polynomial: all complex roots of the characteristic polynomial of a supersolvable arrangement are positive integers. In [**Te86**], Terao proved that supersolvable arrangements are precisely fiber-type arrangements introduced by Falk and Randell [**FR85**], who showed that these arrangements are $K(\pi,1)$. Thus every supersolvable arrangement is a $K(\pi,1)$ arrangement. Note, however, that most arrangements are not $K(\pi,1)$: a result due to Hattori [**Hat75**] implies that generic arrangements are not $K(\pi,1)$, except in trivial cases where the arrangements are Boolean or of rank at most 2.

Although some progress has been made in the search for necessary and sufficient conditions for $K(\pi,1)$-ness (see [**FR00**] for a survey), the $K(\pi,1)$ problem seems to be far from settled at the moment. Even an "easier" variant of this problem which asks whether the $K(\pi,1)$ property is combinatorial is still open [**FR00**]. It is worth mentioning here that a closely related question that whether the fundamental group $\pi_1(\mathsf{M})$ of M is a combinatorial invariant was answered in the negative by Rybnikov [**Ry11**] with a delicate example.

Apart from $K(\pi,1)$ arrangements, another fascinating class of arrangements are free arrangements, which were introduced by Terao [**Te80**]. The arrangement $\mathscr{A}$ is called free if the following module of $\mathscr{A}$-derivations is a free R-module:

$$D(\mathscr{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(R) \mid \theta(\alpha_i) \in \alpha_i R \text{ for } i = 1,\dots,n\},$$

where $\mathrm{Der}_{\mathbb{K}}(R) = \bigoplus_{i=1}^{\ell} R\frac{\partial}{\partial z_i}$ is the module of derivations of R over $\mathbb{K}$. This definition, which is purely algebraic, is a specialization of the notion of free divisors initially studied from analytic point of view by Saito [**Sa75, Sa80**]. The class of free arrangements includes all reflection arrangements [**Te89**] and supersolvable arrangements [**JT84**]. These results provide evidence for a famous conjecture of Saito [**Sa75**] that the classes of $K(\pi,1)$ arrangements and free arrangements coincide. However, this conjecture turned out to be false. In fact, there is no containment between the two classes. Terao [**Te80**] found out that some arrangements in a list of simplicial arrangements given by Grünbaum [**Gr71**] are not free (recall that these arrangements are $K(\pi,1)$ by Deligne's result). Later, Edelman and Reiner [**EdR95**] gave a family of free arrangements which are not $K(\pi,1)$. Several fundamental properties of free arrangements were proved by Terao. His factorization theorem asserts that the characteristic polynomial of a free arrangement $\mathscr{A}$ factorizes over positive integers, with roots being the degrees of a homogeneous basis for $D(\mathscr{A})$ [**Te80, Te81**]. He also proved the addition-deletion theorem which allows one to check the freeness of an arrangement inductively [**Te80**]. It was these results that led Terao [**Te80**] to a prominent conjecture that freeness of hyperplane arrangements is a combinatorial property. This has been one of motivating conjectures for much of the recent work on free arrangements. In several special cases the conjecture is known to be true. For instance, a result due to Stanley (see [**EdR94**]) affirms that a graphic arrangement is free if and only if it is supersolvable. Ziegler [**Zi90**] verified Terao's conjecture for arrangements whose underlying matroid is binary and arrangements over a finite field of at most 4 elements. He also gave an example of a free arrangement and a nonfree arrangement with isomorphic intersection lattices. However, these arrangements are defined over two fields of different characteristics. By now Terao's conjecture remains unsolved for arrangements defined over the same field. Another approach to this conjecture was proposed by Yuzvinsky [**Yu93b**] who proved that free arrangements form a Zariski open set in the space of all arrangements with a fixed intersection lattice. Recently, one promising technique for studying free arrangements and Terao's conjecture which involves multiarrangements introduced by Ziegler [**Zi89**] has been developed. For more information on this fruitful direction, the reader is referred, for example, to work by Abe and Yoshinaga [**Yo04, AY13, Yo14, Ab15**].

Assume that $\mathscr{A}$ is a complex arrangement. Then the homogeneous polynomial $Q = \prod_{i=1}^{n} \alpha_i$ defines a map $Q : \mathsf{M} \to \mathbb{C}^*$. This map is the projection of a fiber bundle, called the

Milnor fibration, and the typical fiber $F = Q^{-1}(1)$ is known as the Milnor fiber of $\mathscr{A}$. A long-standing open question about the Milnor fibration of an arrangement is that whether the homology of the Milnor fiber F is combinatorial (see [**FR00**]). There has been many efforts to tackle this problem. We refer to a paper of Papadima and Suciu [**PSu14**] and references therein for some recent progress on the problem.

Motivated by the idea that $K(\pi,1)$ arrangements should be extremal in some sense, Falk and Randell introduced in [**FR86**] the notion of 2-formal arrangements. An arrangement is called 2-formal if its relation space – which, roughly speaking, consists of dependencies among the hyperplanes of the arrangement – is generated by dependencies involving only 3 hyperplanes. Falk and Randell showed that 2-formality is a condition necessary (but not sufficient) for M to be a $K(\pi,1)$ space and for $\mathscr{A}$ to have a quadratic Orlik–Solomon algebra. They asked whether free arrangements are 2-formal, and whether 2-formality is a combinatorial property. Both questions were later answered by Yuzvinsky [**Yu93a**]. He proved that free arrangements are indeed 2-formal, yet 2-formality is not determined by the intersection lattice. Thus 2-formality is rather interesting, but it cannot be studied solely by combinatorial tools.

At this point it is worth noticing that the Orlik–Terao algebra, which was introduced by Orlik and Terao [**OT94**] to answer a question of Aomoto in the context of hypergeometric functions, can be used to characterize 2-formal arrangements. This fact, proved by Schenck and Tohăneanu [**ST09**], points out that the Orlik–Terao algebra encodes subtle information which is not included in the Orlik–Solomon algebra. Therefore, one might expect that the Orlik–Terao algebra would be a useful tool for studying properties which are not combinatorial (and perhaps those ones which are not yet known to be combinatorial or not). Furthermore, the following features of the Orlik–Terao algebra make it engaging in its own right:

(1) it has a presentation similar to that of the Orlik–Solomon algebra and can be regarded as a commutative analogue of this algebra [**PS06, ST09**],
(2) it is the coordinate ring of an irreducible variety called reciprocal plane, which plays a role in the study of a problem stemming from a model in theoretical neuroscience [**SSV13**],
(3) it degenerates flatly to a broken circuit algebra, i.e., the Stanley–Reisner ring of a broken circuit complex [**PS06**].

In this thesis we investigate the Orlik–Terao algebra and its companion, the broken circuit complex. This complex, on the one hand, serves as our main tool for studying the Orlik–Terao algebra. On the other hand, it deserves particular attention due to its importance.

The notion of broken circuit complexes was introduced by Whitney [**Wh32a**] and developed further by Rota [**Rot64**], Wilf [**Wil76**], and Brylawski [**Bry77**], originally from a combinatorial point of view: the face numbers (which constitute the so-called f-vector) of the broken circuit complex of a matroid are (in reverse order and up to sign) the coefficients of the characteristic polynomial of the matroid. Given that the characteristic polynomial has a large number of diverse applications (such as in the study of the critical problem, linear codes, hyperplane arrangements, separation of points by hyperplanes, series–parallel networks, colorings and flows in graphs, and orientations of graphs; see [**BO92, Za87**] for surveys), the f-vector of the broken circuit complex is one of the most interesting numerical invariants in matroid theory. Other applications of broken circuit complexes were found by Björner [**Bj82, Bj92**] who made use of these complexes to construct bases for the homology of independence complexes and geometric lattice complexes. Notice that the Orlik–Solomon algebra can be described in terms of the homology of the geometric lattice of intersections [**OS80**]. Therefore, the broken circuit complex provides a basis for the Orlik–Solomon algebra. This fact, which was shown independently by Gel'fand and Zelevinsky [**GZ86**] and Jambu and Terao [**JT89**], plays an important role in a lot of research on the Orlik–Solomon algebra; see, e.g., [**BZ91, DY02, Fa02, EPY03, Pe03, KR09**]. A commutative analogue of this fact, as mentioned in (3), will also play a crucial role in our study of the Orlik–Terao algebra.

This thesis is divided into 3 chapters. Chapter 2 provides a brief review of the broken circuit complex, the Orlik–Terao algebra, and other necessary materials.

Chapter 3, which consists of 5 sections, summarizes the main results of [**LR13, Le14, Le16, LM15**]. One basic property of the broken circuit complex is that it is shellable. The broken circuit ideal and the Orlik–Terao ideal are therefore Cohen–Macaulay. It is then natural to ask when they are Gorenstein, complete intersections, or when they have linear resolutions. Answers to these questions have been given in [**LR13, Le14**] and will be presented in Sections 3.1-3.3.

In [**LR13**], we characterized generic arrangements and their cones as those ones whose Orlik–Terao ideal admits a linear resolution. Thus having Orlik–Terao ideal with a linear resolution is a combinatorial property (see Section 3.1). We also showed that the complete intersection property of the Orlik–Terao ideal is combinatorial and characterized those arrangements with this property (see Section 3.2).

In [**Le14**], we studied the Gorenstein property of the broken circuit ideal and the Orlik–Terao ideal. We proved that these ideals are Gorenstein if and only if they are complete intersections. Furthermore, we showed that the Gorenstein/complete intersection property of the Orlik–Terao ideal can be determined by the h-vector – a linear transformation of the f-vector – of the associated broken circuit complex (see Section 3.3).

Motivated by the results of [**Le14**], we investigated in [**Le16**] broken circuit complexes of series–parallel networks, an important class of matroids which contains as a subclass those matroids with a complete intersection broken circuit complex. Our main result is a formula which relates certain entries of the h-vector of the broken circuit complex of a series–parallel network with an ear decomposition of the network (see Section 3.4).

Finally, in [**LM15**], we studied the relationship between the Orlik–Terao algebra and the relation space of a hyperplane arrangement. In particular, we gave a characterization of spanning sets of the relation space in terms of the Orlik–Terao algebra, which generalizes the characterization of 2-formality due to Schenck and Tohăneanu mentioned before (see Section 3.5).

CHAPTER 2

Background

In this chapter we introduce the two main objects of the thesis: the broken circuit complex of a matroid and the Orlik–Terao algebra of a hyperplane arrangement. Certain needed notions and facts from matroid theory and hyperplane arrangements will also be briefly summarized. We begin by recalling the relevant background from combinatorics and algebra.

2.1. Basic notions

We collect here several definitions and results concerning graded rings and simplicial complexes that will be used later. The reader is referred to [**BH98, Sta96**] for more details.

2.1.1. Graded rings, free resolutions, and h-vectors. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ provided with the standard grading (i.e., $\deg x_i = 1$ for every i). We consider a standard graded $\mathbb{K}$-algebra $R = S/I$, where I is a graded ideal of S. Let d denote the Krull dimension of R. Assume that the minimal graded free resolution of R is given by

$$0 \to F_p \to \cdots \to F_1 \to S \to R \to 0,$$

where p is the projective dimension of R. Then the rank of the graded free S-module F_p is called the *type* of R. The ring R is said to be

- *Cohen–Macaulay* if $p = n - d$,
- *level* if R is Cohen–Macaulay and F_p is generated in one degree,
- *Gorenstein* if R is Cohen–Macaulay of type 1,
- a *complete intersection* if I is generated by a regular sequence.

The ideal I is also called Cohen–Macaulay (level, etc.) if the ring R is so. We have the following implications:

$$\text{complete intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{level} \Rightarrow \text{Cohen–Macaulay},$$

in which the first implication is well-known, while the other ones follow immediately from the definitions.

Assume that the Hilbert series of R has the form

$$H_R(t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

with $h_0 + h_1 t + \cdots + h_s t^s \in \mathbb{Z}[t]$ and $h_s \neq 0$. Then we call $(h_0, h_1, \dots, h_s)$ the *h-vector* of R. Classical results of Macaulay completely characterize h-vectors of Cohen–Macaulay and complete intersection standard graded algebras (see [**Sta78**]; the characterization of the Cohen–Macaulay case will be briefly discussed below). However, no characterizations of

h-vectors of Gorenstein or level algebras are known yet. The following result of Stanley [**Sta78**, Theorems 4.1, 4.4] provides information on h-vectors of Gorenstein algebras.

THEOREM 2.1.1. *Let* $(h_0, h_1, \dots, h_s)$ *be the h-vector of R. If R is Gorenstein, then* $h_i = h_{s-i}$ *for* $0 \le i \le \lfloor s/2 \rfloor$. *The converse holds if R is a Cohen–Macaulay domain.*

Let us now recall a notion which is used to characterize h-vectors of Cohen–Macaulay algebras. A nonempty set Γ of monic monomials in a polynomial ring is called an *order ideal* if whenever $u \in \Gamma$ and v divides u, then $v \in \Gamma$. A finite order ideal is *pure* if all of its maximal elements (ordered by divisibility) have the same degree. We call a sequence $(h_0, h_1, \dots, h_s)$ of integers an *O-sequence* (respectively, a *pure O-sequence*) if there exists an order ideal (respectively, a pure order ideal) Γ such that h_i is the number of monomials of Γ of degree i. A result of Macaulay, as mentioned above, identifies O-sequences with h-vectors of Cohen–Macaulay standard graded algebras, and characterizes them by a numerical condition. No similar characterization of pure O-sequences is available. Note that pure O-sequences are exactly h-vectors of artinian monomial level algebras; see, e.g., [**BMM$^+$12, MNZ13**]. We will revisit pure O-sequences in Stanley's conjecture on h-vectors of independence complexes (see Conjecture 2.3.2).

To conclude this section, we recall the definitions of modules with a linear resolution and Koszul algebras. Let U be a vector space over $\mathbb{K}$ with basis $\{x_1, \dots, x_n\}$. Let $T(U)$ denote the tensor algebra over U. Thus $T(U)$ is a $\mathbb{K}$-graded algebra with grading $T(U) = \bigoplus_{k \ge 0} T^k U$, where $T^k U$ is a $\mathbb{K}$-vector space with basis $\{x_{i_1} \cdots x_{i_k} \mid 1 \le i_1, \dots, i_k \le n\}$. We identify $T(U)$ with $\mathbb{K}\langle x_1, \dots, x_n \rangle$, the $\mathbb{K}$-free algebra on $x_1, \dots, x_n$. Let I be a two-sided graded ideal in $\mathbb{K}\langle x_1, \dots, x_n \rangle$ and consider the graded algebra $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$. We may assume that I contains no nonzero elements of degree 1. The graded algebra A is called *quadratic* if I is generated by elements of degree 2. For example, the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ and the exterior algebra $E = \bigwedge(U)$ of U are both quadratic, since

$$S \cong \mathbb{K}\langle x_1, \dots, x_n \rangle / (x_i x_j - x_j x_i \mid i \ne j), \; E \cong \mathbb{K}\langle x_1, \dots, x_n \rangle / (x_i^2, x_i x_j + x_j x_i \mid 1 \le i, j \le n).$$

It is obvious that if A has a presentation $A = B/J$, where B is a quadratic algebra, then A is quadratic if and only if J is generated by elements of degree 2.

Let $A = \mathbb{K}\langle x_1, \dots, x_n \rangle / I$ be a $\mathbb{K}$-graded algebra and let W be a finitely generated graded A-module. Given an integer m, W is said to have an *m-linear resolution* (over A) if its minimal graded free resolution is of the form

$$\cdots \to A(-m-s)^{\beta_s} \to \cdots \to A(-m-1)^{\beta_1} \to A(-m)^{\beta_0} \to W \to 0.$$

In other words, W has an m-linear resolution if W is generated in degree m and all the maps in the free resolution of W are represented by matrices of linear forms. Modules with linear resolutions are interesting for many reasons. For instance, if A is a quotient of a polynomial ring and W has a linear resolution over A, then the Betti numbers of W are determined by its Hilbert series; see, e.g., [**Pe11**, Proposition 17.11]. A remarkable result of Eagon and Reiner [**EaR98**] is that a simplicial complex is Cohen–Macaulay if and only if the Stanley–Reisner ideal of the Alexander dual of this complex has a linear resolution.

The ring A is called *Koszul* if the field $\mathbb{K}$ has a linear resolution as an A-module. For example, the polynomial ring S and the exterior algebra E are both Koszul, because $\mathbb{K}$

is resolved over S by the Koszul complex and over E by the Cartan complex, which are both linear. Koszul algebras were introduced by Priddy [**Pri70**]. They arise naturally and frequently in various mathematical fields. We refer to the book [**PP05**] and survey papers [**CDR13, Fr99**] for more information.

For later usage, we only state the following basic properties of Koszul algebras.

THEOREM 2.1.2. *With notation as above, assume that* $A = \mathbb{K}\langle x_1,\dots,x_n\rangle/I$ *or* $A = S/I$ *or* $A = E/I$*, where in each case* I *is a graded ideal containing no nonzero elements of degree 1. Then the following statements hold.*

(i) *If* A *is Koszul, then* A *is quadratic.*
(ii) *If* I *has a quadratic Gröbner basis, i.e., a Gröbner basis consisting of elements of degree* 2*, then* A *is Koszul.*

The useful criterion for Koszulness in the second statement of the above theorem is essentially due to Fröberg [**Fr75, Fr99**] (see also [**Yu01**, Theorem 6.16]). Concerning this statement we note that Gröbner basis theory has been developed for not only polynomial rings (see, e.g., [**Ei95**]) but also free algebras (see, e.g., [**Mo94**]) and exterior algebras (see [**HH11**]).

2.1.2. Simplicial complexes. A *simplicial complex* on a vertex set of n elements, which will be identified with $[n] = \{1,\dots,n\}$, is a collection Δ of subsets of $[n]$ such that

(i) $\{i\} \in \Delta$ for $i = 1,\dots,n$;
(ii) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

For simplicity, all simplicial complexes considered in this thesis are assumed to be nonempty. Thus we always have $\emptyset \in \Delta$. Elements of Δ are called *faces*. Maximal faces, with respect to inclusion, are called *facets*. It is clear that every simplicial complex is determined by its collection of facets. We will sometimes write $\Delta = \langle F_1,\dots,F_t\rangle$ to indicate that $F_1,\dots,F_t$ are the facets of Δ.

The *dimension* of a face $F \in \Delta$ is $\dim F = |F| - 1$. Define $\dim\Delta = \max\{\dim F \mid F \in \Delta\}$ to be the *dimension* of Δ. We say that Δ is *pure* if all of its facets have the same dimension.

Assume that $\dim\Delta = r-1$. Then the *f-vector* of Δ is the $(r+1)$-tuple $(f_0, f_1,\dots,f_r)$, where f_i is the number of faces of Δ of cardinality i. (Notice that in the literature f_i is frequently defined to be the number of faces of dimension i. This definition differs from ours by a shift of the indices of the f-vector by 1.) The polynomial $f_\Delta(t) = \sum_{i=0}^r f_i t^{r-i}$ is called the *f-polynomial* of Δ. The *reduced Euler characteristic* of Δ is

$$\widetilde{\chi}(\Delta) = (-1)^{r-1} f_\Delta(-1) = \sum_{i=0}^{r} (-1)^{i+1} f_i.$$

Let F be a face of Δ. The *star* and the *link* of F in Δ are defined to be the following simplicial complexes:

$$\begin{aligned}\operatorname{star}_\Delta F &= \{G \in \Delta \mid F \cup G \in \Delta\},\\ \operatorname{link}_\Delta F &= \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\}.\end{aligned}$$

Given a subset W of $[n]$, the *restriction* of Δ to W is the complex

$$\Delta_W = \{G \in \Delta \mid G \subseteq W\}.$$

The *core* of Δ, denoted $\operatorname{core}\Delta$, is the restriction of Δ to $\operatorname{core}[n] := \{i \in [n] \mid \operatorname{star}_\Delta\{i\} \neq \Delta\}$.

Δ is called a *cone* with *apex* j if every facet of Δ contains j. When this is the case, we also say that Δ is a cone over the restriction $\Delta_{[n]-j}$. It is easy to see that Δ is a cone if and only if $[n] \neq \operatorname{core}[n]$ (or in other words, $\Delta \neq \operatorname{core}\Delta$), and in this case every vertex $j \in [n] - \operatorname{core}[n]$ is an apex of Δ.

Let $\mathbb{K}$ be a field. We can associate to Δ the following squarefree monomial ideal in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$:

$$I_\Delta = (x_{i_1} \cdots x_{i_p} \mid \{i_1, \dots, i_p\} \notin \Delta).$$

The ideal I_Δ and the corresponding quotient algebra $K[\Delta] = S/I_\Delta$ are called the *Stanley–Reisner ideal* and the *Stanley–Reisner ring* of Δ, respectively. It is well-known that $\dim \mathbb{K}[\Delta] = \dim \Delta + 1$.

We define the *h-vector* of Δ to be the h-vector of the Stanley–Reisner ring $\mathbb{K}[\Delta]$. It can be shown that if $(h_0, h_1, \dots, h_s)$ is the h-vector of Δ then $s \leq r$, where $r = \dim \Delta + 1$. In the literature the h-vector of Δ is usually defined to be $(h_0, \dots, h_s, h_{s+1}, \dots, h_r)$, in which $h_{s+1} = \cdots = h_r = 0$ if $s < r$. However, for our purposes it is more convenient to consider the h-vector with zero entries at the end removed. Note that if Δ is a cone over $\Delta_{[n]-j}$, then $\mathbb{K}[\Delta] = \mathbb{K}[\Delta_{[n]-j}][x_j]$. It follows that Δ and $\Delta_{[n]-j}$ have the same h-vector.

