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Random Polytopes

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Dipl.-Math. Mareen Beermann
aus Osnabrück (Niedersachsen)

UNIVERSITÄT OSNABRÜCK

Betreuer: Prof. Dr. techn. Matthias Reitzner

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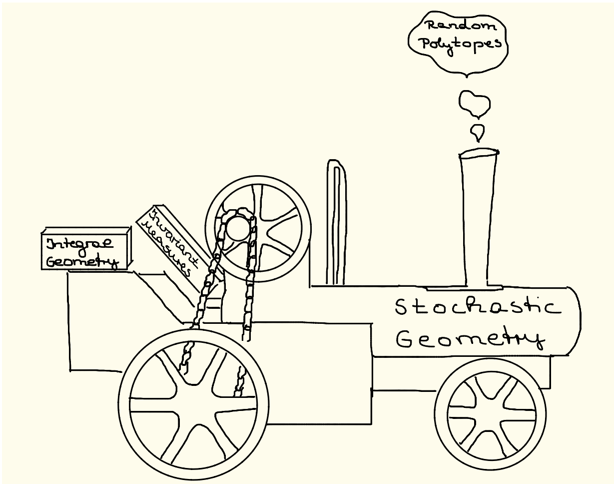
Contents

1. Introduction



Random Polytopes are a subject and product of the theory and machinery of *Stochastic Geometry*. This field of creating mathematical models to deal with random geometric structures was influenced and developed by the progress in *Invariant Measure Theory* and *Integral Geometry*.

The starting point of the theory of stochastic geometry or geometric probabilities shall be deemed to be the pose of a geometric version of gambling games in 1733. The french natural scientist G.L.L. Comte de Buffon asked in a presentation in front of the Académie Francaise for the probability that a dice thrown onto a floor which is devided into a regular mosaic will lay completely in one of these regular parts or will touch an edge of it. An answer to an easier version of this question was given as a part of his more extensive work published in 1977 and became famous as *Buffon's needle* problem.

After Buffon seemed to have given a natural solution to that problem, it appears due to *Bertrand's paradoxon* from 1888 that such a situation of considering random geometric objects might be more problematic than presumed. J.L.F. Bertrand considered three different mechanisms to create random chords in the unit circle and asked for the probability whether the length of the random chord exceeds $\sqrt{3}$. According to the different mechanisms he found three different answers - $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. It turned out that invariance properties of the underlying measures have a decisive impact to the given solution.

Invariant measures had already been studied by Morgan Crofton in 1869, but Wilhelm Blaschke an his group worked out a systematic theory after 1935, which was called *Integral Geometry*. The results of this progress were summarized in the book 'Integral Geometry and Geometric Probability' by Luis Santaló in 1976. A special interest for this topic came up since natural, material and medical scientists applied these techniques in *Stereology* and *Image Analysis*.

Another reason for the awakened interest was the field of *Stochastic Geometry*, which arose at the same time. Since then only a finite number of objects with a fixed shape were treated. By the need for more flexible models the theory of random sets were initiated by D.G. Kendall and G.F.P.M. Matheron independently. The book 'Random Sets and Integral Geometry' by Matheron from 1975 showed the relation between Stochastic Geometry and Integral Geometry. Roger Miles has written his thesis on Poisson processes of geometric objects, which was giving direction to the prospective development of Stochastic Geometry.

One subject of Stochastic Geometry which was studied with increasing interest in the last decades are *random polytopes* or, to be more specific, randomly generated convex polytopes. A convex polytope $P \subset \mathbb{R}^d$ can be understood either as the convex hull of a finite number of points in $\mathbb{R}^d$ or as the intersection of a finite number of closed half spaces of $\mathbb{R}^d$, given that this intersection is bounded. This leads us to look at two certain models, although there are many different constructions of random polytopes. These two are

- convex hulls of random points and
- intersections of random half spaces.

The first kind of random polytopes has a quite direct construction. We choose n points $X_1, \dots, X_n$ independently and according to some distribution function in $\mathbb{R}^d$. The convex hull of these points, denoted by $P_n = [X_1, \dots, X_n]$, is a random polytope. A natural choice are uniformly distributed random points. But this requires a restriction to a bounded set K in $\mathbb{R}^d$. Since we consider the convex hull of these points, it is straightforward to assume K to be convex.

But there are also other distributions which appeared to be interesting in this framework - as for instance normal distributions. In this case a Gaussian sample $X_1, \ldots, X_n$ in $\mathbb{R}^d$ is considered and the convex hull of these points is called a *Gaussian polytope*. Another commonly studied case is the Poisson model. In this case η is a Poisson point process in $\mathbb{R}^d$ with intensity t. Then the intersection of a convex set $K \subset \mathbb{R}^d$ with η consists of uniformly distributed random points $X_1, \ldots, X_M$, where M is a random variable and the convex hull of these points is called a *Poisson polytope*. This is equivalent to consider M uniformly distributed random points in K, where M itself is a Poisson distributed random variable.

Random polytopes constructed as the intersection of random half spaces represent somehow the dual model. Following this direction we come to the theory of *random tessellations*. By a tessellation in $\mathbb{R}^d$ we understand a locally finite partition of the space into compact convex polytopes which do not overlap. An important class of such tessellations are the so called random *hyperplane tessellations*. A random hyperplane process in $\mathbb{R}^d$ divides the whole space in a system of d-dimensional closed sets which have pairwise no interior points in common. These sets are convex polytopes and are called cells. Assuming the hyperplane process to fulfill a Poisson assumption gives better possibilities of calculating geometric parameters and special properties of the tessellation.

There are a lot of questions concerning random polytopes, which were studied in the last years. Some examples regarding convex hulls of random points are expectations, higher moments, limit theorems or large deviation inequalities for functionals of the random polytope as, e.g., the number of k-dimensional faces or the i-th intrinsic volume. Since in most cases there are difficulties to derive general or explicit formulas, the focus lies on the behavior of the random polytope as the number of generating points tends to infinity. Regarding random tessellations there was special interest on parameters of the typical and the zero cell, in particular concerning the shape of large cells, where large was measured in different ways, e.g., by the volume or perimeter.

The field of random polytopes and thus the amount of different issues concerning it are quite large. We are unable to even begin to give a complete overview on this topic. The aim of this thesis is to present some new results about selected topics. The results from Chapters 3 and 4 are mainly based on the following papers, jointly written with Claudia Redenbach, Matthias Reitzner and Christoph Thäle:

Mareen Beermann, Claudia Redenbach and Christoph Thäle
Asymptotic shape of small cells
Math. Nachr., **287**, 737–747 (2014).

Mareen Beermann and Matthias Reitzner
Beyond the Efron-Buchta identities: distributional results for Poisson polytopes
Discrete Comput. Geom., **53**, 226–244 (2015).

This thesis begins with a chapter devoted to background material from different fields, which are related to the main topic and will be used in the following chapters. The main part is divided regarding the two different models of random polytopes studied here. Chapter 3 contains all results concerning random polytopes constructed as intersections of half spaces. Whereas, the second part, comprising Chapters 4 and 5, reveals our results respecting convex hulls of random points.

The background material starts with a short overview of basic notation and outlines basic facts from convex and integral geometry, probability theory, stochastic geometry and analysis. An important concept from convex geometry are the intrinsic volumes provided by the Steiner formula. On the basis of