Let $(h_0, h_1, \dots, h_s)$ be the h-vector of Δ. Then the *h-polynomial* of Δ is defined by $h_\Delta(t) = \sum_{i=0}^{s} h_i t^{r-i}$. The f-vector and h-vector of Δ are intimately related through the polynomial identity $f_\Delta(t-1) = h_\Delta(t)$. In other words, one vector is obtained from the other by a linear transformation:

$$f_i = \sum_{j=0}^{\min\{i,s\}} \binom{r-j}{i-j} h_j, \quad i = 0, \dots, r,$$

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{r-j}{i-j} f_j, \quad i = 0, \dots, s.$$

Thus the f-vector and h-vector encode the same information about the complex Δ. We single out the following special cases of the above equations for later usage:

$$h_0 = f_0 = 1, \quad h_1 = f_1 - r. \tag{2.1}$$

We say that Δ is *Cohen–Macaulay* (*Gorenstein*, etc.) over $\mathbb{K}$ if so is the ring $\mathbb{K}[\Delta]$. Moreover, Δ is called *locally Gorenstein* (respectively, a *locally complete intersection*) over $\mathbb{K}$ if $\operatorname{link}_\Delta\{i\}$ is Gorenstein (respectively, a complete intersection) over $\mathbb{K}$ for every $i = 1, \dots, n$. Note that a simplicial complex is a complete intersection if and only if its minimal non-faces are pairwise disjoint. Thus the (locally) complete intersection property of a simplicial complex is independent of the field $\mathbb{K}$.

In practice, the Cohen–Macaulayness of a simplicial complex is frequently proved using a combinatorial condition called shellability. A pure simplicial complex is said to be *shellable* if there exists an ordering $F_1, \dots, F_t$ of its facets such that for all i, j with $1 \leq i < j \leq t$, there is an element $x \in F_j - F_i$ and some k with $1 \leq k < j$ such that $F_j - F_k = \{x\}$. Both complexes considered mainly in this thesis – the broken circuit complex and the independence complex of a matroid – are shellable. It is well-known that shellable complexes are Cohen–Macaulay (over every field); see, e.g., [**BH98**, Theorem 5.1.13].

In order to characterize the Gorenstein property of the broken circuit complex we will need the following result of Hochster [**Ho77**, Proposition 5.5, Theorem 6.7] (see also [**Sta96**, Theorem 5.1]):

THEOREM 2.1.3. *Let* Δ *be a simplicial complex with* $\dim \Delta \geq 0$. *Let* $\Gamma = \operatorname{core} \Delta$ *and denote by* $\widetilde{\chi}(\Gamma)$ *the reduced Euler characteristic of* Γ. *Then* Δ *is Gorenstein over a given field* $\mathbb{K}$ *if and only if one of the following conditions holds:*

(i) Δ *consists of one or two vertices,*
(ii) Δ *is Cohen–Macaulay over* $\mathbb{K}$ *with* $\dim \Delta \geq 1$, $\widetilde{\chi}(\Gamma) = (-1)^{\dim \Gamma}$, *and for any face* F *with* $\dim \operatorname{link}_\Delta F = 1$, $\operatorname{link}_\Delta F$ *is either an n-gon* ($n \geq 3$), *or a path with at most 3 vertices.*

We will also need the following characterization of (locally) complete intersection complexes due to Terai and Yoshida [**TY09**, Corollary 1.10, Proposition 1.11]:

THEOREM 2.1.4. *Let* $\mathbb{K}$ *be a field. Let* Δ *be a simplicial complex with* $\dim \Delta \geq 1$.

(a) *If* $\dim \Delta = 1$, *then the following conditions are equivalent:*
 (i) Δ *is a locally complete intersection complex;*
 (ii) Δ *is a locally Gorenstein complex over* $\mathbb{K}$*;*
 (iii) Δ *is either an n-gon* ($n \geq 3$), *or an m-vertex path* ($m \geq 2$).
(b) *Assume that* $\dim \Delta \geq 2$ *and* $\mathbb{K}[\Delta]$ *satisfies Serre's condition* (S_2). *Then the following conditions are equivalent:*
 (i) Δ *is a complete intersection complex;*
 (ii) Δ *is a locally complete intersection complex;*
 (iii) *for any face* F *with* $\dim \operatorname{link}_\Delta F = 1$, $\operatorname{link}_\Delta F$ *is a complete intersection complex.*

Recall that a Noetherian ring R is said to *satisfy Serre's condition* (S_2) if $\operatorname{depth} R_P \geq \min\{\dim R_P, 2\}$ for every prime ideal P of R. So for instance, all Cohen–Macaulay rings satisfy (S_2).

2.2. Matroids

Matroids were introduced by Whitney [**Wh35**] as a common generalization of dependence in linear algebra and graph theory. Since then a rich theory of matroids has been developed which provides a framework for approaching many combinatorial problems. In our context, several constructions related to hyperplane arrangements are conveniently expressed in terms of associated matroids. We review here the needed facts and definitions from matroid theory, referring to the seminal book by Oxley [**Ox11**] for more details.

2.2.1. Basic concepts.
A matroid can be defined in many different but equivalent ways; see [**Ox11**, Chapter 1]. The following one is best suited for our purposes.

Definition 2.2.1. A *matroid* $M = (E, \mathfrak{I})$ consists of a finite ground set E and a nonempty collection $\mathfrak{I}$ of subsets of E, called *independent sets*, satisfying the following conditions:

(i) every subset of an independent set is independent,
(ii) if X and Y are independent and $|X| < |Y|$, then there exists $y \in Y - X$ such that $X \cup y$ is independent.

Maximal independent sets of a matroid M are called *bases*. A subset of the ground set E is *dependent* if it is not a member of $\mathfrak{I}$. Minimal dependent sets are called *circuits*, and an m-circuit is a circuit of cardinality m. For any set $X \subseteq E$, all maximal independent subsets of X have the same size, which is called the *rank* of X, denoted by $r(X)$. In particular, the rank of E, which is the common cardinality of the bases of M, is also called the *rank* $r(M)$ of M. The assignment $X \mapsto r(X)$ defines a map from 2^E to the set of nonnegative integers. This map is called the *rank function* of M. A matroid is specified by either its bases, circuits, or rank function. In fact, there are equivalent definitions of matroids in terms of bases, circuits, and rank functions, among many others.

Two matroids $M = (E, \mathfrak{I}), M' = (E', \mathfrak{I}')$ are said to be *isomorphic* if there exists a bijection $\varphi : E \to E'$ such that for every subset X of E, $X \in \mathfrak{I}$ if and only if $\varphi(X) \in \mathfrak{I}'$.

In this thesis we will be mostly concerned with the following types of matroids, the first two of which are prototypical examples:

(i) *Linear/representable matroids*: Let V be a vector space over a field $\mathbb{K}$. If E is a finite subset of V, then the linearly independent subsets of E form a matroid, called the *linear matroid* of E. When the vectors of E are given by the columns of a matrix A, then the linear matroid of E is also called the linear matroid of A and is denoted by $M[A]$. A matroid is *representable* over $\mathbb{K}$ or $\mathbb{K}$-*representable* if it is isomorphic to the linear matroid of a matrix over $\mathbb{K}$.

(ii) *Cycle/graphic matroids*: Let G be a graph with edge set E. Let $\mathfrak{I}$ be the collection of subsets of E which contain no cycles. Then $M(G) = (E, \mathfrak{I})$ is a matroid, called the *cycle matroid* of G. A basis of $M(G)$ is a spanning tree of G, and a circuit of $M(G)$ is a cycle of G. Any matroid that is isomorphic to the cycle matroid of a graph is called a *graphic matroid*.

(iii) *Uniform matroids*: Let m, n be nonnegative integers with $m \leq n$. Let E be an n-element set and $\mathfrak{I}$ the collection of all subsets of E of cardinality at most m. Then $U_{m,n} = (E, \mathfrak{I})$ is a matroid, called the *uniform matroid* of rank m on an n-element set. A basis of $U_{m,n}$ is a subset of E of cardinality m, and a circuit of $U_{m,n}$ is a subset of E of cardinality $m+1$. In particular, when $m = n-1$, the matroid $U_{n-1,n}$ has a unique n-circuit $C_n = E$. For convenience we will make no distinction between $U_{n-1,n}$ and C_n.

2.2.2. Loopless and simple matroids. Let M be a matroid on the ground set E. An element $e \in E$ is called a *loop* if $\{e\}$ is a circuit of M. In other words, e is a loop if and only if it is not contained in any basis of M. We say that M is *loopless* if it has no loops.

Two elements $e, f \in E$ are said to be *parallel* if they form a circuit of M. A *parallel class* of M is a maximal subset of E in which any two distinct elements are parallel and no element is a loop. A parallel class is *non-trivial* if it contains at least two elements. A matroid is called *simple* if it has no loops and no non-trivial parallel classes. In other words, a matroid is simple if all of its circuits have cardinality at least 3.

Given an arbitrary matroid M we may associate to it a simple matroid by first deleting all the loops from M and then, for every parallel class X, deleting all but one distinguished element of X. The matroid obtained, denoted by $\overline{M}$, is uniquely determined up to the choice of the distinguished elements and is called the *simplification* of M. Evidently, one may also construct the *simplification* $\overline{G}$ of a given graph G in the same manner as above, and moreover, one has $\overline{M(G)} = M(\overline{G})$ (see Figure 2.1).

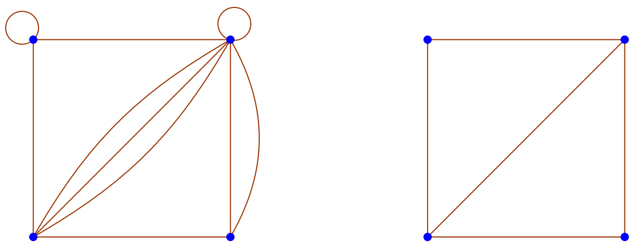


FIGURE 2.1. A graph and its simplification

2.2.3. Duality and minors. Let M be a matroid on the ground set E. Let $\mathscr{B}$ be the family of bases of M. Then $\mathscr{B}^* = \{E - B \mid B \in \mathscr{B}\}$ is also the family of bases of a matroid. We denote this matroid by M^* and call it the *dual* of M. For example, $U^*_{m,n} = U_{n-m,n}$. It is well-known that if G is a planar graph, then $M(G)^* \cong M(G^*)$, where G^* is a geometric dual of G. Moreover, for any matrix A, $M[A]^* = M[A^*]$ where A^* is a matrix whose row space is the orthogonal space of the row space of A.

The loops of M^* are called *coloops* of M. Clearly, e is a coloop of M if and only if it is contained in every basis of M. We say that M is *coloopless* if it contains no coloops.

Let X be a subset of E. The *deletion* of X from M, denoted $M - X$, is the matroid on the ground set $E - X$ whose independent sets are the independent sets of M that are contained in $E - X$. The *contraction* of X from M is defined to be $M/X = (M^* - X)^*$. Note that deletion and contraction for matroids generalize the corresponding operations for graphs, i.e., $M(G) - X = M(G - X)$ and $M(G)/X = M(G/X)$ for any graph G and any set X of edges of G. Note also that the operations of deletion and contraction commute, i.e., $(M - X)/Y = M/Y - X$ for disjoint subsets X and Y of E.

A *minor* of M is a matroid which can be obtained from M by a sequence of deletions and contractions. So every minor of M has the form $(M - X)/Y$, where X, Y are disjoint subsets of E. We call a minor of M *proper* if it is different from M.

Many important classes of graphs/matroids can be characterized by certain finite lists of excluded minors. Here an *excluded minor* of a class of graphs/matroids means a graph/matroid outside the class whose proper minors are all in the class. For example, a classical result of Kuratowski [**Ku30**] states that the class of planar graphs has two excluded minors, namely, $K_{3,3}$ and K_5. Another famous result, due to Tutte [**Tu58**], establishes that $U_{2,4}$ is the unique excluded minor for *binary matroids*, i.e., those matroids that are representable over the field of two elements. At the International Congress of Mathematicians in 1970, Rota [**Rot71**] posed a prominent conjecture that for each finite field F there are only finitely many excluded minors for the class of F-representable matroids. A lengthy proof of this conjecture, the writing process of which will probably take a few years, has recently been announced by Geelen, Gerards, and Whittle [**GGW14**].

2.2.4. Connectivity. Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2. Their *direct sum* $M_1 \oplus M_2$ is the matroid on the ground set $E_1 \cup E_2$ whose independent sets are the unions of an independent set of M_1 and an independent set of M_2. In other words, a circuit of $M_1 \oplus M_2$ is either a circuit of M_1 or a circuit of M_2, and vice versa. The direct sum of a finite collection of matroids can be defined similarly.

A matroid is called *connected* if it is not the direct sum of two smaller matroids. One can check that a matroid is connected if and only if, for every pair of distinct elements of the ground set, there is a circuit of the matroid containing both. The notion of connectivity for matroids extends that of nonseparability for graphs. Recall that a graph is *nonseparable* if it is connected and cannot be decomposed as the union of two proper connected subgraphs which have just one vertex in common; otherwise it is *separable*. For a connected graph G, it is well-known that the cycle matroid $M(G)$ is connected if and only if G is nonseparable.

An arbitrary matroid M can be decomposed uniquely (up to oder) as a direct sum $M = M_1 \oplus \cdots \oplus M_k$, where $M_1, \ldots, M_k$ are connected matroids. We call $M_1, \ldots, M_k$ the *connected components* of M.

2.2.5. Lattice of flats, the characteristic and Tutte polynomials. Let M be a matroid on the ground set E with rank function r. Extending the notion of subspaces of a vector space, one can define the *closure* of a subset X of E as follows

$$\overline{X} = \{e \in E \mid r(X \cup e) = r(X)\}.$$

If $X = \overline{X}$, then X is called a *flat* of M. Let $L(M)$ denote the poset of flats of M ordered by inclusion. For $X, Y \in L(M)$, their *meet* $X \wedge Y$ and their *join* $X \vee Y$ are defined by

$$X \wedge Y = X \cap Y,$$
$$X \vee Y = \overline{X \cup Y}.$$

Then $(L(M), \wedge, \vee)$ forms a lattice, called the *lattice of flats* of M.

Define the *Möbius function* of $L(M)$, $\mu : L(M) \times L(M) \to \mathbb{Z}$, by the conditions:

$$\begin{aligned} \mu(X,X) &= 1 && \text{for all } X \in L(M), \\ \mu(X,Y) &= -\sum_{X \leq Z < Y} \mu(X,Z) && \text{for all } X,Y \in L(M), X < Y, \\ \mu(X,Y) &= 0 && \text{otherwise.} \end{aligned}$$

The minimum element of $L(M)$ is $\hat{0} := \overline{\emptyset}$. By definition, $\hat{0}$ is the set of loops of M, thus $\hat{0} = \emptyset$ if M is loopless. For every $X \in L(M)$ we write $\mu(X) = \mu(\hat{0}, X)$. The *characteristic polynomial* of M is defined to be

$$\chi(M,t) = \sum_{X \in L(M)} \mu(X) t^{r(M)-r(X)}.$$

An example of a characteristic polynomial will be given later; see Example 2.4.1.

The characteristic polynomial is an evaluation of the *Tutte polynomial*, defined by

$$T(M;x,y) = \sum_{X \subseteq E} (x-1)^{r(M)-r(X)} (y-1)^{|X|-r(X)}.$$

It is well-known that

$$\chi(M,t) = (-1)^{r(M)} T(M; 1-t, 0).$$

The Tutte polynomial might be the most important enumerative invariant of a matroid. Many applications of this polynomial have been found. Apart from applications of the characteristic polynomial mentioned in Chapter 1, the Tutte polynomial can be used to interpret the reliability probability of a network, the partition function of the Potts model of statistical physics, the Jones polynomial of an alternating knot, colorings and flows in random graphs, lattice point enumeration and the Ehrhart polynomial, and configurations arising in chip firing games. See [**BO92, We99**] for surveys.

2.2.6. Series–parallel networks. Series–parallel networks may serve as a model of series–parallel electrical circuits. Graphs of this type are of considerable interest in computational complexity theory, because a large number of graph problems, which are NP-complete or NP-hard for general graphs, are linear-time computable for the class of series–parallel graphs; see [**TNS82**]. There are several ways to define series–parallel networks. In this section we discuss their construction through the operations of series and parallel connection and give some of their characterizations.

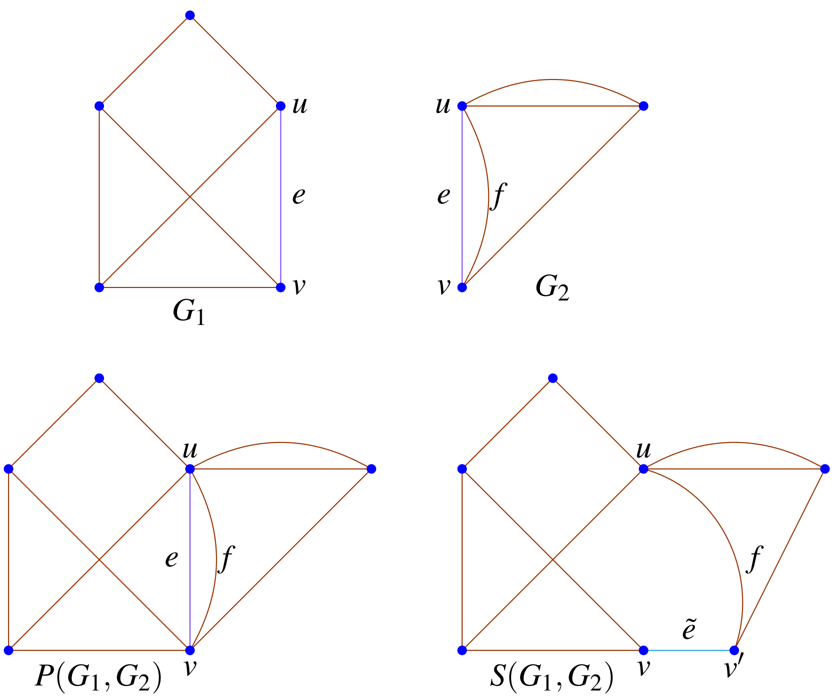


FIGURE 2.2. Series and parallel connections of graphs

Let us first recall the definitions of series and parallel connections of two graphs. Let G_i be a graph with vertex set V_i and edge set E_i for $i = 1,2$. Assume that G_1 and G_2 have only one common edge e and two common vertices u,v which are the end vertices of e. Then the *parallel connection* of G_1 and G_2 with respect to the *base edge* e is merely the union of G_1 and G_2, i.e., the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. We denote this graph by $P(G_1,G_2)$. To define the series connection of G_1 and G_2, we first form a copy $G_2' = (V_2', E_2')$ of G_2 by just renaming the vertex v to v' and the edge e to e'. Then we remove the edge e from G_1 and the edge e' from G_2'. Finally, we add a new edge $\tilde{e}$ joining

v and v'. The *series connection* of G_1 and G_2 with respect to e, denoted $S(G_1,G_2)$, is the graph with vertex set $V_1 \cup V_2'$ and edge set $(E_1 - e) \cup (E_2' - e') \cup \tilde{e}$ (see Figure 2.2).

Next we extend the previous constructions to matroids, following Brylawski [**Bry71**]. Let M_1 and M_2 be matroids on ground sets E_1 and E_2 with $E_1 \cap E_2 = \{e\}$. Assume that e is neither a loop nor a coloop of M_1 or M_2. Let $\mathscr{C}(M)$ denote the family of circuits of a matroid M. The *series connection* $S(M_1,M_2)$ and the *parallel connection* $P(M_1,M_2)$ of M_1,M_2 with respect to the *basepoint* e are the matroids on the ground set $E_1 \cup E_2$ whose families of circuits are respectively:

$$\mathscr{C}(S(M_1,M_2)) = \mathscr{C}(M_1 - e) \cup \mathscr{C}(M_2 - e) \cup \{C_1 \cup C_2 : e \in C_i \in \mathscr{C}(M_i) \text{ for } i = 1,2\},$$
$$\mathscr{C}(P(M_1,M_2)) = \mathscr{C}(M_1) \cup \mathscr{C}(M_2) \cup \{C_1 \cup C_2 - e : e \in C_i \in \mathscr{C}(M_i) \text{ for } i = 1,2\}.$$

Moreover, if e is a loop or coloop of M_1 we set

$$S(M_1,M_2) = \begin{cases} (M_1/e) \oplus M_2 \text{ if } e \text{ is a loop of } M_1, \\ M_1 \oplus (M_2 - e) \text{ if } e \text{ is a coloop of } M_1; \end{cases}$$

$$P(M_1,M_2) = \begin{cases} M_1 \oplus (M_2/e) \text{ if } e \text{ is a loop of } M_1, \\ (M_1 - e) \oplus M_2 \text{ if } e \text{ is a coloop of } M_1. \end{cases}$$

One may check that $S(M_1,M_2) = S(M_2,M_1)$ and $P(M_1,M_2) = P(M_2,M_1)$ in all cases.

Note that the operations of series and parallel connection for matroids generalize the corresponding ones for graphs: if G_1,G_2 are graphs which have only one common edge, then

$$P(M(G_1),M(G_2)) \cong M(P(G_1,G_2)), \quad S(M(G_1),M(G_2)) \cong M(S(G_1,G_2)).$$

It is possible to define series and parallel connections of more than two matroids, just by iterating the above constructions. Let $M_1,\dots,M_n$ be matroids on ground sets $E_1,\dots,E_n$ such that $E_{i+1} \cap (\bigcup_{j=1}^{i} E_j) = \{e_i\}$ for $i = 1,\dots,n-1$, in which $e_1,\dots,e_{n-1}$ need not be distinct. Then we can form $P(M_1,M_2)$, $P(P(M_1,M_2),M_3)$, and so on. The last matroid obtained in this way, denoted by $P(M_1,\dots,M_n)$, is called the *(iterated) parallel connection* of $M_1,\dots,M_n$ with respect to the basepoints $e_1,\dots,e_{n-1}$. The series connection of $M_1,\dots,M_n$ is defined similarly. Of course, one can also define iterated series and parallel connections of graphs in an analogous way.

Special cases of series and parallel connections are series and parallel extensions. Let M, N be matroids. We say that M is a *series extension* of N and N a *series contraction* of M if $M = S(N,C_2)$, where C_2 is a 2-circuit. On the other hand, M is a *parallel extension* of N and N a *parallel deletion* of M if $M = P(N,C_2)$. Moreover, we call N a *parallel minor* of M if N can be obtained from M by a sequence of contractions and parallel deletions. Series extension and parallel extension for graphs mean subdividing an edge and duplicating an edge, respectively.

Example 2.2.2. Let M be a loopless matroid. Then M is an iterated parallel extension of its simplification $\overline{M}$, thus $\overline{M}$ is a parallel minor of M. More generally, one can show that if M is the parallel connection of loopless matroids $M_1,\dots,M_n$, then each M_i is a parallel minor of M.

Assume M is a connected matroid on the ground set E. Then M is called *parallel irreducible at* $e \in E$ if either M is trivial (i.e., $|E| = 1$) or M is not a parallel connection of two non-trivial matroids with respect to the basepoint e. We say that M is *parallel irreducible* if it is parallel irreducible at every element of E.

Parallel irreducible matroids are useful in certain inductive arguments. The following proposition characterizes parallel irreducible matroids and shows that every connected matroid admits a decomposition as an iterated parallel connection of parallel irreducible matroids.

PROPOSITION 2.2.3. *Let M be a non-trivial connected matroid on the ground set E. Then the following statements hold.*

(i) [**Bry71**, Proposition 5.8] *M is parallel irreducible at $e \in E$ if and only if M/e is connected. Hence M is parallel irreducible if and only if M/e is connected for every $e \in E$.*
(ii) [**Le16**, Lemma 2.1] *M can be decomposed as an iterated parallel connection $M = P(N_1, \dots, N_s)$, where N_i is non-trivial and parallel irreducible. Moreover, if M is simple, then the decomposition is unique up to a permutation of the components.*

Remark 2.2.4. (i) If a connected matroid M is not simple, then the parallel irreducible decomposition of M as in Proposition 2.2.3(ii) is no longer unique. For instance, if M contains a 2-circuit $C_2 = \{e_1, e_2\}$, then $M = P(M_1, C_2) = P(M_2, C_2)$, where $M_i = M - e_i$ for $i = 1, 2$. Nevertheless, it is not difficult to show that if $M = P(N_1, \dots, N_s)$ and $M = P(N'_1, \dots, N'_t)$ are two parallel irreducible decompositions of M, then $s = t$ and (after reindexing) $N_i \cong N'_i$ for $i = 1, \dots, s$.

(ii) One can easily define *parallel irreducible graphs* or a *parallel irreducible decomposition* of a graph. Therefore, we will use these notions without further explanation.

Let us now turn to the definition of series–parallel networks. A graph G is called a *series–parallel network* (or a *series–parallel graph*) if it is nonseparable and can be obtained from the complete graph K_2 by subdividing and duplicating edges. For instance, the graph G_2 in Figure 2.2 is a series–parallel network, while the graph G_1 is not. Extending the notion to matroids, we call a connected matroid M a *series–parallel network* if it can be obtained from a coloop by a sequence of series and parallel extensions. Clearly, a matroid is a series–parallel network if and only if it is the cycle matroid of a series–parallel graph.

Several characterizations of series–parallel networks are given below. For a matroid M on the ground set E we define $\beta(M) := (-1)^{r(M)} \sum_{X \subseteq E} (-1)^{|X|} r(X)$ to be the *beta invariant* of M. This invariant was introduced by Crapo [**Cr67**] and discussed further, e.g., in [**Bry71, Ox82, Za87**].

THEOREM 2.2.5. *Let M be a loopless connected matroid and G a loopless nonseparable graph with at least one edge. Then*

(i) [**Du65**, Theorem 1] *G is a series–parallel network if and only if it contains no subgraph that is a subdivision of K_4.*
(ii) [**Bry71**, Theorem 7.6] *The following conditions are equivalent:*
(a) M is a series–parallel network;

(b) M has no minor isomorphic to $U_{2,4}$ or $M(K_4)$;
(c) $\beta(M) = 1$.

2.3. The independence and broken circuit complexes

The independence and broken circuit complexes are two simplicial complexes that arise naturally from a matroid. They are interrelated in an appealing way. The broken circuit complex is a subcomplex of the independence complex, while the cone over the independence complex is the broken circuit complex of another matroid. In this thesis we mainly focus our attention on the broken circuit complex. Nevertheless, since several results and conjectures on the independence complex lead to interesting problems on the broken circuit complex, we also briefly review the independence complex here.

2.3.1. The independence complex. Let M be a matroid of rank r on the ground set E. Then the collection $IN(M)$ of all independent sets in M clearly forms a pure $(r-1)$-dimensional complex, called the *independence complex* of M. This complex was shown to be shellable by Provan [**Pro77**]. Thus it is Cohen–Macaulay and therefore its h-vector is an O-sequence. Moreover, Stanley [**Sta77**] proved:

THEOREM 2.3.1. *$IN(M)$ is a level complex.*

Recall that h-vectors of artinian monomial level algebras can be identified with pure O-sequences; see [**BMM+12, MNZ13**]. The above result led Stanley to the following challenging conjecture, which remains wide open.

CONJECTURE 2.3.2. *The h-vector of $IN(M)$ is a pure O-sequence.*

Note that a pure O-sequence $(h_0, h_1, \ldots, h_s)$ with $h_s \neq 0$ has the following properties

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor s/2 \rfloor}, \tag{2.2}$$

$$h_i \leq h_{s-i} \text{ for } i = 0, \ldots, \lfloor s/2 \rfloor. \tag{2.3}$$

This was proved by Hibi [**Hi89**], who later proposed the following weaker version of Stanley's conjecture [**Hi92**]:

CONJECTURE 2.3.3. *Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $IN(M)$. Then it satisfies inequalities* (2.2) *and* (2.3).

If M has a coloop e, then e is contained in every basis of M. So $IN(M)$ is a cone over $IN(M-e)$. It follows that $IN(M)$ and $IN(M-e)$ have the same h-vector. Thus when discussing the h-vector of $IN(M)$ one may assume that M is coloopless.

Hibi's conjecture was resolved by Chari [**Ch97**]. The main idea in his proof is the introduction of the notion of *convex ear decompositions* for simplicial complexes, which can be viewed as a higher-dimensional analogue of the notion of ear decompositions for graphs (see Section 3.4). He showed that the h-vectors of simplicial complexes that admit a convex ear decomposition satisfy inequalities (2.2) and (2.3), and that the independence complex of every coloopless matroid admits such a decomposition, thereby settling Hibi's conjecture.

Before turning to another topic let us mention that the h-polynomial of $IN(M)$ can be expressed in terms of the Tutte polynomial of M as follows (see [**Bj92**]):

$$h_{IN(M)}(t) = T(M;t,1).$$

Regarding Theorem 2.3.1 one may ask when the complex $IN(M)$ is Gorenstein or a complete intersection. The answer to the complete intersection question is rather simple. Since the minimal non-faces of $IN(M)$ are the circuits of M, $IN(M)$ is a complete intersection if and only if the circuits of M are pairwise disjoint, which means that M is a direct sum of circuits and coloops. The following result of Stokes [**Sto08**, Theorem 4.4.10] shows that there are no non-trivial Gorenstein independence complexes.

THEOREM 2.3.4. *$IN(M)$ is Gorenstein if and only if it is a complete intersection.*

One may also characterize the Gorenstein/complete intersection property of $IN(M)$ in terms of the h-vector $(h_0, h_1, \ldots, h_s)$ of this complex. Since $IN(M)$ is level, the type of the Stanley–Reisner ring of $IN(M)$ is equal to h_s; see, e.g., [**Sta96**, p. 91]. So from Equation (2.1) and Theorem 2.3.4 we immediately get

COROLLARY 2.3.5. *Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $IN(M)$. Then $IN(M)$ is a Gorenstein/complete intersection complex if and only if $h_0 = h_s$.*

2.3.2. The broken circuit complex. We keep the assumption that M is a matroid of rank r on the ground set E. Assume further that there is a linear order $<$ on E. In this case, $(M,<)$ is called an *ordered matroid.*

Definition 2.3.6. A *broken circuit* of $(M,<)$ is a subset of E of the form $C-e$, where C is a circuit of M and e is the least element of C. The *broken circuit complex* of $(M,<)$, denoted $BC_<(M)$ (or briefly $BC(M)$ if no confusion may arise), is defined by

$$BC(M) = \{F \subseteq E \mid F \text{ contains no broken circuit}\}.$$

Note that if M contains a loop, then $\emptyset$ is a broken circuit, and so $BC(M) = \emptyset$. Hence it is enough to consider broken circuit complexes of loopless matroids. Moreover, if necessary, one may even restrict attention to simple matroids because the broken circuit complex of a loopless matroid is isomorphic to that of its simplification; see [**Bj92**, Proposition 7.4.1].

From now on assume that M is loopless. Then $BC(M)$ is evidently a simplicial complex which is contained in the independence complex $IN(M)$. For brevity, the Stanley–Reisner ideal and Stanley–Reisner ring of $BC(M)$ will be called the *broken circuit ideal* and *broken circuit algebra* of M, respectively.

Let e_0 be the least element of E. The restriction of $BC(M)$ to $E-e_0$, denoted by $\overline{BC}(M)$, is called the *reduced broken circuit complex* of M. Several basic properties of the broken circuit complex were shown by Brylawski [**Bry77**]:

PROPOSITION 2.3.7. *With the above notation and assumption, the following statements hold:*

(i) *$BC(M)$ is a pure complex of dimension $r-1$.*
(ii) *$BC(M)$ is a cone over $\overline{BC}(M)$ with apex e_0.*
(iii) *There exists a matroid M' such that $IN(M) = \overline{BC}(M')$.*

Note that the matroid M' in the last statement of Proposition 2.3.7 can be constructed explicitly. Moreover, the converse of that statement is not true, i.e., not every reduced broken circuit complex can be represented by some independence complex. We refer to the paper of Brylawski [**Bry77**] for details.

Example 2.3.8. Let $M = M(G)$ be the cycle matroid of the graph G depicted in Figure 2.3(a). This matroid has 3 circuits: $123, 145, 2345$. With respect to the natural ordering $<$ of the edges of G, the broken circuits of M are $23, 45, 345$. It follows that $BC_<(M) = \langle 124, 134, 135, 125\rangle$ and $\overline{BC}_<(M) = \langle 24, 34, 35, 25\rangle$. With respect to the ordering $\prec$: $5 \prec 4 \prec 3 \prec 2 \prec 1$, the broken circuits are $12, 14, 234$. Thus $BC_\prec(M) = \langle 135, 235, 245, 345\rangle$ and $\overline{BC}_\prec(M) = \langle 13, 23, 24, 34\rangle$. One may notice that the complexes obtained in both cases are shellable. The two reduced complexes are illustrated in Figure 2.3(b), (c).

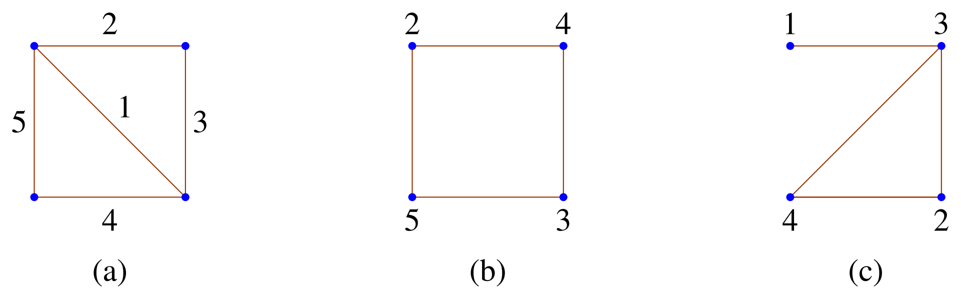


FIGURE 2.3. (a) Graph G, (b) $\overline{BC}_<(M)$, (c) $\overline{BC}_\prec(M)$

The previous example points out that different orderings may yield non-isomorphic broken circuit complexes. However, a number of important properties of the broken circuit complex, such as those ones listed in Proposition 2.3.7, the shellability, the f-vector and h-vector (see below), do not depend on a specified order.

The shellability of the broken circuit complex, like that of the independence complex, was established by Provan [**Pro77**] (see also [**PB80**] and [**Bj92**]).

THEOREM 2.3.9. *The complex $BC(M)$ is shellable.*

From this result it follows that the broken circuit algebra is Cohen–Macaulay, which in turn implies the Cohen–Macaulayness of the Orlik–Terao algebra (see Theorem 2.5.2). It is then natural to ask when these algebras are level, Gorenstein, or complete intersections. We will discuss the answers to these questions in the next chapter (see Sections 3.2, 3.3).

Apart from shellability, another interesting feature of the broken circuit complex, which primarily makes this complex important, is that it carries the 'chromatic' property of the matroid: the f-vector of $BC(M)$ encodes the coefficients of the characteristic polynomial of M. The following formula, which extends Whitney's formula for the chromatic polynomial of a graph [**Wh32a**], was proved by Rota [**Rot64**].

THEOREM 2.3.10. *Let $(f_0, f_1, \ldots, f_r)$ be the f-vector of $BC(M)$. Then*

$$\chi(M,t) = \sum_{i=0}^{r} (-1)^i f_i t^{r-i}.$$

This result, on the one hand, implies that the f-vector and h-vector of $BC(M)$ are independent of the ordering given to the ground set E. On the other hand, since the characteristic polynomial has many applications (see Chapter 1), it confirms that the f-vector and h-vector of $BC(M)$ are among the most interesting numerical invariants in matroid theory.

In this thesis we give some applications of the h-vector of the broken circuit complex. We prove that if M is the underlying matroid of an arrangement $\mathscr{A}$, then the h-vector of $BC(M)$ captures the Gorenstein/complete intersection property of the Orlik–Terao algebra of $\mathscr{A}$ (see Theorem 3.3.7). In the case that M is a series–parallel network, the h-vector of $BC(M)$ is shown to have significant implications for the structure of M (see Section 3.4). For the moment we note the following basic properties of this vector; see [**Bj92, BO92**].

THEOREM 2.3.11. *Let $(h_0, h_1, \ldots, h_s)$ and $h_{BC(M)}(t) = \sum_{i=0}^{s} h_i t^{r-i}$ be the h-vector and the h-polynomial of $BC(M)$, respectively. Let $T(M;x,y)$ be the Tutte polynomial of M. Then the following statements hold.*

(i) $h_{BC(M)}(t) = T(M;t,0)$.
(ii) *$s = r - c$, where c is the number of connected component of M.*
(iii) *If M is connected, then $h_{r-1} = \beta(M)$, where $\beta(M)$ is the beta invariant of M.*

We also note the following characterization of series–parallel networks, which follows directly from the above result, Equation (2.1) and Theorem 2.2.5.

PROPOSITION 2.3.12. *Assume M is connected. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(M)$. Then M is a series–parallel network if and only if $h_0 = h_s$.*

We conclude this section with the following major open problem in matroid theory which is closely related to Conjecture 2.3.3 (see [**Sw03**]):

CONJECTURE 2.3.13. *Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(M)$. Then $h_i \le h_{s-i}$ for $i = 0, \ldots, \lfloor s/2 \rfloor$.*

Note that the collection of h-vectors of independence complexes is strictly contained in the collection of h-vectors of broken circuit complexes, as is easily seen from Proposition 2.3.7 (and the discussion following it). Consequently, Conjecture 2.3.13 is stronger than inequality (2.3) in Conjecture 2.3.3. (We remark that inequality (2.2) for the h-vectors of broken circuit complexes would follow from Conjecture 2.3.13 and unimodality of those h-vectors. A recent remarkable result of Huh [**Hu15**] confirms log-concavity (hence unimodality) for the h-vectors of broken circuit complexes of matroids representable over a field of characteristic zero.) It is worth mentioning that, as Chari noted in his paper [**Ch97**], the broken circuit complex does not in general admit a convex ear decomposition. Therefore, Chari's method does not establish Conjecture 2.3.13. We will provide evidence supporting this conjecture in the case M is a series–parallel network (see Theorem 3.4.16).

2.4. Hyperplane arrangements

In this section several classes of arrangements relevant to our work will be briefly reviewed. Some constructions related to hyperplane arrangements will also be recalled.

We refer to the classical book by Orlik and Terao [**OT92**] as well as the lecture notes of Stanley [**Sta07**] for more details.

Throughout the section let $\mathscr{A} = \{H_1, \ldots, H_n\}$ be an arrangement of n hyperplanes in an ℓ-dimensional vector space V over some field $\mathbb{K}$. We will call $\mathscr{A}$ an *ℓ-arrangement* in case we want to emphasize the dimension of V. Note that $\mathscr{A}$ is always assumed to be *central*, i.e., each hyperplane H_i contains the origin of V. Let $\{z_1, \ldots, z_\ell\}$ be a basis for the dual space V^* of V. Then we may identify the symmetric algebra of V^* with the polynomial ring $R = \mathbb{K}[z_1, \ldots, z_\ell]$. So each hyperplane H_i is the kernel of a unique (up to a constant) linear form $\alpha_i \in R$. The polynomial $Q(\mathscr{A}) := \prod_{i=1}^n \alpha_i$ is called a *defining polynomial* of $\mathscr{A}$.

2.4.1. The underlying matroid and the intersection lattice. The combinatorics of an arrangement can be encoded by either its underlying matroid or intersection lattice. These objects carry the same information about the arrangement and determine each other.

The *underlying matroid* of $\mathscr{A}$, denoted $M(\mathscr{A})$, is defined on the ground set $\mathscr{A}$ with the usual notion of independence of hyperplanes, i.e., a subset $\mathscr{B} = \{H_{i_1}, \ldots, H_{i_p}\}$ of $\mathscr{A}$ is *independent* if the corresponding linear polynomials $\alpha_{i_1}, \ldots, \alpha_{i_p}$ are linearly independent. Clearly, $M(\mathscr{A})$ is a simple matroid. The rank function of $M(\mathscr{A})$ is given by $r(\mathscr{B}) = \operatorname{codim}(\bigcap_{j=1}^p H_{i_j})$ for $\mathscr{B} = \{H_{i_1}, \ldots, H_{i_p}\} \subseteq \mathscr{A}$. The rank of $M(\mathscr{A})$, which is $\operatorname{codim}(\bigcap_{i=1}^n H_i)$, is called the *rank* of $\mathscr{A}$ and denoted by $r(\mathscr{A})$. We say that $\mathscr{A}$ is *essential* if $r(\mathscr{A}) = \ell$. Thus $\mathscr{A}$ is essential if and only if $\bigcap_{i=1}^n H_i$ consists of only the origin.

Recall that the *intersection lattice* $L(\mathscr{A})$ of $\mathscr{A}$ is the collection of all intersections of hyperplanes in $\mathscr{A}$:

$$L(\mathscr{A}) = \{\bigcap_{j=1}^{p} H_{i_j} \mid \{i_1, \ldots, i_p\} \subseteq [n]\}.$$

This is a poset ordered by *reverse inclusion*, i.e., $X \leq Y$ in $L(\mathscr{A})$ if and only if $Y \subseteq X$. Note that the ambient space V, regarded as the intersection of the empty set of hyperplanes, is the minimum element of $L(\mathscr{A})$. The maximum element of $L(\mathscr{A})$ is $\bigcap_{i=1}^n H_i$. With the meet and join operations defined by

$$X \wedge Y = \bigcap_{X \cup Y \subseteq Z \in L(\mathscr{A})} Z,$$
$$X \vee Y = X \cap Y,$$

$L(\mathscr{A})$ is indeed a lattice. This lattice is determined by the underlying matroid $M(\mathscr{A})$. In fact, $L(\mathscr{A})$ is isomorphic to the lattice of flats of $M(\mathscr{A})$, the isomorphism being given by assigning to each flat of $M(\mathscr{A})$ the intersection of the hyperplanes in that flat. Since the bigger the flat the smaller the intersection and since the lattice of flats of $M(\mathscr{A})$ has been ordered by inclusion, this isomorphism is the primary reason for ordering $L(\mathscr{A})$ by reverse inclusion rather than inclusion. Note that, conversely, $M(\mathscr{A})$ is determined by $L(\mathscr{A})$, because every simple matroid can be restored from its lattice of flats; see, e.g., [**Ox11**, Section 1.7] or [**Sta07**, Theorem 3.8].

The characteristic polynomial of $M(\mathscr{A})$ is called the *characteristic polynomial* of $\mathscr{A}$, denoted by $\chi(\mathscr{A},t)$. Thus

$$\chi(\mathscr{A},t) = \sum_{X\in L(\mathscr{A})} \mu(X)t^{r(\mathscr{A})-r(X)} = t^{r(\mathscr{A})-\ell} \sum_{X\in L(\mathscr{A})} \mu(X)t^{\dim X}.$$

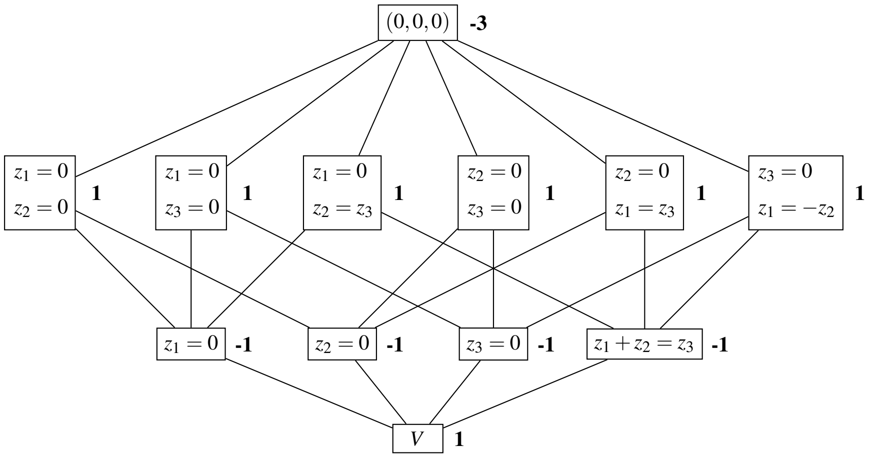


FIGURE 2.4. The intersection lattice of $Q(\mathscr{A}) = z_1z_2z_3(z_1+z_2-z_3)$ and its Möbius function

Example 2.4.1. Let $\mathscr{A}$ be a 3-arrangement defined by $Q(\mathscr{A}) = z_1z_2z_3(z_1+z_2-z_3)$. The intersection lattice of $\mathscr{A}$ and its Möbius function are shown in Figure 2.4. We have

$$\chi(\mathscr{A},t) = t^3-4t^2+6t-3 = (t-1)(t^2-3t+3).$$

2.4.2. Cone and product. The *cone* over $\mathscr{A}$, denoted $c\mathscr{A}$, is the $(\ell+1)$-arrangement with defining polynomial $Q(c\mathscr{A}) = z_0Q(\mathscr{A}) \in R[z_0]$. In other words, $c\mathscr{A}$ can be viewed as an arrangement in $V\oplus\mathbb{K}$ consisting of the "coned" hyperplanes $H_1\oplus\mathbb{K},\dots,H_n\oplus\mathbb{K}$ and the additional hyperplane $H_0 = \ker(z_0)$.

Assume that we have another arrangement $\mathscr{A}'$ in a vector space V'. Then the *product* arrangement $\mathscr{A}\times\mathscr{A}'$ is defined in $V\oplus V'$ as follows

$$\mathscr{A}\times\mathscr{A}' = \{H\oplus V' \mid H\in\mathscr{A}\}\cup\{V\oplus H' \mid H'\in\mathscr{A}'\}.$$

For example, the cone over $\mathscr{A}$ is the product of $\mathscr{A}$ with the arrangement $\mathscr{A}' = \{0\}$ in $V' = \mathbb{K}$. The following relation among the underlying matroids is clear

$$M(\mathscr{A}\times\mathscr{A}') \cong M(\mathscr{A})\oplus M(\mathscr{A}').$$

2.4.3. The Orlik–Solomon algebra. Let $\mathscr{K}$ be an arbitrary commutative ring and let $E_1 = \bigoplus_{i=1}^n \mathscr{K} e_i$ be a free $\mathscr{K}$-module which has a basis $e_1,\dots,e_n$ corresponding to the hyperplanes of $\mathscr{A}$. Denote by $E = \bigwedge(E_1)$ the exterior algebra of E_1. Then $E = \bigoplus_{p=0}^n E_p$ is a graded $\mathscr{K}$-algebra. For each p, $E_p = \bigwedge^p(E_1)$ is a free $\mathscr{K}$-module with a basis consisting of the monomials $e_{\mathscr{B}} = e_{i_1}\cdots e_{i_p}$, where $\mathscr{B} = \{H_{i_1},\dots,H_{i_p}\}$ ranges over all subsets of $\mathscr{A}$ of cardinality p and $i_1<\cdots<i_p$. Note that the multiplication in E is anticommutative:

$uv = (-1)^{pq}vu$ for $u \in E_p$, $v \in E_q$. Define a $\mathscr{K}$-linear map $\partial : E \to E$ by $\partial 1 = 0$, $\partial e_i = 1$ and for $p \geq 2$,

$$\partial e_{\mathscr{B}} = \sum_{j=1}^{p} (-1)^{j-1} e_{\mathscr{B}-\{H_{i_j}\}}$$

for all $\mathscr{B} = \{H_{i_1}, \ldots, H_{i_p}\} \subseteq \mathscr{A}$.

Definition 2.4.2. The *Orlik–Solomon ideal* of $\mathscr{A}$ is the ideal $J(\mathscr{A})$ of E generated by

$$\{\partial e_{\mathscr{B}} \mid \mathscr{B} \subseteq \mathscr{A} \text{ is dependent}\}.$$

The quotient algebra $\mathbf{A}(\mathscr{A}) = E/J(\mathscr{A})$ is called the *Orlik–Solomon algebra* of $\mathscr{A}$.

The Orlik–Solomon algebra was introduced by Orlik and Solomon in [**OS80**]. They proved that if $\mathscr{A}$ is a complex arrangement and $\mathscr{K} = \mathbb{C}$, then $\mathbf{A}(\mathscr{A})$ is isomorphic as a graded algebra to the cohomology ring $H^*(\mathsf{M};\mathbb{C})$ of the complement M of $\mathscr{A}$ [**OS80**, Theorem 5.2]. A proof of the isomorphism for the case with integer coefficients appeared in [**OT92**] and [**Yu01**]. Since the Orlik–Solomon algebra is evidently determined by the intersection lattice, this celebrated result indicates that the cohomology ring of the complement of a complex arrangement is combinatorial.

Note that $J(\mathscr{A})$ is a graded ideal in E. So $\mathbf{A}(\mathscr{A})$ is a graded algebra and we may write $\mathbf{A}(\mathscr{A}) = \bigoplus_{p=0}^{n} \mathbf{A}(\mathscr{A})_p$, where $\mathbf{A}(\mathscr{A})_p = E_p/(J(\mathscr{A}) \cap E_p)$. It is known that $\mathbf{A}(\mathscr{A})$ and all of its graded components are free $\mathscr{K}$-modules. A canonical basis for $\mathbf{A}(\mathscr{A})$, called the *no broken circuit basis*, was constructed independently by Gel'fand and Zelevinsky [**GZ86**] and Jambu and Terao [**JT89**]. In fact, the broken circuit complex of the underlying matroid $M(\mathscr{A})$ provides a *universal Gröbner basis* (i.e., a Gröbner basis with respect to every monomial order) for $J(\mathscr{A})$ (see [**Bj92, Yu01, CF05**]):

THEOREM 2.4.3. *The set*

$$\{\partial e_C \mid C \text{ is a circuit of } M(\mathscr{A})\}$$

is a universal Gröbner basis for $J(\mathscr{A})$. In particular, for any ordering of $\mathscr{A}$, the set

$$\{e_{\mathscr{B}} \mid \mathscr{B} \in BC(M(\mathscr{A})), |\mathscr{B}| = p\}$$

forms a $\mathscr{K}$-basis for $\mathbf{A}(\mathscr{A})_p$.

The *Poincaré polynomial* of $\mathscr{A}$ is defined to be

$$\pi(\mathscr{A},t) = \sum_{p=0}^{n} \operatorname{rank} \mathbf{A}(\mathscr{A})_p t^p.$$

By Theorem 2.4.3, $\operatorname{rank} \mathbf{A}(\mathscr{A})_p = f_p$ is the number of faces of cardinality p of the broken circuit complex $BC(M(\mathscr{A}))$. So from Theorem 2.3.10 we obtain the following formula which was also proved by Orlik and Solomon [**OS80**, Theorem 5.2]:

$$\pi(\mathscr{A},t) = (-t)^{r(\mathscr{A})} \chi(\mathscr{A}, -t^{-1}) = \sum_{X \in L(\mathscr{A})} \mu(X)(-t)^{r(X)}.$$

Note that when $\mathscr{A}$ is a complex arrangement, $\pi(\mathscr{A},t)$ coincides with the *Poincaré polynomial* of the complement M defined by $P(\mathsf{M},t) = \sum_{p=0}^{n} \dim H^p(\mathsf{M};\mathbb{C}) t^p$. Theorem

2.4.3 then implies that all topological *Betti numbers* $\beta_p(M) := \dim H^p(\mathsf{M};\mathbb{C})$ of M are given by the f-vector of the complex $BC(M(\mathscr{A}))$.

2.4.4. Graphic arrangements. Let G be a simple graph with vertex set $V(G) = \{1,2,\dots,\ell\}$ and edge set $E(G)$. Then the *graphic arrangement* associated to G is an arrangement in $\mathbb{K}^\ell$ defined by

$$\mathscr{A}_G = \{\ker(z_i - z_j) \mid \{i,j\} \in E(G)\}.$$

For instance, if $\mathbb{K} = \mathbb{C}$ and $G = K_\ell$ is the complete graph on ℓ vertices, then $\mathscr{A}_G$ is the braid arrangement $\mathscr{A}_{\ell-1}$ mentioned in Chapter 1. It is clear that the underlying matroid of $\mathscr{A}_G$ is isomorphic to the cycle matroid of G. Thus $r(\mathscr{A}_G)$ is the size of a spanning tree of G, which is $\ell - c(G)$ where $c(G)$ is the number of connected components of G.

There are a number of interesting connections between invariants of G and invariants of $\mathscr{A}_G$. For example, the chromatic polynomial of G is essentially the characteristic polynomial of $\mathscr{A}_G$. Recall that a *coloring* of G with t colors is a map $\gamma : V(G) \to \{1,2,\dots,t\}$ such that $\gamma(i) \neq \gamma(j)$ whenever $\{i,j\} \in E(G)$. The *chromatic function* $\chi(G,t)$ counts the number of colorings of G with t colors. Then $\chi(G,t)$ is a polynomial of degree ℓ which is related to the characteristic polynomial of $\mathscr{A}_G$ as follows (see [**Za87**]):

$$\chi(G,t) = t^{c(G)}\chi(\mathscr{A}_G,t). \tag{2.4}$$

Another invariant of G that is closely connected with $\mathscr{A}_G$ is its number of acyclic orientations. An *orientation* of G is an assignment of a direction to each edge of G. We say that an orientation is *acyclic* if it has no directed cycles. A beautiful result of Stanley [**Sta73**] states that the number of acyclic orientations of G is $(-1)^\ell\chi(G,-1)$. Thus by Zaslavsky's region counting theorem [**Za75**], this is exactly the number of regions of the arrangement $\mathscr{A}_G$.

2.4.5. Generic arrangements. We say that $\mathscr{A}$ is *generic* if every subset $\mathscr{B} \subseteq \mathscr{A}$ with $|\mathscr{B}| = r(\mathscr{A})$ is independent. Thus $\mathscr{A}$ is generic if and only if its underlying matroid is a uniform matroid. In particular, we call an arrangement *Boolean* if its underlying matroid is of the form $U_{\ell,\ell}$. Choosing suitable coordinates, a Boolean arrangement $\mathscr{A}$ is usually given by the defining polynomial $Q(\mathscr{A}) = z_1 \cdots z_\ell$.

Example 2.4.4. Every 2-arrangement is generic. The 3-arrangement $\mathscr{A}$ in Example 2.4.1 is also generic. But not every 3-arrangement is generic, e.g., the arrangement $\mathscr{B}$ defined by $Q(\mathscr{B}) = z_1z_2z_3(z_1-z_2)(z_1+z_2-z_3)$ is not generic.

The homotopy type of a generic arrangement was determined by Hattori [**Hat75**]. This result implies that non-Boolean generic arrangements of rank at least 3 are not $K(\pi,1)$.

In the next chapter we will characterize generic arrangements and their cones as those arrangements whose Orlik–Terao ideal admits a linear resolution (see Theorem 3.1.5).

2.4.6. Supersolvable arrangements. In [**Sta72**], Stanley introduced the notion of supersolvable lattices that provides a unified explanation for the factorization of the characteristic polynomial in many particular cases. This leads to the definition of supersolvable arrangements which are one of the nicest and best understood classes of hyperplane arrangements.

Let $L(\mathscr{A})$ be the intersection lattice of the arrangement $\mathscr{A}$. An element $X \in L(\mathscr{A})$ is called *modular* if for all $Y \in L(\mathscr{A})$ and all $Z \in L(\mathscr{A})$ with $Z \leq Y$,

$$Z \vee (X \wedge Y) = (Z \vee X) \wedge Y.$$

It is easy to show that X is modular if and only if each of the following conditions holds:

(i) $r(X) + r(Y) = r(X \wedge Y) + r(X \vee Y)$ for all $Y \in L(\mathscr{A})$;
(ii) $X + Y \in L(\mathscr{A})$ for all $Y \in L(\mathscr{A})$.

For instance, the ambient space V, all the hyperplanes $H_i \in \mathscr{A}$, and the maximum element $T = T(\mathscr{A}) := \bigcap_{i=1}^{n} H_i$ are modular in $L(\mathscr{A})$.

Let $r = r(\mathscr{A})$. We say that $\mathscr{A}$ is *supersolvable* if $L(\mathscr{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_r = T.$$

As one readily sees, supersolvability is a combinatorial property. There are many important arrangements which are supersolvable.

Example 2.4.5. (i) Every 2-arrangement is supersolvable, for $V < H < T$ is a maximal chain of modular elements, in which H is an arbitrary hyperplane of the arrangement.

(ii) The Boolean arrangement $\mathscr{A}$ defined by $Q(\mathscr{A}) = z_1 \cdots z_\ell$ is supersolvable because every element of the intersection lattice is modular. The braid arrangement $\mathscr{A}_{\ell-1}$ is also supersolvable with the following maximal chain of modular elements

$$V < \{z_1 = z_2\} < \{z_1 = z_2 = z_3\} < \cdots < \{z_1 = z_2 = \cdots = z_\ell\} = T.$$

(iii) Terao [**Te86**] proved that supersolvable arrangements are nothing but fiber-type arrangements introduced by Falk and Randell [**FR85**].

(iv) Let G be a simple graph. Then the graphic arrangement $\mathscr{A}_G$ is supersolvable if and only if G is *chordal*, i.e., G has no induced k-cycles for $k \geq 4$. This result is essentially due to Stanley [**Sta72**]; see also [**EdR94**].

(v) The arrangement $\mathscr{A}$ considered in Example 2.4.1 is not supersolvable because it has no modular element of rank 2. More generally, every generic ℓ-arrangement for $\ell \geq 3$ is not supersolvable.

The following factorization theorem of Stanley [**Sta72**, Theorem 4.1] is one of the main reasons for the introduction of supersolvability.

THEOREM 2.4.6. *Let $\mathscr{A}$ be a supersolvable arrangement with the following maximal chain of modular elements*

$$V = X_0 < X_1 < \cdots < X_r = T.$$

Let $d_i = |\{H \in \mathscr{A} \mid H \leq X_i,\ H \not\leq X_{i-1}\}|$. Then

$$\chi(\mathscr{A}, t) = \prod_{i=1}^{r} (t - d_i).$$

The next useful characterization of supersolvability was given by Björner and Ziegler in [**BZ91**, Theorem 2.8].

THEOREM 2.4.7. *Let $M(\mathscr{A})$ be the underlying matroid of $\mathscr{A}$. Then $\mathscr{A}$ is supersolvable if and only if there exists an ordering of $\mathscr{A}$ such that the minimal broken circuits of $M(\mathscr{A})$ all have size* 2.

This result and Theorem 2.4.3 imply that the Orlik–Solomon ideal of a supersolvable arrangement has a quadratic Gröbner basis. Peeva [**Pe03**, Theorem 4.3] showed that the converse holds as well.

THEOREM 2.4.8. *$\mathscr{A}$ is supersolvable if and only if the Orlik–Solomon ideal $J(\mathscr{A})$ has a quadratic Gröbner basis.*

A direct consequence of the previous theorem and Theorem 2.1.2(ii) is the following result due to Shelton and Yuzvinsky [**SY97**, Theorem 4.6].

THEOREM 2.4.9. *If $\mathscr{A}$ is supersolvable, then the Orlik–Solomon algebra $\mathbf{A}(\mathscr{A})$ is Koszul.*

This leads to the following question which is one of the major open problems in hyperplane arrangement theory; see [**Yu01**, Problem 6.7.1].

PROBLEM 2.4.10. *Does the Koszulness of $\mathbf{A}(\mathscr{A})$ imply that $\mathscr{A}$ is supersolvable?*

Affirmative answers to this question are known for graphic arrangements [**JP98**] and hypersolvable arrangements [**SS02**]. Note that the class of hypersolvable arrangements, introduced by Jambu and Papadima [**JP98**], contains all supersolvable arrangements and generic arrangements.

Concerning the Koszulness of $\mathbf{A}(\mathscr{A})$ it is desirable to know when $\mathbf{A}(\mathscr{A})$ is quadratic; see Theorem 2.1.2(i). Several necessary conditions for $\mathbf{A}(\mathscr{A})$ being quadratic were given by Falk [**Fa02**] and Denham and Yuzvinsky [**DY02**]. Nevertheless, no characterization of the quadraticity of $\mathbf{A}(\mathscr{A})$ is known yet. Notice that this is a combinatorial property.

PROBLEM 2.4.11. *Characterize arrangements (or underlying matroids) for which the Orlik–Solomon algebra is quadratic.*

2.4.7. $K(\pi,1)$ arrangements. Recall that the arrangement $\mathscr{A}$ is called a $K(\pi,1)$ *arrangement* if its complement M is a $K(\pi,1)$ space, i.e., the higher homotopy groups $\pi_i(\mathsf{M})$ vanish for all $i>1$.

As mentioned in Chapter 1, the class of $K(\pi,1)$ arrangements includes supersolvable arrangements [**FR85, Te86**], complexifications of simplicial arrangements [**De72**] (which contain all Coxeter arrangements), and complex reflection arrangements [**Na83, OS88, Be15**]. However, most arrangements are not $K(\pi,1)$. For example, non-Boolean generic arrangements of rank at least 3 do not have this property [**Hat75**].

The problem of determining when an arrangement is $K(\pi,1)$, known as the $K(\pi,1)$ *problem*, has been a major focus of study in the topological theory of arrangements. The following more precise form of this problem remains unsolved; see [**FR00**].

PROBLEM 2.4.12. *Decide whether the $K(\pi,1)$ property is combinatorial.*

2.4.8. Free arrangements. A *derivation* of the polynomial ring $R=\mathbb{K}[z_1,\dots,z_\ell]$ is a $\mathbb{K}$-linear map $\theta : R\to R$ satisfying Leibniz's rule

$$\theta(pq)=p\theta(q)+q\theta(p)\quad \text{for all } p,q\in R.$$

Let $\mathrm{Der}_{\mathbb{K}}(R)$ be the set of all derivations of R. Then $\mathrm{Der}_{\mathbb{K}}(R)$ is a free R-module with basic $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_\ell}$. Moreover, $\mathrm{Der}_{\mathbb{K}}(R)$ is a graded module, where a derivation $\theta = \sum_{k=1}^{\ell} p_k \frac{\partial}{\partial z_k}$ is homogeneous of degree m if p_k is a homogeneous polynomial of degree m for $k = 1, \ldots, \ell$.

The *module of $\mathscr{A}$-derivations* is defined to be the submodule

$$D(\mathscr{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(R) \mid \theta(\alpha_i) \in \alpha_i R \text{ for } i = 1, \ldots, n\}.$$

For instance, the Euler derivation $\theta_E = \sum_{k=1}^{\ell} z_k \frac{\partial}{\partial z_k}$ is an element of $D(\mathscr{A})$.

The arrangement $\mathscr{A}$ is called *free* if $D(\mathscr{A})$ is a free R-module. Free arrangements were introduced by Terao [**Te80**], based on the notion of free divisors defined by Saito [**Sa75**]. Examples of free arrangements include supersolvable arrangements [**JT84**] and reflection arrangements [**Te89**].

Free arrangements seemed to have relations with $K(\pi, 1)$ arrangements. In [**Sa75**], Saito conjectured that an arrangement is free if and only if it is $K(\pi, 1)$. However, this conjecture is not true. Terao [**Te80**] found several non-free $K(\pi, 1)$ arrangements (which are in fact simplicial), while a family of free arrangements which are not $K(\pi, 1)$ was given by Edelman and Reiner [**EdR95**].

If $\mathscr{A}$ is free, then $D(\mathscr{A})$ has rank ℓ. Furthermore, there exists a homogeneous basis $\theta_1, \ldots, \theta_\ell$ for $D(\mathscr{A})$ whose degrees $d_1, \ldots, d_\ell$ depend only on $\mathscr{A}$. We call $d_1, \ldots, d_\ell$ the *exponents* of $\mathscr{A}$.

The following useful criterion of freeness is due to Saito [**Sa80**].

THEOREM 2.4.13. *Let $\theta_1, \ldots, \theta_\ell \in D(\mathscr{A})$. Then $D(\mathscr{A})$ is free with basis $\theta_1, \ldots, \theta_\ell$ if and only if the determinant of the matrix $(\theta_k(z_j))_{1 \le k,j \le \ell}$ satisfies*

$$\det(\theta_k(z_j))_{1 \le k,j \le \ell} = cQ(\mathscr{A})$$

for some $c \in \mathbb{K}^*$, where $Q(\mathscr{A})$ is the defining polynomial of $\mathscr{A}$.

For example, using Saito's criterion one can easily verify that the Boolean arrangement defined by $Q(\mathscr{A}) = z_1 \cdots z_\ell$ is free with basis $z_1 \frac{\partial}{\partial z_1}, \ldots, z_\ell \frac{\partial}{\partial z_\ell}$.

Free arrangements have many beautiful properties. The following ones, sometimes called Terao's factorization theorem and the addition-deletion theorem respectively, were both proved by Terao [**Te80**]. Terao's factorization theorem generalizes Stanley's factorization theorem (Theorem 2.4.6).

THEOREM 2.4.14. *Let $\mathscr{A}$ be a free arrangement of rank $r = r(\mathscr{A})$. Then there are exactly r exponents $d_1, \ldots, d_r$ of $\mathscr{A}$ that are nonzero, and the characteristic polynomial of $\mathscr{A}$ is*

$$\chi(\mathscr{A}, t) = \prod_{i=1}^{r} (t - d_i).$$

THEOREM 2.4.15. *Let $\mathscr{A}$ be an arrangement and H_0 a hyperplane of $\mathscr{A}$. Let*

$$\mathscr{A}' = \mathscr{A} - \{H_0\} \quad \textit{and} \quad \mathscr{A}'' = \{H \cap H_0 \mid H \in \mathscr{A}'\}.$$

Then any two of the following statements imply the third:

(i) *$\mathscr{A}$ is free with exponents $d_1, \ldots, d_{\ell-1}, d_\ell$;*
(ii) *$\mathscr{A}'$ is free with exponents $d_1, \ldots, d_{\ell-1}, d_\ell - 1$;*
(iii) *$\mathscr{A}''$ is free with exponents $d_1, \ldots, d_{\ell-1}$.*

Theorem 2.4.14 shows that the exponents of a free arrangement are determined by combinatorial data. Moreover, Theorem 2.4.15 leads to the definition of a subclass of free arrangements, called *inductively free arrangements*, which can be constructed combinatorially; see [**OT92**] for details. These facts motivate the following prominent conjecture of Terao [**Te80**].

CONJECTURE 2.4.16. *Freeness of arrangements is a combinatorial property.*

In [**Zi90**], Ziegler gave an example of a free arrangement over a field $\mathbb{K}$ of $\mathrm{char}(\mathbb{K}) \neq 3$ and a non-free arrangement over a field $\mathbb{K}'$ of $\mathrm{char}(\mathbb{K}') = 3$ with isomorphic intersection lattices. Thus the freeness of an arrangement may depend on the field characteristic. Until now Terao's conjecture is still open for arrangements over a fixed field.

2.4.9. The relation space and 2-formal arrangements. Falk and Randell [**FR86**] introduced the notion of 2-formal arrangements with the motivation to study $K(\pi,1)$ arrangements. It turns out that 2-formality is a necessary condition for an arrangement to be $K(\pi,1)$ or free. Moreover, 2-formality is also necessary for an arrangement to have a quadratic Orlik–Solomon algebra. Nonetheless, this property is not combinatorial.

We begin with the definition of the relation space, introduced by Brandt and Terao [**BT94**]. For convenience in representing the Orlik–Terao ideal, we will regard relations as linear polynomials. Let $S = \mathbb{K}[x_1,\dots,x_n]$ be a polynomial ring in n variables over $\mathbb{K}$ (here n is the number of hyperplanes of $\mathscr{A}$). Let $S_1 = \bigoplus_{i=1}^n \mathbb{K}x_i$ be the $\mathbb{K}$-vector space of linear forms in S. The *relation space* $F(\mathscr{A})$ of $\mathscr{A}$ is the kernel of the following $\mathbb{K}$-linear map

$$S_1 = \bigoplus_{i=1}^{n} \mathbb{K}x_i \to V^*,\ x_i \mapsto \alpha_i \ \text{ for } \ i = 1,\dots,n.$$

Note that $\dim_{\mathbb{K}} F(\mathscr{A}) = n - r(\mathscr{A})$, where $r(\mathscr{A})$ is the rank of $\mathscr{A}$. Elements of $F(\mathscr{A})$ are called *relations*. Evidently, relations come from dependencies among the hyperplanes of $\mathscr{A}$: if $\{H_{i_1},\dots,H_{i_m}\}$ is dependent and $a_t \in \mathbb{K}$ are constants such that $\sum_{t=1}^m a_t\alpha_{i_t} = 0$, then $r = \sum_{t=1}^m a_t x_{i_t}$ is a relation. In particular, every circuit C of the underlying matroid $M(\mathscr{A})$ gives rise to a unique (up to a constant) relation r_C. It can be shown that

$$\{r_C \mid C \text{ is a circuit of } M(\mathscr{A})\}$$

forms a generating set for $F(\mathscr{A})$.

Given a relation $r = \sum_{i=1}^n a_i x_i$, the *support* of r is defined to be $\mathrm{supp}(r) = \{i \mid a_i \neq 0\}$. The *length* of r, denoted $\mathrm{length}(r)$, is the cardinality of its support.

Example 2.4.17. Let $\mathscr{A}$ be the arrangement of the following four lines in $\mathbb{C}^2$

$$H_1 = \{y_1 = 0\},\ H_2 = \{y_2 = 0\},\ H_3 = \{y_1 + y_2 = 0\},\ H_4 = \{y_1 - y_2 = 0\}.$$

Then $M(\mathscr{A})$ has 4 circuits $C_i = \mathscr{A} - \{H_i\}$ for $i = 1,\dots,4$. The space $F(\mathscr{A})$ has dimension 2 and is generated, e.g., by the relations $r_{C_4} = x_3 - x_1 - x_2$ and $r_{C_3} = x_4 - x_1 + x_2$.

We now turn to the definition of 2-formal arrangements. For later usage, we slightly generalize [**BT94**, Definition 3.3].

Definition 2.4.18. Let $\mathfrak{R}$ be a subset of $F(\mathscr{A})$. We say that $\mathscr{A}$ is $\mathfrak{R}$-*generated* if $F(\mathscr{A})$ is spanned by $\mathfrak{R}$. When this is the case and $\mathfrak{R}$ is the set of relations of length at most $k+1$

for some $k \geq 2$, then $\mathscr{A}$ is called *k-generated*. 2-generated arrangements are also called *2-formal* arrangements.

For instance, the arrangement in Example 2.4.17 is 2-formal, while the arrangement in Example 2.4.1 is not 2-formal, but 3-generated. The following result was shown by Falk and Randell [**FR86**]; see also [**Fa02**, Theorem 3.3].

THEOREM 2.4.19. *$\mathscr{A}$ is 2-formal if $\mathscr{A}$ is $K(\pi,1)$ or the Orlik–Solomon algebra $\mathbf{A}(\mathscr{A})$ is quadratic.*

Falk and Randell [**FR86**] also asked whether free arrangements are necessarily 2-formal. Yuzvinsky [**Yu93a**, Corollary 2.5] answered this question in the affirmative.

THEOREM 2.4.20. *If $\mathscr{A}$ is free, then $\mathscr{A}$ is 2-formal.*

Brandt and Terao [**BT94**] defined the notion of k-formal arrangements for $k \geq 2$. They generalized the previous result by showing that free arrangements are k-formal for all $k \geq 2$.

Since 2-formality is necessary for freeness and $K(\pi,1)$-ness, it is interesting to know whether this property is combinatorial. This was asked by Falk and Randell [**FR86**]. In [**Yu93a**, Example 2.2], Yuzvinsky constructed the following counterexample.

Example 2.4.21. Define two arrangements $\mathscr{A}_1$ and $\mathscr{A}_2$, each consisting of 9 hyperplanes in $\mathbb{C}^3$, by

$$\begin{aligned} Q(\mathscr{A}_1) =& z_1z_2z_3(z_1+z_2+z_3)(2z_1+z_2+z_3)(2z_1+3z_2+z_3)(2z_1+3z_2+4z_3)\\ &(3z_1+5z_3)(3z_1+4z_2+5z_3),\\ Q(\mathscr{A}_2) =& z_1z_2z_3(z_1+z_2+z_3)(2z_1+z_2+z_3)(2z_1+3z_2+z_3)(2z_1+3z_2+4z_3)\\ &(z_1+3z_3)(z_1+2z_2+3z_3). \end{aligned}$$

It is easy to see that these arrangements have isomorphic intersection lattices. However, as shown by Yuzvinsky, $\mathscr{A}_1$ is 2-formal, while $\mathscr{A}_2$ is not.

Examining the previous example, Schenck and Tohăneanu [**ST09**] realized that the arrangements $\mathscr{A}_1$, $\mathscr{A}_2$ have different Orlik–Terao ideals, and they were led to a characterization of 2-formality in terms of the Orlik–Terao ideal. We will present their result in the next section.

2.5. The Orlik–Terao algebra

We keep the notation and assumptions of Section 2.4. Thus, in particular, $\mathscr{A}$ is an ℓ-arrangement of n hyperplanes with defining polynomial $Q(\mathscr{A}) = \prod_{i=1}^n \alpha_i$ in $R = \mathbb{K}[z_1,\dots,z_\ell]$. We denote the quotient field of R by $\mathbb{K}(z_1,\dots,z_\ell)$. Let $S = \mathbb{K}[x_1,\dots,x_n]$ be a polynomial ring in n variables over $\mathbb{K}$.

Definition 2.5.1. The *Orlik–Terao algebra* of $\mathscr{A}$ is the $\mathbb{K}$-subalgebra of $\mathbb{K}(z_1,\dots,z_\ell)$ generated by reciprocals of the α_i:

$$\mathbf{C}(\mathscr{A}) := \mathbb{K}[1/\alpha_1,\dots,1/\alpha_n].$$

The kernel of the surjection

$$S \to \mathbf{C}(\mathscr{A}),\ x_i \mapsto 1/\alpha_i \ \text{ for } \ i = 1,\dots,n$$

is called the *Orlik–Terao ideal* of $\mathscr{A}$, denoted by $I(\mathscr{A})$.

The Orlik–Terao algebra was introduced by Orlik and Terao in [**OT94**]. Since then it has been the topic of a significant amount of research. Terao [**Te02**] interpreted the Hilbert series of $\mathbf{C}(\mathscr{A})$ in terms of the Poincaré polynomial of $\mathscr{A}$. Proudfoot and Speyer [**PS06**] found a universal Gröbner basis for $I(\mathscr{A})$ in terms of the circuits of the underlying matroid $M(\mathscr{A})$, which implies that $\mathbf{C}(\mathscr{A})$ degenerates to the broken circuit algebra of $M(\mathscr{A})$, and thus $\mathbf{C}(\mathscr{A})$ is a Cohen–Macaulay ring. As mentioned before, the Orlik–Terao ideal was used by Schenck and Tohăneanu [**ST09**] to characterize 2-formal arrangements. The Orlik–Terao algebra also plays an important role in studying hypertoric varieties: Braden and Proudfoot [**BP09**] proved that the intersection cohomology of a hypertoric variety is isomorphic to the Orlik–Terao algebra of a hyperplane arrangement; see also [**MP14**] for further results in this direction. In [**Sche11**], Schenck established a connection between $\mathbf{C}(\mathscr{A})$ and the first resonance variety of $\mathscr{A}$. Note that the first resonance variety is related to the fundamental group of the arrangement complement by Suciu's conjectures [**Su01**]. Recently, Denham, Garrousian and Tohăneanu [**DGT14**] gave a characterization of modular flats of $\mathscr{A}$ in terms of a certain decomposition of $\mathbf{C}(\mathscr{A})$. The reciprocal plane, an algebraic variety whose coordinate ring is the Orlik–Terao algebra of an arrangement, has also been studied in various contexts; see [**HK12**], [**SSV13**] and [**EPW14**] for the role played by this variety in understanding the characteristic polynomial, the entropic discriminant and the Kazhdan–Lusztig polynomial, respectively.

Since the results of Proudfoot and Speyer, Schenck and Tohăneanu mentioned above are relevant to our work, we will describe them in more details. Let us first define a map $\iota : S \to S$ which is in some sense a commutative analogue of the map ∂ defined in Section 2.4.3. For a polynomial $f \in S$, let $\Lambda(f)$ be the least common multiple of the monomials of f. (By convention, $\Lambda(0) = 1$.) Then the evaluation $f(x_1^{-1}, \dots, x_n^{-1})$ of f at $(x_1^{-1}, \dots, x_n^{-1})$ can be uniquely written in the form

$$f(x_1^{-1}, \dots, x_n^{-1}) = \frac{\iota(f)}{\Lambda(f)} \quad \text{with} \quad \iota(f) \in S.$$

Thus we obtain a map $\iota : S \to S,\ f \mapsto \iota(f)$. For example, if $r = \sum_{i=1}^n a_i x_i$ is a relation in the relation space $F(\mathscr{A})$, then $\Lambda(r) = x_{\mathrm{supp}(r)}$ and

$$\iota(r) = \sum_{i \in \mathrm{supp}(r)} a_i x_{\mathrm{supp}(r)-i}.$$

Here, we write $x_\Gamma = \prod_{i \in \Gamma} x_i$ for a subset Γ of $[n]$.

We are now ready to state the result of Proudfoot and Speyer [**PS06**, Theorem 4]. This result, which is a commutative analogue of Theorem 2.4.3, relates the Orlik–Terao algebra to the broken circuit complex. Recall that every circuit C of the underlying matroid $M(\mathscr{A})$ corresponds to a unique (up to a constant) relation $r_C \in F(\mathscr{A})$.

THEOREM 2.5.2. *The set*

$$\{\iota(r_C) \mid C \text{ is a circuit of } M(\mathscr{A})\}$$

is a universal Gröbner basis for the Orlik–Terao ideal $I(\mathscr{A})$. Thus for any ordering $<$ of the hyperplanes of $\mathscr{A}$ and any induced monomial order on S, we have

$$\operatorname{in}_<(I(\mathscr{A})) = \mathscr{I}_<(M(\mathscr{A})),$$

where $\mathscr{I}_<(M(\mathscr{A}))$ is the broken circuit ideal of $(M(\mathscr{A}),<)$. In particular, the Orlik–Terao algebra $\mathbf{C}(\mathscr{A})$ is a Cohen–Macaulay ring.

Example 2.5.3. We reconsider the arrangement $\mathscr{A}$ in Example 2.4.17. By Theorem 2.5.2,

$$\begin{aligned} I(\mathscr{A}) &= (\iota(r_{C_i}) \mid i = 1,\dots,4) \\ &= (x_1x_2 - x_1x_3 - x_2x_3, x_1x_2 + x_1x_4 - x_2x_4, x_1x_3 + x_1x_4 - 2x_3x_4), \end{aligned}$$

and for any monomial order $<$ with $x_4 < x_3 < x_2 < x_1$,

$$\operatorname{in}_<(I(\mathscr{A})) = (\operatorname{in}_<(\iota(r_{C_i})) \mid i = 1,\dots,4) = (x_1x_2, x_1x_3, x_2x_3).$$

Note that the last ideal is the broken circuit ideal of $M(\mathscr{A})$ with respect to the ordering $H_4 < H_3 < H_2 < H_1$.

From Theorem 2.5.2 it is easy to derive the following results, which provide further analogies between the Orlik–Terao algebra and the Orlik–Solomon algebra; cf. Theorems 2.4.8, 2.4.9.

THEOREM 2.5.4. *$\mathscr{A}$ is supersolvable if and only if the Orlik–Terao ideal $I(\mathscr{A})$ has a quadratic Gröbner basis.*

THEOREM 2.5.5. *If $\mathscr{A}$ is supersolvable, then $\mathbf{C}(\mathscr{A})$ is a Koszul algebra.*

The latter theorem leads to the following open problems (see [**DGT14**]), which are analogous to Problems 2.4.10, 2.4.11.

PROBLEM 2.5.6. *Does the Koszulness of $\mathbf{C}(\mathscr{A})$ imply that $\mathscr{A}$ is supersolvable?*

PROBLEM 2.5.7. *Characterize arrangements for which the Orlik–Terao algebra is quadratic.*

We would like to emphasize that Problem 2.5.7 is in fact equivalent to Problem 2.4.11:

PROPOSITION 2.5.8. *Let $\mathscr{D}$ be a subset of the set of circuits of $M(\mathscr{A})$. Then the set $\{\iota(r_C) \mid C \in \mathscr{D}\}$ generates $I(\mathscr{A})$ if and only if the set $\{\partial(e_C) \mid C \in \mathscr{D}\}$ generates $J(\mathscr{A})$. In particular, $\mathbf{C}(\mathscr{A})$ is quadratic if and only if $\mathbf{A}(\mathscr{A})$ is quadratic.*

Since this result appears to be new, we will sketch the idea of its proof. Let $C = \{H_{i_1},\dots,H_{i_k}\}$ be a circuit of $M(\mathscr{A})$ and let $C_j = C - \{H_{i_j}\}$ for $i = 1,\dots,k$. Then the linear forms $\alpha_{i_1},\dots,\alpha_{i_k}$ generate a subspace of V^* of dimension $k-1$, a basis of which is any $(k-1)$-subset of $\{\alpha_{i_1},\dots,\alpha_{i_k}\}$. We fix a basis of this subspace, e.g., $\alpha_{i_1},\dots,\alpha_{i_{k-1}}$. Let $\varphi(C_j)$ be the determinant of the vectors $\alpha_{i_1},\dots,\hat{\alpha}_{i_j},\dots,\alpha_{i_k}$ (in that order) with respect to the chosen basis. Then it is easily seen that $\sum_{j=1}^k (-1)^{j-1}\varphi(C_j)\alpha_{i_j} = 0$. Thus one may take $r_C = \sum_{j=1}^k (-1)^{j-1}\varphi(C_j)x_{i_j}$ and $\iota(r_C) = \sum_{j=1}^k (-1)^{j-1}\varphi(C_j)x_{C_j}$, where $x_{C_j} = x_{i_1}\cdots\hat{x}_{i_j}\cdots x_{i_k}$. Now we define

$$\kappa(r_C) = \sum_{j=1}^k (-1)^{j-1} x_{C_j}.$$

One should compare the above expression of $\kappa(r_C)$ with the following one of $\partial(e_C)$: $\partial(e_C) = \sum_{j=1}^{k}(-1)^{j-1}e_{C_j}$. Set

$$\tilde{I}(\mathscr{A}) = (\kappa(r_C) \mid C \text{ is a circuit of } M(\mathscr{A})).$$

Then one can show that $I(\mathscr{A}) = (\iota(r_C) \mid C \in \mathscr{D})$ if and only if $\tilde{I}(\mathscr{A}) = (\kappa(r_C) \mid C \in \mathscr{D})$. So in order to prove Proposition 2.5.8 we need to show further that for any circuit D of $M(\mathscr{A})$, there exist polynomials $f_C \in S$ such that $\kappa(r_D) = \sum_{C\in\mathscr{D}} f_C \kappa(r_C)$ if and only if there exist $g_C \in E$ such that $\partial(e_D) = \sum_{C\in\mathscr{D}} g_C \partial(e_C)$. Note that if $\kappa(r_D) = \sum_{C\in\mathscr{D}} f_C \kappa(r_C)$, then one can choose f_C so that every term of f_C is squarefree. Thus in order to switch from the equation $\kappa(r_D) = \sum_{C\in\mathscr{D}} f_C \kappa(r_C)$ to the equation $\partial(e_D) = \sum_{C\in\mathscr{D}} g_C \partial(e_C)$ one merely needs to replace $\kappa(r_D)$ (respectively, $\kappa(r_C)$) with $\partial(e_D)$ (respectively, $\partial(e_C)$) and each monomial of f_C (which is squarefree) with the corresponding monomial in E multiplied by 1 or -1 (due to the anticommutativity of the multiplication in E). The following example illustrates what we just discussed.

Example 2.5.9. Let $\mathscr{A}$ be the non-Fano arrangement defined by

$$Q(\mathscr{A}) = z_3(z_1+z_2)(z_1-z_2)(z_1+z_3)(z_1-z_3)(z_2+z_3)(z_2-z_3).$$

To simplify notation a subset $\{H_{i_1},\dots,H_{i_k}\}$ of $\mathscr{A}$ will be identified with $i_1\dots i_k$. We have the following expression of $\iota(r_{1236})$ in terms of $\iota(r_{167})$, $\iota(r_{346})$, $\iota(r_{247})$, $\iota(r_{1237})$:

$$\begin{aligned}
\iota(r_{1236}) =& -x_1x_2x_3 - \frac{1}{2}x_1x_2x_6 + \frac{1}{2}x_1x_3x_6 + x_2x_3x_6\\
=&(-x_1x_2x_3 - \frac{1}{2}x_1x_2x_7 + \frac{1}{2}x_1x_3x_7 - x_2x_3x_7) + (-\frac{x_2}{2} + \frac{x_3}{2})(x_1x_6 - x_1x_7 + 2x_6x_7)\\
&+(-x_2+x_7)(x_3x_4 - x_3x_6 + x_4x_6) + (x_3+x_6)(x_2x_4 + x_2x_7 - x_4x_7)\\
=&\iota(r_{1237}) + (-\frac{x_2}{2} + \frac{x_3}{2})\iota(r_{167}) + (-x_2+x_7)\iota(r_{346}) + (x_3+x_6)\iota(r_{247}).
\end{aligned}$$

This corresponds to the following expression of $\kappa(r_{1236})$:

$$\kappa(r_{1236}) = \kappa(r_{1237}) + (x_2-x_3)\kappa(r_{167}) + (-x_2-x_7)\kappa(r_{346}) + (x_3+x_6)\kappa(r_{247}),$$

which in turn corresponds to the following expression of $\partial(e_{1236})$:

$$\partial(e_{1236}) = \partial(e_{1237}) + (-e_2+e_3)\partial(e_{167}) + (-e_2+e_7)\partial(e_{346}) + (-e_3+e_6)\partial(e_{247}).$$

Note that by using Macaulay2 [**GS**] one finds that minimal sets of generators of $I(\mathscr{A})$ and $J(\mathscr{A})$ are

$$\{\iota(r_{167}),\iota(r_{346}),\iota(r_{247}),\iota(r_{347}),\iota(r_{256}),\iota(r_{145}),\iota(r_{1237})\},$$
$$\{\partial(e_{167}),\partial(e_{346}),\partial(e_{247}),\partial(e_{347}),\partial(e_{256}),\partial(e_{145}),\partial(e_{1237})\},$$

respectively. Thus $\mathbf{C}(\mathscr{A})$ and $\mathbf{A}(\mathscr{A})$ are not quadratic.

Before turning to the result of Schenck and Tohăneanu, let us note the following useful properties of the Orlik–Terao ideal which follow immediately from the definition and Theorem 2.5.2; see also [**ST09**, Proposition 2.1, Corollary 2.2] for details.

PROPOSITION 2.5.10. *Let $r(\mathscr{A})$ be the rank of $\mathscr{A}$. Then $I(\mathscr{A})$ is prime ideal containing no linear forms and* $\operatorname{codim} I(\mathscr{A}) = n - r(\mathscr{A})$.

Now assume that $\mathscr{A}$ is a complex arrangement. Then it follows from the preceding proposition that the variety $\mathbf{V}(I(\mathscr{A}))$ is a nondegenerate, irreducible variety of codimension $n-r(\mathscr{A})$ in $\mathbb{C}^n$. The characterization of 2-formality due to Schenck and Tohăneanu [**ST09**, Theorem 2.3] is as follows.

THEOREM 2.5.11. *Let $\mathscr{A}$ be a complex arrangement. Denote by $I_{\langle 2\rangle}(\mathscr{A})$ the ideal generated by the quadratic elements of the Orlik–Terao ideal $I(\mathscr{A})$. Then $\mathscr{A}$ is 2-formal if and only if*

$$\operatorname{codim}\mathbf{V}(I(\mathscr{A})) = \operatorname{codim}(\mathbf{V}(I_{\langle 2\rangle}(\mathscr{A}))\cap(\mathbb{C}^*)^n).$$

Example 2.5.12. Let us revisit Example 2.4.21. By Proposition 2.5.10, we know that

$$\operatorname{codim}\mathbf{V}(I(\mathscr{A}_1)) = \operatorname{codim}\mathbf{V}(I(\mathscr{A}_2)) = 9-3=6.$$

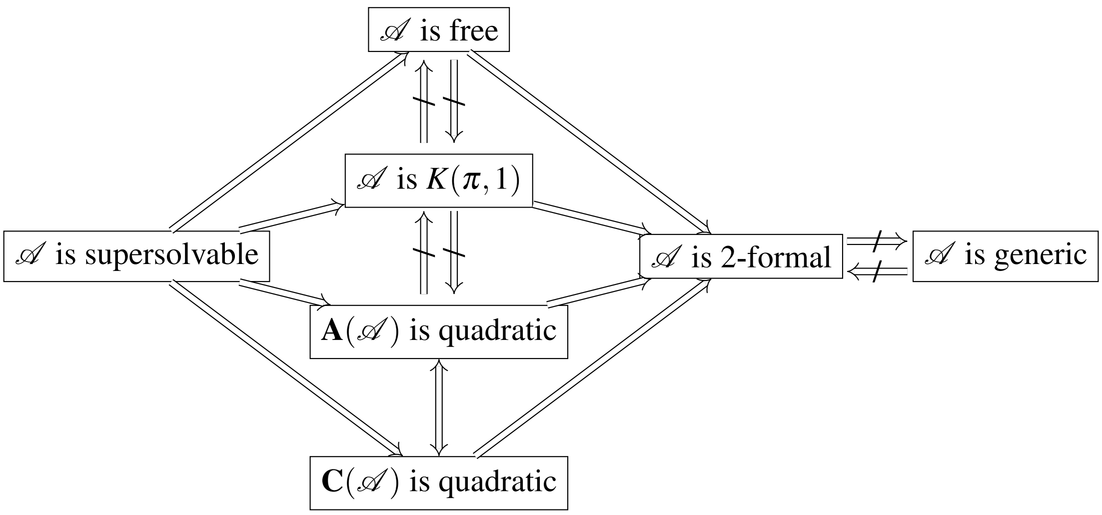
Computations using Macaulay2 give

$$\operatorname{codim}(\mathbf{V}(I_{\langle 2\rangle}(\mathscr{A}_1))\cap(\mathbb{C}^*)^9)=6, \quad \operatorname{codim}(\mathbf{V}(I_{\langle 2\rangle}(\mathscr{A}_2))\cap(\mathbb{C}^*)^9)=5.$$

Thus Theorem 2.5.11 offers another proof that $\mathscr{A}_1$ is 2-formal and $\mathscr{A}_2$ is not.

Since 2-formality is in fact a property of the relation space, Theorem 2.5.11 shows a connection between the Orlik–Terao ideal and the relation space of an arrangement. Further connections between the two objects, including a generalization of Theorem 2.5.11, will be given in the next chapter (see Section 3.5).

So far we have been concerned with several classes of hyperplane arrangements. For the convenience of the reader, we briefly summarize their relationships in the following diagram:



A few explanations are in order. We mentioned before that there exist free arrangements which are not $K(\pi,1)$ [**EdR95**] and $K(\pi,1)$ arrangements which are not free [**Te80**]. The non-Fano arrangement $\mathscr{A}$ considered in Example 2.5.9 is both free and $K(\pi,1)$ (see [**FR86**]), but as we have seen, $\mathbf{A}(\mathscr{A})$ and $\mathbf{C}(\mathscr{A})$ are not quadratic. On the other hand, the Orlik–Solomon and Orlik–Terao algebras of the arrangement X_2 defined by $Q=z_3(z_1+z_3)(z_1-z_3)(z_2+z_3)(z_2-z_3)(z_1+z_2+2z_3)(z_1+z_2-2z_3)$ are quadratic, but this arrangement is neither free nor $K(\pi,1)$; see [**Fa02**, Example 3.16]. It is easy to create a 2-formal arrangement which is not generic and a generic arrangement which is not 2-formal. In fact, for non-Boolean arrangements of rank at least three, 2-formality implies non-genericity, and conversely, genericity implies non-2-formality.

CHAPTER 3

Main results

In this chapter we summarize the main results of [**LR13, Le14, Le16, LM15**]. As we have seen in the last chapter, the Orlik–Terao ideal and the broken circuit ideal are always Cohen–Macaulay. So one may naturally ask the following questions:

(Q1) When are they Gorenstein?
(Q2) When are they complete intersections?
(Q3) When do they have linear resolutions?

These questions have been answered in [**LR13, Le14**]. Our strategy is based on the following fundamental facts:

(F1) Every initial ideal of the Orlik–Terao ideal of an arrangement $\mathscr{A}$ is an broken circuit ideal of the underlying matroid $M(\mathscr{A})$; see Theorem 2.5.2.
(F2) A graded ideal in a polynomial ring inherits from any of its initial ideals each of the following properties: being Gorenstein, a complete intersection, having a linear resolution; see, e.g., [**HH11**, Corollary 3.3.3, Theorem 3.3.4].

Thus in order to answer each of the above questions we essentially follow three steps:

Step 1: Answer the question for the broken circuit ideal.
Step 2: Prove that a partial converse to (F2) holds for the Orlik–Terao ideal: if the Orlik–Terao ideal has one of the properties listed in (F2), then one of its initial ideals has the same property.
Step 3: Deduce the answer for the Orlik–Terao ideal from Step 1 and Step 2.

In [**LR13**], answers to questions (Q2) and (Q3) are provided. In particular, it is shown that generic arrangements and their cones are exactly those arrangements whose Orlik–Terao ideal has a linear resolution (see Section 3.1). Moreover, arrangements with complete intersection Orlik–Terao ideal are characterized combinatorially: the underlying matroid of such an arrangement is a direct sum of parallel connections of circuits and coloops (see Section 3.2).

Answers to question (Q1) are given in [**Le14**]. It is proved that for the Orlik–Terao ideal and the broken circuit ideal, the Gorenstein and complete intersection properties coincide. In addition, the Gorenstein/complete intersection property of the Orlik–Terao ideal can be determined by two conditions on the h-vector of the broken circuit complex of the underlying matroid. More precisely, let $(h_0, h_1, \ldots, h_s)$ be the h-vector of the broken circuit complex of a matroid M. Then there is an ordering of the ground set of M such that the corresponding broken circuit complex is a complete intersection if and only if $h_0 = h_s$ and $h_1 = h_{s-1}$. Applying this result to the case when M is the underlying matroid of an arrangement yields the mentioned characterization of the Gorenstein/complete intersection property of the Orlik–Terao ideal (see Section 3.3).

Recall from Proposition 2.3.12 that if M is a connected matroid, then the condition $h_0 = h_s$ is equivalent to M being a series–parallel network. Hence when M is a series–parallel network, the aforementioned result of [**Le14**] leads to the idea that the difference $h_{s-1} - h_1$ might have some significance. In [**Le16**], this idea is made precise by a formula for $h_{s-1} - h_1$ in terms of an ear decomposition of M. This formula has several interesting consequences and related results. In particular, an excluded minor characterization for the case $h_{s-1} - h_1 = 0$ is provided. Evidence supporting Conjecture 2.3.13 is also given (see Section 3.4).

Finally, the relationship between the relation space and the Orlik–Terao ideal of an arrangement is studied in [**LM15**]. Our main result is a characterization of spanning sets of the relation space in terms of the Orlik–Terao ideal, which generalizes Theorem 2.5.11. Minimal prime ideals of certain subideals of the Orlik–Terao ideal that correspond to subsets of the relation space are also described (see Section 3.5).

Along the chapter several open problems related to our work will also be discussed.

3.1. Linear resolutions

Eisenbud, Popescu and Yuzvinsky [**EPY03**] proved that generic arrangements and their cones are exactly those arrangements for which the Orlik–Solomon ideal admits a linear resolution. This result was extended to matroids by Kämpf and Römer [**KR09**]. In this section, we give analogous characterizations of Orlik–Terao ideals and broken circuit ideals with linear resolutions. Our results presented here are part of [**LR13**].

Throughout the section let $\mathbb{K}$ be an infinite field and $S = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring over $\mathbb{K}$. Let $\mathscr{A}$ be an arrangement of n hyperplanes over $\mathbb{K}$ with underlying matroid $M(\mathscr{A})$ and Orlik–Terao ideal $I(\mathscr{A}) \subset S$. For an ordered matroid $(M, <)$, its broken circuit ideal will be denoted by $\mathscr{I}_<(M)$.

First, note that for a graded Cohen–Macaulay ideal in S, one can determine whether it admits a linear resolution from its Hilbert function, by virtue of the following result of Cavaliere, Rossi, and Valla [**CRV90**, Proposition 2.1] (see also Rentería and Villarreal [**RV92**, Theorem 3.2]).

PROPOSITION 3.1.1. *Let $I = \bigoplus_{j \geq p} I_j$ with $I_p \neq 0$ be a graded Cohen-Macaulay ideal in S of codimension c. Then I has a p-linear resolution if and only if $H(I, p) = \binom{p+c-1}{p}$, where $H(I, \bullet)$ denotes the Hilbert function of I.*

Since passing to initial ideals preserves the Hilbert function, the previous result and Theorem 2.5.2 yield:

COROLLARY 3.1.2. *The following conditions are equivalent:*

(i) *$I(\mathscr{A})$ has a linear resolution;*
(ii) *$\mathscr{I}_<(M(\mathscr{A}))$ has a linear resolution for some ordering $<$ of $\mathscr{A}$;*
(iii) *$\mathscr{I}_<(M(\mathscr{A}))$ has a linear resolution for every ordering $<$ of $\mathscr{A}$.*

Thus characterizing the Orlik–Terao ideal with a linear resolution becomes a purely combinatorial problem on the broken circuit ideal. To solve this problem we need the following lower bound for the f-vector of the broken circuit complex which was shown by Heron [**He72**] and Björner [**Bj92**, Proposition 7.5.6].

PROPOSITION 3.1.3. *Let M be a simple matroid of rank r on an n-element ground set. Let $(f_0, f_1, \ldots, f_r)$ be the f-vector of $BC(M)$. Denote by g the smallest size of a circuit of M, i.e., $g = \min\{|C| : C \text{ is a circuit of } M\}$. Then*

$$f_k \geq \sum_{i=0}^{g-2} \binom{n-r+i-1}{i} \binom{r-i}{k-i} \quad \text{for } k = 0, 1, \ldots, r,$$

and the following conditions are equivalent:

(i) *equality holds for some $k \geq g-1$;*
(ii) *equality holds for all k;*
(iii) *M is isomorphic to $U_{g-1,n-r+g-1} \oplus U_{r-g+1,r-g+1}$.*

Based on Propositions 3.1.1 and 3.1.3 we obtain the following characterization of broken circuit ideals with a linear resolution.

THEOREM 3.1.4 ([**LR13**, Theorem 3.3]). *Let $(M,<)$ be an ordered simple matroid of rank r on an n-element ground set. Let p be an integer with $2 \leq p \leq r$. Then $\mathscr{I}_<(M)$ has a p-linear resolution if and only if M is isomorphic to $U_{p,n-r+p} \oplus U_{r-p,r-p}$.*

Now those arrangements for which the Orlik–Terao ideal admits a linear resolution can be characterized as follows.

THEOREM 3.1.5 ([**LR13**, Theorem 3.5]). *Let $r = r(\mathscr{A})$ and let p be an integer with $2 \leq p \leq r$. Then the following conditions are equivalent:*

(i) *$I(\mathscr{A})$ has a p-linear resolution;*
(ii) *$M(\mathscr{A})$ is isomorphic to $U_{p,n-r+p} \oplus U_{r-p,r-p}$;*
(iii) *$\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$, where $\mathscr{A}_1$ is a rank p generic arrangement of $n-r+p$ hyperplanes and $\mathscr{A}_2$ is a rank $(r-p)$ Boolean arrangement;*
(iv) *$\mathscr{A}$ is obtained by successively coning a rank p generic arrangement of $n-r+p$ hyperplanes;*
(v) *the Orlik–Solomon ideal of $\mathscr{A}$ has a p-linear resolution.*

An important generalization of the notion of ideals with linear resolutions is that of componentwise linear ideals. We say that a graded ideal $I \subset S$ is *componentwise linear* if $I_{\langle j \rangle}$ has a linear resolution for every j, where $I_{\langle j \rangle}$ denotes the ideal generated by all homogeneous elements of I of degree j. It is well-known that ideals with linear resolutions are componentwise linear; see, e.g., [**HH11**, Lemma 8.2.10]. So the following problem arises naturally from Theorems 3.1.4, 3.1.5.

PROBLEM 3.1.6. *Characterize the componentwise linearity of the Orlik–Terao ideal and the broken circuit ideal.*

3.2. Complete intersections

In this section we characterize the complete intersection property of the broken circuit ideal and the Orlik–Terao ideal. Somewhat surprisingly, if the Orlik–Terao ideal of an arrangement is a complete intersection, then some broken circuit ideal of the underlying matroid is also a complete intersection. Thus the complete intersection property of the Orlik–Terao ideal depends only on the underlying matroid and is therefore combinatorial.

Along the way, we improve upper bounds for coefficients of the chromatic polynomial of a maximal planar graph given by Wilf [**Wil76**]. Moreover, in the special case when the Orlik–Terao ideal is a complete intersection, we show that Problems 2.4.10, 2.4.11 can be easily resolved. The results of this section are part of [**LR13**].

Let us start with the complete intersection property of the broken circuit ideal. Let $(M,<)$ be an ordered loopless matroid. It is apparent that the minimal non-faces of $BC_<(M)$ are exactly the minimal broken circuits of M. Therefore, the broken circuit ideal $\mathscr{I}_<(M)$ is minimally generated by the monomials indexed by those minimal broken circuits. It follows that $\mathscr{I}_<(M)$ is a complete intersection if and only if the minimal broken circuits of M are pairwise disjoint. This observation leads to consideration of matroids which are parallel connections of circuits.

Recall from Section 2.2.6 that a graph $G=(V(G),E(G))$ is the parallel connection of the cycles $C_1,\dots,C_k$ with respect to the base edges $e_1,\dots,e_{k-1}$ if the following conditions are satisfied:

(i) G is the union of $C_1,\dots,C_k$, i.e., $V(G)=\bigcup_{i=1}^k V(C_i)$ and $E(G)=\bigcup_{i=1}^k E(C_i)$;
(ii) $E(C_{j+1})\cap(\bigcup_{i=1}^j E(C_i))=\{e_j\}$ for $j=1,\dots,k-1$.

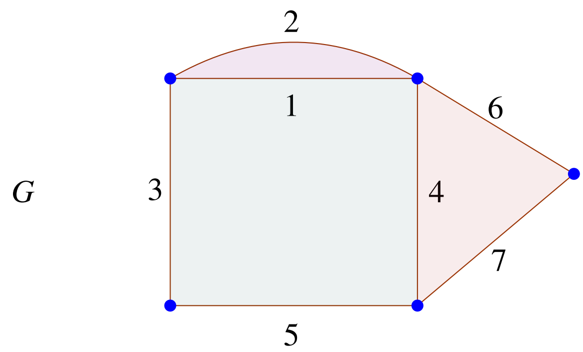


FIGURE 3.1. A parallel connection of cycles

Definition 3.2.1. Assume that G is the parallel connection of the cycles $C_1,\dots,C_k$ with respect to the base edges $e_1,\dots,e_{k-1}$. A linear ordering $<$ of $E(G)$ is said to be *relevant* to G if $e_j=\min_< E(C_{j+1})$ or $e_j=\min_< \bigcup_{i=1}^j E(C_i)$ for $j=1,\dots,k-1$.

Example 3.2.2. Let G be the graph depicted in Figure 3.1. Then G is the parallel connection of 3 cycles $C_1=\{1,2\}$, $C_2=\{1,3,5,4\}$, $C_3=\{4,6,7\}$ with respect to the base edges 1 and 4. Evidently, the ordinary ordering $1<2<\cdots<7$ is relevant to G. With this ordering the minimal broken circuits of the cycles matroid $M(G)$ are pairwise disjoint and $\mathscr{I}_<(M(G))=(x_2,x_3x_4x_5,x_6x_7)$ is a complete intersection.

One can easily generalize the above example: if $M=M(G)$ where G is a parallel connection of cycles and $<$ is relevant to G, then the minimal broken circuits of $(M,<)$ are pairwise disjoint and $\mathscr{I}_<(M)$ is a complete intersection. We are able to show that the converse holds as well:

THEOREM 3.2.3 ([**LR13**, Theorem 4.1]). *Let $(M,<)$ be an ordered loopless matroid with broken circuit ideal $\mathscr{I}_<(M)$. Then the following conditions are equivalent:*

(i) *$\mathscr{I}_<(M)$ is a complete intersection;*
(ii) *the minimal broken circuits of M are pairwise disjoint;*
(iii) *each connected component of M is a cycle matroid of a loopless graph G, where G is either the complete graph K_2 or a parallel connection of cycles and $<$ is relevant to G.*

As an application of the above result, we will give upper bounds for the coefficients of the chromatic polynomial of a maximal planar graph, improving the bounds given by Wilf [**Wil76**]. Note that Wilf was the first one who considered the broken circuit complex (of a graph) as a simplicial complex. This viewpoint led him to several inequalities for coefficients of chromatic polynomials of graphs, including the bounds just mentioned.

A simple graph G is called *maximal planar* if G is planar and adding any edge (on the vertex set of G) results in a non-planar graph. Recall that any plane drawing of a planar graph divides the plane into regions, called *faces* of the graph. Evidently, maximal planar graphs are connected and every face of a maximal planar graph is bouded by a triangle. Moreover, a maximal planar graph on $\ell \geq 3$ vertices has exactly $3\ell-6$ edges and $2\ell-4$ faces; see, e.g., [**Har69**, Corollary 11.1(b)].

Let G be a maximal planar graph on ℓ vertices with chromatic polynomial $\chi(G,t) = \sum_{i=0}^{\ell-1}(-1)^i f_i t^{\ell-i}$. Notice that $(f_0,f_1,\ldots,f_{\ell-1})$ is the f-vector of $BC(M(G))$; see Theorem 2.3.10 and Equation (2.4). In [**Wil76**], Wilf observed that the coefficients of $\chi(G,t)$ are dominated by the corresponding coefficients of the function

$$t^{-2\ell+6+m}(t+1)^{3\ell-6-2m}(t+2)^m,$$

where m is the cardinality of a set of pairwise disjoint broken circuits of $M(G)$. To obtain the sharpest bounds we need that m is as large as possible. Wilf [**Wil76**, Theorem 3] showed that there exists an ordering of the edges of G such that $M(G)$ has at least $\ell-2$ pairwise disjoint broken circuits. Consequently, the coefficients of $\chi(G,t)$ are bounded above by the corresponding coefficients of the function $t^{-\ell+4}(t+1)^{\ell-2}(t+2)^{\ell-2}$, i.e.,

$$f_i \leq \sum_{k=0}^{\min\{i,\ell-2\}} \binom{\ell-2}{i-k}\binom{\ell-2}{k}2^k, \quad i=0,\ldots,\ell-1;$$

see [**Wil76**, Theorem 4]. Based on Theorem 3.2.3, we get better lower bounds for the maximal number of pairwise disjoint broken circuits of $M(G)$ as follows.

PROPOSITION 3.2.4 ([**LR13**, Proposition 4.11]). *Let G be a maximal planar graph on $\ell \geq 3$ vertices. Then there exists an ordering of the edges of G such that $M(G)$ has at least $\ell-2+\lfloor \ell/4 \rfloor$ pairwise disjoint broken circuits. Moreover, if the dual graph of G contains no triangles, then this lower bound can be improved to $\ell-3+\lceil \ell/3 \rceil$.*

Thus we obtain the following improvement of [**Wil76**, Theorem 4].

THEOREM 3.2.5 ([**LR13**, Theorem 4.12]). *Let $\chi(G,t) = \sum_{i=0}^{\ell-1}(-1)^i f_i t^{\ell-i}$ be the chromatic polynomial of a maximal planar graph G. Then the coefficients of $\chi(G,t)$ are bounded above by the corresponding coefficients of the function*

$$\eta(t) = t^{-\ell+4+\lfloor \ell/4 \rfloor}(t+1)^{\ell-2-2\lfloor \ell/4 \rfloor}(t+2)^{\ell-2+\lfloor \ell/4 \rfloor},$$

or explicitly,

$$f_i \leq \sum_{k=0}^{\min\{i,\ell-2+\lfloor \ell/4\rfloor\}} \binom{\ell-2-2\lfloor \ell/4\rfloor}{i-k}\binom{\ell-2+\lfloor \ell/4\rfloor}{k} 2^k, \quad i=0,\dots,\ell-1.$$

If the dual graph of G has no triangles, then the function $\eta(t)$ can be replaced by

$$\theta(t) = t^{-\ell+3+\lceil \ell/3\rceil}(t+1)^{\ell-2\lceil \ell/3\rceil}(t+2)^{\ell-3+\lceil \ell/3\rceil},$$

and we have

$$f_i \leq \sum_{k=0}^{\min\{i,\ell-3+\lceil \ell/3\rceil\}} \binom{\ell-2\lceil \ell/3\rceil}{i-k}\binom{\ell-3+\lceil \ell/3\rceil}{k} 2^k, \quad i=0,\dots,\ell-1.$$

Let us now turn to the complete intersection property of the Orlik–Terao ideal. We are able to show that if the Orlik–Terao ideal is a complete intersection, then at least one of its initial ideals is also a complete intersection. Thus from Theorem 3.2.3 we obtain the following characterization.

THEOREM 3.2.6 ([**LR13**, Theorem 4.16]). *Let $\mathscr{A}$ be an arrangement with Orlik–Terao ideal $I(\mathscr{A})$ and underlying matroid $M(\mathscr{A})$. Then the following conditions are equivalent:*

- (i) *$I(\mathscr{A})$ is a complete intersection;*
- (ii) *there exists an ordering $<$ of $\mathscr{A}$ such that the broken ideal $\mathscr{I}_<(M(\mathscr{A}))$ is a complete intersection;*
- (iii) *each connected component of $M(\mathscr{A})$ is a cycle matroid of a simple graph G, where G is either K_2 or a parallel connection of cycles.*

Arrangements with complete intersection Orlik–Terao ideal are quite special. For these arrangements several important properties coincide. In particular, it is easy to solve Problems 2.4.10, 2.4.11 within this class of arrangements.

THEOREM 3.2.7 ([**LR13**, Corollary 4.18]). *Let $\mathscr{A}$ be an arrangement of n hyperplanes. Assume that the Orlik–Terao ideal $I(\mathscr{A})$ of $\mathscr{A}$ is a complete intersection. Let $q_1,\dots,q_r$ be the degree sequence of a minimal system of homogeneous generators of $I(\mathscr{A})$. Then the Poincaré polynomial of $\mathscr{A}$ is*

$$\pi(\mathscr{A},t) = (t+1)^{n-\sum_{i=1}^r q_i}\prod_{i=1}^r\big((t+1)^{q_i}-t^{q_i}\big).$$

Moreover, the following conditions are equivalent:

- (i) *$\pi(\mathscr{A},t)$ factors completely over $\mathbb{Z}$;*
- (ii) *$\mathbf{C}(\mathscr{A})$ is quadratic, i.e., $q_i=2$ for all $i=1,\dots,r$;*
- (iii) *$\mathbf{C}(\mathscr{A})$ is Koszul;*
- (iv) *$\mathscr{A}$ is supersolvable;*
- (v) *$\mathscr{A}$ is free;*
- (vi) *$\mathscr{A}$ is $K(\pi,1)$;*
- (vii) *$\mathscr{A}$ is 2-formal;*
- (viii) *$\mathbf{A}(\mathscr{A})$ is Koszul.*

It should be noted here that when the Orlik–Terao ideal is quadratic, its complete intersection property has been independently studied by Denham, Garrousian and Tohăneanu [**DGT14**] with a different method. In particular, they obtained a variation of Theorem 3.2.7; see [**DGT14**, Corollary 5.12].

3.3. The Gorenstein property

Having characterized the complete intersection property of the broken circuit ideal and the Orlik–Terao ideal we now present our results on their Gorensteinness, following [**Le14**]. We show that each of these ideals is Gorenstein if and only if it is a complete intersection. Furthermore, the Gorenstein/complete intersection property of the Orlik–Terao ideal can be determined by (at most) the last two entries of the h-vector of the broken circuit complex of the underlying matroid.

For the Gorensteinness of the broken circuit ideal, based on Theorems 2.1.3 and 2.1.4, we obtain the following characterization. It implies, in particular, that the Gorensteinness of the broken circuit ideal does not depend on the base field.

THEOREM 3.3.1 ([**Le14**, Theorem 1.1]). *Let M be an ordered loopless matroid with broken circuit complex $BC(M)$. Let $\mathbb{K}$ be an arbitrary field. Then the following conditions are equivalent:*

(i) *$BC(M)$ is Gorenstein over $\mathbb{K}$;*
(ii) *$BC(M)$ is locally Gorenstein over $\mathbb{K}$;*
(iii) *for any face F of $BC(M)$ with $\dim \operatorname{link}_{BC(M)} F = 1$ one has that $\operatorname{link}_{BC(M)} F$ is Gorenstein over $\mathbb{K}$;*
(iv) *for any face F of $BC(M)$ with $\dim \operatorname{link}_{BC(M)} F = 1$ one has that $\operatorname{link}_{BC(M)} F$ is a complete intersection complex;*
(v) *$BC(M)$ is a locally complete intersection complex;*
(vi) *$BC(M)$ is a complete intersection complex.*

Apart from Theorems 2.1.3 and 2.1.4, the following properties of the broken circuit complex are essential in our proof of the above result.

LEMMA 3.3.2 ([**Le14**, Lemmas 3.4, 3.5]). *Let M be an ordered loopless matroid. The following statements hold:*

(i) *If F is a face of $BC(M)$ such that $\operatorname{link}_{BC(M)} F$ is an n-gon, then $n \le 4$.*
(ii) *If $BC(M)$ is an m-vertex path, then $m \le 3$.*

Let us discuss some related results of Theorem 3.3.1. A classical theorem due to Cowsik and Nori [**CN76**] implies that the Stanley–Reisner ideal I_Δ of a simplicial complex Δ is a complete intersection if and only if all of its powers I_Δ^m are Cohen–Macaulay. In [**TT12**], Terai and Trung gave a refinement of this result: they showed that I_Δ is a complete intersection if and only if I_Δ^m is Cohen–Macaulay for some $m \ge 3$, which is the case precisely when I_Δ^m satisfies Serre's condition (S_2) for some $m \ge 3$. They also pointed out that there are simplicial complexes Δ for which I_Δ^2 is Cohen–Macaulay but I_Δ^m is not Cohen–Macaulay for every $m \ge 3$ (e.g., a 5-gon). This phenomenon, however, does not happen for the broken circuit complex.

PROPOSITION 3.3.3 ([**Le14**, Proposition 3.6]). *Let M be an ordered loopless matroid with broken circuit ideal $\mathscr{I}(M)$. Then the following conditions are equivalent:*

(i) *$\mathscr{I}(M)$ is a complete intersection;*
(ii) *$\mathscr{I}(M)^m$ is Cohen–Macaulay for every $m \geq 1$;*
(iii) *$\mathscr{I}(M)^m$ is Cohen–Macaulay for some $m \geq 2$;*
(iv) *$\mathscr{I}(M)^m$ satisfies (S_2) for some $m \geq 2$.*

Theorem 3.3.1 and Proposition 3.3.3 have applications to the independence complex. Let M be a matroid. Recall from Proposition 2.3.7 that the cone over the independence complex $IN(M)$ is the broken circuit complex $BC(M')$ of another matroid M'. Since the Stanley–Reiner ring of $BC(M')$ is then a polynomial ring over the Stanley–Reiner ring of $IN(M)$, it follows that $BC(M')$ is Gorenstein (respectively, a complete intersection) if and only if so is $IN(M)$. Thus from Theorem 3.3.1 and Proposition 3.3.3 we easily get the following corollary, which recovers Theorem 2.3.4.

COROLLARY 3.3.4 ([**Le14**, Corollary 3.7]). *Let M be a matroid with independence complex $IN(M)$. Let $\mathbb{K}$ be an arbitrary field. Denote by $I_{IN(M)}$ the Stanley–Reisner ideal of $IN(M)$ over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *$IN(M)$ is Gorenstein over $\mathbb{K}$;*
(ii) *$IN(M)$ is a complete intersection complex;*
(iii) *$I_{IN(M)}^m$ is Cohen–Macaulay for every $m \geq 1$;*
(iv) *$I_{IN(M)}^m$ is Cohen–Macaulay for some $m \geq 2$;*
(v) *$I_{IN(M)}^m$ satisfies (S_2) for some $m \geq 2$.*

Moreover, if $\dim IN(M) \geq 1$, then each of the above conditions is equivalent to any one of the following:

(vi) *$IN(M)$ is locally Gorenstein over $\mathbb{K}$;*
(vii) *for any face F of $IN(M)$ with $\dim \operatorname{link}_{IN(M)} F = 1$ one has that $\operatorname{link}_{IN(M)} F$ is Gorenstein over $\mathbb{K}$;*
(viii) *for any face F of $IN(M)$ with $\dim \operatorname{link}_{IN(M)} F = 1$ one has that $\operatorname{link}_{IN(M)} F$ is a complete intersection complex;*
(ix) *$IN(M)$ is a locally complete intersection complex.*

Note that in the preceding result, if $\dim IN(M) = 0$, then conditions (i)-(v) may not be equivalent to conditions (vi)-(ix). The reason is that $IN(M)$ is not always a cone.

Example 3.3.5. Let M be the uniform matroid $U_{1,n}$ with $n \geq 3$. Then $IN(M)$ consists of n vertices. So $IN(M)$ is locally Gorenstein (and a locally complete intersection), but not Gorenstein by Theorem 2.1.3(ii). Note that in this case, conditions (vii), (viii) in Corollary 3.3.4 hold vacuously.

We now turn to the Gorensteinness of the Orlik–Terao ideal. Let $\mathscr{A}$ be an arrangement with Orlik–Terao ideal $I(\mathscr{A})$, Orlik–Terao algebra $\mathbf{C}(\mathscr{A})$, and underlying matroid $M(\mathscr{A})$. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of the broken circuit complex of $M(\mathscr{A})$. Note that $(h_0, h_1, \ldots, h_s)$ is also the h-vector of $\mathbf{C}(\mathscr{A})$, by virtue of Theorem 2.5.2. Since $\mathbf{C}(\mathscr{A})$ is a Cohen–Macaulay domain, it follows from Theorem 2.1.1 that $\mathbf{C}(\mathscr{A})$ is Gorenstein if and only if the h-vector $(h_0, h_1, \dots, h_s)$ is symmetric. The latter condition can be characterized for general matroids as follows; cf. Corollary 2.3.5.

THEOREM 3.3.6 ([**Le14**, Theorem 1.2]). *Let $(h_0, h_1, \dots, h_s)$ be the h-vector of the broken circuit complex of a loopless matroid M. The following conditions are equivalent:*

(i) *The h-vector $(h_0, h_1, \dots, h_s)$ is symmetric, i.e., $h_i = h_{s-i}$ for $i = 0, \dots, s$;*
(ii) *$h_0 = h_s$ and $h_1 = h_{s-1}$;*
(iii) *each connected component of M is a cycle matroid of a loopless graph G, where G is either K_2 or a parallel connection of cycles;*
(iv) *there exists an ordering $<$ of the ground set of M such that the broken circuit ideal $\mathscr{I}_<(M)$ is a complete intersection.*

This result reveals a somewhat unexpected implication of the h-vector of the broken circuit complex of a matroid on the structure of the matroid. We will explore further this implication for series–parallel networks in the next section. Note that our proof of the above result relies heavily on the theory of series–parallel networks developed by Brylawski [**Bry71**].

From Theorem 3.3.6 we obtain the following characterization of the Gorensteinness of the Orlik–Terao ideal.

THEOREM 3.3.7 ([**Le14**, Theorem 1.3]). *Let $\mathscr{A}$ be an arrangement with Orlik–Terao ideal $I(\mathscr{A})$ and underlying matroid $M(\mathscr{A})$. Let $(h_0, h_1, \dots, h_s)$ be the h-vector of the broken circuit complex of $M(\mathscr{A})$. Then the following conditions are equivalent:*

(i) *$I(\mathscr{A})$ is Gorenstein;*
(ii) *$h_0 = h_s$ and $h_1 = h_{s-1}$;*
(iii) *$I(\mathscr{A})$ is a complete intersection.*

By Equation (2.1) we know that $h_0 = 1$ and $h_1 = n - r$ if the arrangement $\mathscr{A}$ contains n hyperplanes and has rank r. So the above theorem implies that the Gorenstein/complete intersection property of $I(\mathscr{A})$ is determined by h_s and h_{s-1}.

We conclude this section with two open problems. The first one arises naturally from the levelness of the independence complex (Theorem 2.3.1) and our characterizations of the Gorenstein and complete intersection properties of the broken circuit algebra and the Orlik–Terao algebra.

PROBLEM 3.3.8. *Characterize the levelness of the broken circuit algebra and the Orlik–Terao algebra.*

By now, as far as we are aware, there is only one result on the levelness of the broken circuit algebra due to Llamas, Martínez-Bernal and Merino [**LMM10**]. They showed that the cycle matroid of the cone over a simple graph admits a level broken circuit algebra.

Since the broken circuit and Orlik–Terao algebras are both Cohen–Macaulay, they have canonical modules. The following problem might be of interest.

PROBLEM 3.3.9. *Determine the canonical modules of the broken circuit algebra and the Orlik–Terao algebra.*

3.4. Series–parallel networks

This section serves as a summary of [**Le16**]. Let M be a series–parallel network. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of the broken circuit complex of M. Then $h_0 = h_s$ by Proposition 2.3.12. So it follows from Theorem 3.3.6 that M admits a complete intersection broken circuit complex if and only if $h_{s-1} - h_1 = 0$. This observation suggests that the difference $h_{s-1} - h_1$ might have significant implications for the structure of M. In order to make this idea more precise we give a formula for $h_{s-1} - h_1$ in terms of an ear decomposition of the underlying graph of M. Several applications and related results of this formula are then discussed. In particular, we derive some bounds for $h_{s-1} - h_1$. We also provide an excluded minor characterization of the class of matroids for which $h_0 = h_s$ and $h_{s-1} = h_1$, i.e., those matroids that have been characterized by Theorem 3.3.6. Finally, we give evidence for Conjecture 2.3.13.

Let us start by recalling the notion of ear decompositions of graphs.

Definition 3.4.1. Let G be a loopless connected graph. An *ear decomposition* of G is a partition of the edges of G into a sequence of *ears* $\pi_1, \pi_2, \ldots, \pi_n$ such that:

(ED1) π_1 is a cycle and each π_i is a simple path (i.e., a path that does not intersect itself) for $i \geq 2$,
(ED2) each end vertex of π_i, $i \geq 2$, is contained in some π_j with $j < i$,
(ED3) no internal vertex of π_i is in π_j for any $j < i$.

Whitney [**Wh32b**] proved that a graph with at least 2 edges admits an ear decomposition if and only if it is nonseparable. He also showed that for a nonseparable graph $G = (V(G), E(G))$, the number n of ears in any ear decomposition of G is independent of the decomposition and coincides with the *nullity* (or *cyclomatic number*) of G, i.e., $n = |E(G)| - |V(G)| + 1$.

Let $\Pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be an ear decomposition of a graph G. Then the ear π_i is called *nested* in π_j, $j < i$, if both end vertices of π_i belong to π_j and at least one of them is an internal vertex of π_j. When π_i is nested in π_j, the *nest interval* of π_i in π_j is the path in π_j between the two end vertices of π_i. Here we adopt the convention that the nest interval of an ear in π_1 is the shorter path, and that if π_1 is divided into two paths of equal length then at most one of them could be a nest interval. The ear decomposition Π is called *nested* if the following conditions hold:

(N1) for each $i > 1$ there exists $j < i$ such that π_i is nested in π_j,
(N2) if π_i and π_k are both nested in π_j, then either their nest intervals in π_j are disjoint, or one nest interval contains the other.

Eppstein [**Ep92**, Theorem 1] gave the following characterization of series–parallel networks in terms of ear decompositions.

THEOREM 3.4.2. *Let G be a nonseparable graph with at least 2 edges. Then the following conditions are equivalent:*

(i) *G is a series–parallel network;*
(ii) *G has a nested ear decomposition;*
(iii) *every ear decomposition of G is nested.*

As mentioned before, the number of ears in any ear decomposition is independent of the decomposition. Naturally, one may ask whether the number of nest intervals in a nested ear decomposition is also independent of the decomposition. It turns out that this question is not as simple as it seems. In fact, we can only achieve an affirmative answer to the question through a subtle analysis of the set of nest intervals.

Let $\Pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be a nested ear decomposition of a series–parallel network G. By abuse of notation, we will also regard Π as the set $\{\pi_1, \pi_2, \ldots, \pi_n\}$. For a nest interval I appearing in Π, consider the following collection of paths

$$\Sigma(I) = \{\pi_i \in \Pi : I \text{ is the nest interval of } \pi_i\} \cup \{I\}.$$

Denote by $\lambda(I)$ the minimal length of a path in $\Sigma(I)$.

Definition 3.4.3. With assumptions and notation as above, we define $p_1(G;\Pi)$, $p_2(G;\Pi)$ to be the numbers of nest intervals I such that $\lambda(I) = 1$ and $\lambda(I) > 1$, respectively.

We will see that $p_1(G;\Pi)$ and $p_2(G;\Pi)$ not only are independent of the decomposition Π but also encode interesting combinatorial information about the graph G. Therefore, in particular, the number of nest intervals appearing in Π is an invariant of G.

Example 3.4.4. Let G be the graph depicted in Figure 3.2. Then $\Pi = (\pi_1, \ldots, \pi_5)$ is a nested ear decomposition of G, where

$$\pi_1 = \{1,2,3,4,5\},\ \pi_2 = \{6\},\ \pi_3 = \{7\},\ \pi_4 = \{8,9,10\},\ \pi_5 = \{11,12\}.$$

This decomposition has 3 nest intervals: $I_1 = \{3\}, I_2 = \{4,5\}, I_3 = \{9,10\}$. Since $\lambda(I_1) = \lambda(I_2) = 1$ and $\lambda(I_3) = 2$, we have $p_1(G;\Pi) = 2$ and $p_2(G;\Pi) = 1$.

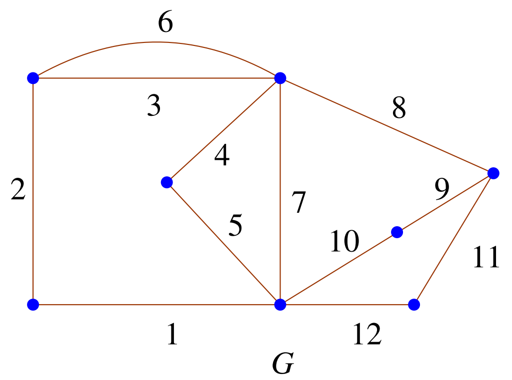


FIGURE 3.2. A series–parallel network

The following result shows that $p_1(G;\Pi)$ is independent of Π and that this number is related to parallel irreducible decompositions of G.

THEOREM 3.4.5 ([**Le16**, Theorem 4.2]). *Let G be a series–parallel network with at least 2 edges. Denote by $\overline{G}$ the simplification of G. Let $F(G)$ be the set of all edges e of $\overline{G}$ such that G/e is separable. Then for any nested ear decomposition Π of G we have $p_1(G;\Pi) = |F(G)|$. In particular, if G is simple, then $p_1(G;\Pi)$ is the number of distinct base edges of the parallel irreducible decomposition of G.*

Example 3.4.6. Consider the graph G depicted in Figure 3.2. Then we may choose $\overline{G} = G - 6$. It is easy to see that $F(G) = \{3,7\}$. Thus $|F(G)| = 2 = p_1(G;\Pi)$, where Π is the ear decomposition of G given in Example 3.4.4. Note that G has a parallel irreducible decomposition $G = P(G_1,G_2,G_3,G_4)$ with respect to the base edges $3,7,7$, where the edge sets of the graphs G_i are respectively

$$\{1,2,3,7\},\ \{3,6\},\ \{4,5,7\},\ \{7,8,9,10,11,12\}.$$

From Theorem 3.4.5 and Proposition 2.2.3(i) we immediately obtain the following interesting characterization of parallel irreducibility of series-parallel networks in terms of ear decompositions.

COROLLARY 3.4.7 ([**Le16**, Corollary 4.7]). *Let G be a series-parallel network with at least 2 edges. Then G is parallel irreducible if and only if $p_1(G;\Pi) = 0$ for an/every ear decomposition Π of G.*

The next theorem, which clarifies the connection between the h-vector of the broken circuit complex and the structure of a series–parallel network, is the main result of [**Le16**].

THEOREM 3.4.8 ([**Le16**, Theorem 4.8]). *Let G be a series–parallel network with at least 2 edges. Let $(h_0,h_1,\ldots,h_s)$ be the h-vector of the broken circuit complex of the cycle matroid $M(G)$. Then for any ear decomposition Π of G we have $h_{s-1} - h_1 = p_2(G;\Pi)$. In particular, $p_2(G;\Pi)$ does not depend on Π.*

Example 3.4.9. Let G be the graph in Figure 3.2. Clearly, the cycle matroid $M(G)$ has rank 7. So the h-vector of $BC(M(G))$ has the form $(h_0,h_1,\ldots,h_6)$; see Theorem 2.3.11. Now by Theorem 3.4.8, $h_5 - h_1 = p_2(G;\Pi) = 1$, where Π is the ear decomposition of G given in Example 3.4.4.

We now can confirm that the number of nest intervals in a nested ear decomposition does not depend on the decomposition. The next consequence follows from Theorems 3.4.5 and 3.4.8.

COROLLARY 3.4.10 ([**Le16**, Corollary 4.11]). *Let G be a series-parallel network with at least 2 edges. Let $(h_0,h_1,\ldots,h_s)$ be the h-vector of $BC(M(G))$. Then the number p of nest intervals appearing in a nested ear decomposition of G is independent of the decomposition. In addition, $h_{s-1} - h_1 \le p$ with equality if and only if $M(G)$ is parallel irreducible.*

In the remaining part of this section, we will provide some applications and related results of Theorem 3.4.8. At first, we give a formula for h_{s-1} and some bounds for the difference $h_{s-1} - h_1$, where $(h_0,h_1,\ldots,h_s)$ is the h-vector of the broken circuit complex of a series–parallel network G. Let $\overline{G}$ be the simplification of G. It is easily seen from Equation (2.1) that $h_1 = n(\overline{G})$, where $n(\overline{G})$ is the nullity of $\overline{G}$.

COROLLARY 3.4.11 ([**Le16**, Corollary 4.15, Proposition 4.16]). *We keep the notation and assumptions as above. Let $d_3(G)$ be the number of vertices of G of degree at least 3. Then the following statements hold.*

(i) *$h_{s-1}(M) = p_2(G;\Pi) + n(\overline{G})$ for any ear decomposition Π of G.*

(ii) $h_{s-1} - h_1 \le h_1 - 1$, *or equivalently,* $h_{s-1} \le 2h_1 - 1$. *If equality holds, then G is parallel irreducible.*

(iii) $h_{s-1} - h_1 \le 2d_3(G) - 3$ *when* $d_3(G) > 0$. *If equality holds, then G is parallel irreducible.*

(iv) *Suppose G is parallel irreducible. Then* $h_{s-1} - h_1 \ge d_3(G)/2$.

Example 3.4.12. Again consider the graph G depicted in Figure 3.2 and its simplification $\overline{G} = G - 6$. We have $h_1 = n(\overline{G}) = 4$ and $h_5 = p_2(G;\Pi) + n(\overline{G}) = 5$. Note that $d_3(G) = 4$. So the inequality $h_5 - h_1 \ge d_3(G)/2$ does not hold. The reason is that, as we have seen in Example 3.4.6, G is not parallel irreducible.

Next we give an exclude minor characterization of the class of matroids characterized by Theorem 3.3.6. For this, we need the following.

PROPOSITION 3.4.13 ([**Le16**, Propositions 5.1, 5.4]). *Let* $(h_0, h_1, \dots, h_s)$ *be the h-vector of the broken circuit complex of a series–parallel network M. Then the following statements hold.*

(i) $h_{s-1} - h_1 = 1$ *if and only if M has a parallel irreducible decomposition*

$$M = P(N_1, \dots, N_k),$$

where some N_i *is isomorphic to the cycle matroid of a subdivision of* $K_{2,m}$ *with* $m \ge 3$, *and* N_j *is a non-loop circuit for* $j \ne i$.

(ii) $h_{s-1} - h_1 \ge 1$ *if and only if M has a parallel minor isomorphic to* $M(K_{2,m})$ *for some* $m \ge 3$.

Example 3.4.14. Let us revisit Examples 3.4.6 and 3.4.9. We know that $h_5 - h_1 = 1$ and G has a parallel irreducible decomposition $G = P(G_1, G_2, G_3, G_4)$, where $G_4 \cong K_{2,3}$ and G_i is a cycle for $i = 1, 2, 3$.

From Proposition 3.4.13 and Theorem 2.2.5 we now get the following exclude minor characterization of those matroids which admit a complete intersection broken circuit complex.

THEOREM 3.4.15 ([**Le16**, Theorem 5.7]). *Let* $(h_0, h_1, \dots, h_s)$ *be the h-vector of the broken circuit complex of a loopless matroid M. The following conditions are equivalent:*

(i) $h_0 = h_s$ *and* $h_1 = h_{s-1}$;

(ii) *M has no minor isomorphic to* $U_{2,4}$ *or* $M(K_4)$ *and no parallel minor isomorphic to* $M(K_{2,m})$ *for all* $m \ge 3$.

Finally, as evidence for Conjecture 2.3.13 we have

THEOREM 3.4.16 ([**Le16**, Theorem 6.1, Proposition 6.2]). *Let* $(h_0, h_1, \dots, h_s)$ *be the h-vector of the broken circuit complex of a series–parallel network. Then the following statements hold.*

(i) *If* $h_{s-1} - h_1 = 1$, *then* $h_i \le h_{s-i}$ *for* $i = 0, \dots, \lfloor s/2 \rfloor$.

(ii) $h_2 \le h_{s-2}$ *whenever* $s \ge 4$.

To conclude this section we propose the following problem, which is motivated by Theorem 3.4.8 and Conjecture 2.3.13.

PROBLEM 3.4.17. *Let $(h_0, h_1, \dots, h_s)$ be the h-vector of the broken circuit complex of a loopless matroid M. Find combinatorial interpretations of $h_{s-i} - h_i$ for $i = 0, \dots, \lfloor s/2 \rfloor$.*

3.5. The Orlik–Terao ideal and the relation space

In this section we explore some connections between the Orlik–Terao ideal and the relation space of an arrangement. First of all, we characterize spanning sets of the relation space in terms of the Orlik–Terao ideal, thereby generalizing Theorem 2.5.11. Then we study minimal prime ideals of certain subideals of the Orlik–Terao ideal that correspond to subsets of the relation space. Finally, we show that the codimension of the Orlik–Terao ideal of a 2-formal arrangement is not necessarily equal to that of its subideal generated by the quadratic elements. The results of this section are from [**LM15**].

Let $\mathscr{A}$ be an arrangement of n hyperplanes in a vector space V over a field $\mathbb{K}$. Let $I(\mathscr{A})$ and $F(\mathscr{A})$ be the Orlik–Terao ideal and the relation space of $\mathscr{A}$, respectively. Assume that $I(\mathscr{A})$ and $F(\mathscr{A})$ are both defined in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. Let $\mathfrak{R}$ be a subset of $F(\mathscr{A})$. Consider the following subideal of $I(\mathscr{A})$:

$$J(\mathfrak{R}) = (\iota(r) \mid r \in \mathfrak{R}) \subset S,$$

where the map $\iota : S \to S$ was introduced in Section 2.5. Then spanning sets of $F(\mathscr{A})$ can be characterized as follows.

THEOREM 3.5.1 ([**LM15**, Theorem 3.1, Corollary 3.4]). *Let $\mathscr{A}$ be an arrangement and let $\mathfrak{R} \subseteq F(\mathscr{A})$. Then the following conditions are equivalent:*

(i) *$\mathscr{A}$ is $\mathfrak{R}$-generated;*
(ii) *$I(\mathscr{A}) = J(\mathfrak{R}) : x_{[n]}$, where $x_{[n]} = x_1 \cdots x_n$;*
(iii) $\operatorname{codim}(I(\mathscr{A})) = \operatorname{codim}(J(\mathfrak{R}) : x_{[n]})$.

If, in addition, the field $\mathbb{K}$ is algebraically closed, then each of the above conditions is equivalent to the following one:

(iv) $\operatorname{codim} \mathbf{V}(I(\mathscr{A})) = \operatorname{codim}(\mathbf{V}(J(\mathfrak{R})) \cap (\mathbb{K}^*)^n)$.

If $\mathfrak{R}$ is the subset of $F(\mathscr{A})$ consisting of all relations of length at most $k+1$ for some $k \geq 2$, then $J(\mathfrak{R}) = I_{\leq k}(\mathscr{A})$, where $I_{\leq k}(\mathscr{A})$ is the subideal of $I(\mathscr{A})$ generated by all elements of $I(\mathscr{A})$ of degree at most k. Thus from Theorem 3.5.1 one can easily derive a characterization of k-generated arrangements; see [**LM15**, Corollary 3.6]. Especially, when $k = 2$ we obtain the following characterization of 2-formal arrangements, which recovers Theorem 2.5.11.

COROLLARY 3.5.2 ([**LM15**, Corollary 3.7]). *Let $I_{\langle 2 \rangle}(\mathscr{A})$ be the ideal generated by the quadratic elements of $I(\mathscr{A})$. Then the following conditions are equivalent:*

(i) *$\mathscr{A}$ is 2-formal;*
(ii) $I(\mathscr{A}) = I_{\langle 2 \rangle}(\mathscr{A}) : x_{[n]}$;
(iii) $\operatorname{codim} I(\mathscr{A}) = \operatorname{codim}(I_{\langle 2 \rangle}(\mathscr{A}) : x_{[n]})$.

If the field $\mathbb{K}$ is algebraically closed, then each of the above conditions is equivalent to the following one:

(iv) $\operatorname{codim} \mathbf{V}(I(\mathscr{A})) = \operatorname{codim}(\mathbf{V}(I_{\langle 2 \rangle}(\mathscr{A})) \cap (\mathbb{K}^*)^n)$.

Our proof of Theorem 3.5.1 is based on a construction going back to Yuzvinsky [**Yu93a**], which associates each subset $\mathfrak{R}$ of $F(\mathscr{A})$ to an $\mathfrak{R}$-generated arrangement. Let us now describe this construction, following the presentation of Falk [**Fa02**]. By abuse of notation, we will identify $S_1 = \bigoplus_{i=1}^n \mathbb{K}x_i$ with the dual space $(\mathbb{K}^n)^*$ of $\mathbb{K}^n$. Then every relation $r \in F(\mathscr{A})$ defines a function on $\mathbb{K}^n$. For a subset $\mathfrak{R}$ of $F(\mathscr{A})$ let $Z(\mathfrak{R})$ be the zero locus of $\mathfrak{R}$, i.e.,

$$Z(\mathfrak{R}) = \bigcap_{r\in\mathfrak{R}} \ker(r) = \{v \in \mathbb{K}^n \mid r(v) = 0 \text{ for all } r \in \mathfrak{R}\}.$$

Evidently, $Z(\mathfrak{R})$ is a linear subspace of $\mathbb{K}^n$. Intersecting the coordinate hyperplanes in $\mathbb{K}^n$ with $Z(\mathfrak{R})$ we obtain the following hyperplane arrangement in $Z(\mathfrak{R})$:

$$\mathscr{A}_{\mathfrak{R}} = \{H_{1,\mathfrak{R}}, \dots, H_{n,\mathfrak{R}}\}, \quad \text{where} \quad H_{i,\mathfrak{R}} = \ker(x_i) \cap Z(\mathfrak{R}).$$

Note that $H_{i,\mathfrak{R}}$ are indeed hyperplanes in $Z(\mathfrak{R})$: one has $Z(\mathfrak{R}) \nsubseteq \ker(x_i)$ because $F(\mathscr{A})$ contains no variables.

Under the above construction, the given arrangement $\mathscr{A}$ can be identified with $\mathscr{A}_{F(\mathscr{A})}$. Indeed, let $Q(\mathscr{A}) = \prod_{i=1}^n \alpha_i$ be the defining polynomial of $\mathscr{A}$. Then the linear mapping

$$\Phi = (\alpha_1, \dots, \alpha_n) : V \to \mathbb{K}^n$$

maps V isomorphically to $Z(F(\mathscr{A}))$ and H_i to $H_{i,F(\mathscr{A})}$. Note that for every subset $\mathfrak{R}$ of $F(\mathscr{A})$, the arrangement $\mathscr{A}_{F(\mathscr{A})}$ is a *section* of the arrangement $\mathscr{A}_{\mathfrak{R}}$, in the sense that $\mathscr{A}_{F(\mathscr{A})}$ is obtained by intersecting the hyperplanes of $\mathscr{A}_{\mathfrak{R}}$ with the subspace $Z(F(\mathscr{A}))$ of $Z(\mathfrak{R})$. Thus one may regard the original arrangement $\mathscr{A}$ as a section of the arrangement $\mathscr{A}_{\mathfrak{R}}$.

From the above construction we get the following result, which plays an important role in the proof of Theorem 3.5.1.

PROPOSITION 3.5.3 ([**LM15**, Proposition 2.6, Corollary 3.3]). *The relation space of the arrangement $\mathscr{A}_{\mathfrak{R}}$ is $F(\mathscr{A}_{\mathfrak{R}}) = \mathbb{K}\mathfrak{R}$, where $\mathbb{K}\mathfrak{R}$ is the $\mathbb{K}$-subspace of $F(\mathscr{A})$ spanned by $\mathfrak{R}$. Thus $\mathscr{A}_{\mathfrak{R}}$ is an $\mathfrak{R}$-generated arrangement and its Orlik–Terao ideal is given by*

$$I(\mathscr{A}_{\mathfrak{R}}) = (\iota(r) \mid r \in \mathbb{K}\mathfrak{R}).$$

In addition, we have

$$I(\mathscr{A}_{\mathfrak{R}}) = \sqrt{J(\mathfrak{R})} : x_{[n]} = J(\mathfrak{R}) : x_{[n]}.$$

Hence $I(\mathscr{A}_{\mathfrak{R}})$ is the unique associated prime ideal of $J(\mathfrak{R})$ which does not contain any variables.

Theorem 3.5.1 and Proposition 3.5.3 raise our interest in the ideal $J(\mathfrak{R})$. In particular, it is natural to ask what the associated prime ideals of $J(\mathfrak{R})$ are. Although we do not know the full answer to this question, we are able to describe the minimal prime ideals of $J(\mathfrak{R})$. By Proposition 3.5.3, $I(\mathscr{A}_{\mathfrak{R}})$ is the only minimal prime ideal of $J(\mathfrak{R})$ that does not contain variables. In order to determine other minimal prime ideals of $J(\mathfrak{R})$ we need some more notation. We call a subset Γ of $[n]$ an $\mathfrak{R}$-*cover* if either $|\Gamma \cap \operatorname{supp}(r)| = 0$ or $|\Gamma \cap \operatorname{supp}(r)| \geq 2$ for all $r \in \mathfrak{R}$. The set of all $\mathfrak{R}$-covers is denoted by $\operatorname{co}(\mathfrak{R})$. Suppose $\Gamma \in \operatorname{co}(\mathfrak{R})$. We set

$$\mathfrak{R}_0(\Gamma) = \{r \in \mathfrak{R} \mid \ |\Gamma \cap \operatorname{supp}(r)| = 0\}$$

and consider the ideal

$$Q_\Gamma(\mathfrak{R}) = (x_i \mid i \in \Gamma) + (\iota(r) \mid r \in \mathbb{K}\mathfrak{R}_0(\Gamma)) = (x_i \mid i \in \Gamma) + I(\mathscr{A}_{\mathfrak{R}_0(\Gamma)}) \subset S.$$

For example, $\emptyset \in \mathrm{co}(\mathfrak{R})$ and $Q_\emptyset(\mathfrak{R}) = I(\mathscr{A}_\mathfrak{R})$. Note that $J(\mathfrak{R}) \subseteq Q_\Gamma(\mathfrak{R})$ because for any $r \in \mathfrak{R} - \mathfrak{R}_0(\Gamma)$ we have $|\Gamma \cap \mathrm{supp}(r)| \geq 2$, and so $\iota(r) \in (x_i \mid i \in \Gamma)$. In addition, $Q_\Gamma(\mathfrak{R})$ is a prime ideal since it is the sum of two prime ideals in disjoint sets of variables. Thus $Q_\Gamma(\mathfrak{R})$ is a candidate for a minimal prime ideal of $J(\mathfrak{R})$. In fact, we have

THEOREM 3.5.4 ([**LM15**, Theorem 4.1]). *Let $\mathfrak{R}$ be a subset of the relation space $F(\mathscr{A})$. Then the minimal prime ideals of $J(\mathfrak{R})$ are exactly the minimal elements of the set $\{Q_\Gamma(\mathfrak{R}) \mid \Gamma \in \mathrm{co}(\mathfrak{R})\}$. Thus, in particular,*

$$\sqrt{J(\mathfrak{R})} = \bigcap_{\Gamma \in \mathrm{co}(\mathfrak{R})} Q_\Gamma(\mathfrak{R}), \text{ and}$$

$$\mathrm{codim}\, J(\mathfrak{R}) = \min\{|\Gamma| + \dim_\mathbb{K} \mathbb{K}\mathfrak{R}_0(\Gamma) \mid \Gamma \in \mathrm{co}(\mathfrak{R})\}.$$

So, when is $Q_\Gamma(\mathfrak{R})$ minimal? A characterization of minimality of $Q_\Gamma(\mathfrak{R})$ will be given below.

Recall that every circuit C of the underlying matroid $M(\mathscr{A})$ may be assigned to a unique relation $r_C \in F(\mathscr{A})$. Let $\mathscr{C}(F(\mathscr{A})) = \{r_C \mid C \text{ is a circuit of } M(\mathscr{A})\}$. Notice that $\mathscr{C}(F(\mathscr{A}))$ is a finite set. For a subset $\mathfrak{R}$ of $F(\mathscr{A})$, set $\mathscr{C}(\mathbb{K}\mathfrak{R}) = \mathscr{C}(F(\mathscr{A})) \cap \mathbb{K}\mathfrak{R}$. We say that a relation $r \in \mathbb{K}\mathfrak{R}$ is *induced* from $\mathfrak{R}$ if there exist $r_i \in \mathfrak{R}$ and $a_i \in \mathbb{K}$ for $i = 1, \ldots, m$ such that

$$r = \sum_{i=1}^m a_i r_i, \quad \text{and} \quad |\mathrm{supp}(r_i) \cap (\bigcup_{j=1}^{i-1} \mathrm{supp}(r_j))| \leq 1 \text{ for } i = 2, \ldots, m.$$

Now denote by $\mathscr{C}_0(\mathbb{K}\mathfrak{R})$ the subset of $\mathscr{C}(\mathbb{K}\mathfrak{R})$ consisting of the relations which are not induced from $\mathfrak{R}$. Then we have the following characterization of a minimal prime ideal of $J(\mathfrak{R})$.

PROPOSITION 3.5.5 ([**LM15**, Proposition 4.4]). *Let $\mathfrak{R}$ be a subset of $F(\mathscr{A})$ and let $\Gamma \in \mathrm{co}(\mathfrak{R})$. Then the following conditions are equivalent:*

(i) *$Q_\Gamma(\mathfrak{R})$ is a minimal prime ideal of $J(\mathfrak{R})$;*
(ii) *$Q_{\Gamma'}(\mathfrak{R}) \not\subset Q_\Gamma(\mathfrak{R})$ for every $\Gamma' \in \mathrm{co}(\mathfrak{R})$ with $\Gamma' \subset \Gamma$;*
(iii) *there exists $r \in \mathscr{C}_0(K\mathfrak{R}_0(\Gamma')) - \mathscr{C}(K\mathfrak{R}_0(\Gamma))$ such that $|\Gamma \cap \mathrm{supp}(r)| \leq 1$ for every $\Gamma' \in \mathrm{co}(\mathfrak{R})$ with $\Gamma' \subset \Gamma$.*

Despite a somewhat ugly statement, the previous proposition could be very useful in particular cases. For a concrete application of this proposition the reader is referred to [**LM15**, Example 4.5], where the minimal prime ideals of an ideal $J(\mathfrak{R})$ are computed explicitly.

To conclude this section we would like to emphasize that a minor modification of Corollary 3.5.2 is not true. More precisely, there are 2-formal arrangements $\mathscr{A}$ for which the equality $\mathrm{codim}\, I(\mathscr{A}) = \mathrm{codim}\, I_{\langle 2 \rangle}(\mathscr{A})$ does not hold. Examples of such arrangements are presented in [**LM15**, Examples 5.1, 5.2, 5.3]. Thus the ideal quotient in Corollary

3.5.2 cannot be removed. Rather interestingly, [**LM15**, Example 5.3] leads to the following result, which implies that there are no linear bounds for $\operatorname{codim} I(\mathscr{A})$ in terms of $\operatorname{codim} I_{\langle 2\rangle}(\mathscr{A})$ even when $\mathscr{A}$ is a 2-formal arrangement.

PROPOSITION 3.5.6 ([**LM15**, Corollary 5.4]). *For every integer $k \geq 1$, there exists a 2-formal arrangement $\mathscr{A}$ such that*

$$\operatorname{codim} I(\mathscr{A}) = k \cdot \operatorname{codim} I_{\langle 2\rangle}(\mathscr{A}).$$

We have seen that 2-formality can be characterized by the Orlik–Terao ideal. It would be of great interest to know whether there are characterizations of freeness and $K(\pi,1)$-ness in terms of the Orlik–Terao ideal.

PROBLEM 3.5.7. *If $\mathscr{A}$ is a free or $K(\pi,1)$ arrangement, is it true that* $\operatorname{codim} I(\mathscr{A}) = \operatorname{codim} I_{\langle 2\rangle}(\mathscr{A})$*? More generally, find connections between freeness, $K(\pi,1)$-ness and the Orlik–Terao algebra.*

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Notation

$\mathscr{A}$ hyperplane arrangement 9
$\mathscr{A} \times \mathscr{A}'$ product of the arrangements $\mathscr{A}, \mathscr{A}'$ 31
$\mathbf{A}(\mathscr{A})$ Orlik–Solomon algebra of $\mathscr{A}$ 32
$\mathscr{A}_G$ graphic arrangement associated to G 33
$\mathscr{A}_{\ell-1}$ braid arrangement 10
$\mathscr{A}_{\mathfrak{R}}$ 59

$BC_<(M)$ broken circuit of M with respect to $<$ 27
$\overline{BC}(M)$ reduced broken circuit of M 27
$\beta(M)$ beta invariant of M 25

$\mathbb{C}$ complex numbers 10
$\mathbb{C}^*$ nonzero complex numbers 11
$\mathscr{C}_0(\mathbb{K}\mathfrak{R})$ 60
$\mathbf{C}(\mathscr{A})$ Orlik–Terao algebra of $\mathscr{A}$ 38
$\widetilde{\chi}(\Delta)$ reduced Euler characteristic of the simplicial complex Δ 17
$\chi(M,t)$ characteristic polynomial of M 22
$\chi(\mathscr{A},t)$ characteristic polynomial of $\mathscr{A}$ 30
$\chi(G,t)$ chromatic polynomial of G 33
$\mathscr{C}(\mathbb{K}\mathfrak{R})$ 60
$\mathscr{C}(M)$ set of circuits of M 24
$\mathrm{co}(\mathfrak{R})$ set of all $\mathfrak{R}$-covers 59
$\mathrm{core}\,\Delta$ core of Δ 17

$D(\mathscr{A})$ module of $\mathscr{A}$-derivations 36
Δ_W restriction of Δ to W 17
$\mathrm{Der}_{\mathbb{K}}(R)$ module of derivations of R 35
$\dim \Delta$ dimension of Δ 17

$(f_0, f_1, \ldots, f_r)$ f-vector of a simplicial complex 17
$F(\mathscr{A})$ relation space of $\mathscr{A}$ 37
$f_\Delta(t)$ f-polynomial of Δ 17

$\overline{G}$ simplification of the graph G 20

$(h_0, h_1, \ldots, h_s)$ h-vector of a graded ring or a simplicial complex 15, 18
$h_\Delta(t)$ h-polynomial of Δ 18
$H^*(\mathsf{M})$ cohomology ring of M 10

ι 39
$I(\mathscr{A})$ Orlik–Terao ideal of $\mathscr{A}$ 38
I_Δ Stanley–Reisner ideal of Δ 18
$\mathscr{I}_<(M)$ broken circuit ideal of M with respect to $<$ 46
$\mathrm{in}_<(I)$ initial ideal of the ideal I with respect to $<$ 39
$IN(M)$ independence complex of M 26

$J(\mathscr{A})$ Orlik–Solomon ideal of $\mathscr{A}$ 32
$J(\mathfrak{R})$ 58

$\mathbb{K}$ field 9
$\mathbb{K}^*$ nonzero elements of the field $\mathbb{K}$ 36
$K[\Delta]$ Stanley–Reisner ring of Δ 18
$K_{m,n}$ complete bipartie graph on m and n vertices 21
K_n complete graph on n vertices 21
$K(\pi,1)$ 10

$L(\mathscr{A})$ intersection lattice of $\mathscr{A}$ 30
$\mathrm{link}_\Delta F$ link of the face F in Δ 17
$L(M)$ lattice of flats of M 22

$\overline{M}$ simplification of the matroid M 20
M^* dual of the matroid M 21
$\mathsf{M}(\mathscr{A})$ complement of the hyperplanes of $\mathscr{A}$ 9
$M(\mathscr{A})$ underlying matroid of $\mathscr{A}$ 30
$M(G)$ cycle matroid of the graph G 20

μ Möbius function 22
$M - X$ deletion of X from M 21
M/X contraction of X from M 21

$[n]$ $\{1,\ldots,n\}$ 17

∂ 31
$p_1(G;\Pi)$ 55
$p_2(G;\Pi)$ 55
$\pi(\mathscr{A},t)$ Poincaré polynomial of $\mathscr{A}$ 32
$P(M_1,\ldots,M_n)$ parallel connection of the matroids $M_1,\ldots,M_n$ 24

$Q(\mathscr{A})$ defining polynomial of $\mathscr{A}$ 30
$Q_\Gamma(\mathfrak{R})$ 59

$\mathfrak{R}_0(\Gamma)$ 59
$r(\mathscr{A})$ rank of $\mathscr{A}$ 30
r_C relation corresponding to the circuit C 37
$r(M)$ rank of M 19
$r(X)$ rank of X 19

$S(M_1,M_2)$ series connection of the matroids M_1,M_2 24
$\mathrm{star}_\Delta F$ star of the face F in Δ 17
$\mathrm{supp}(r)$ support of the relation r 37
$\mathrm{Sym}(\ell)$ symmetric group on $\{1,\ldots,\ell\}$ 10

$T(M;x,y)$ Tutte polynomial of M 22

$U_{m,n}$ uniform matroid of rank m on an n-element set 20

V vector space 9
V^* dual space of the vector space V 9
$\mathbf{V}(I)$ affine or projective variety of the ideal I 41

$\mathbb{Z}$ integers 15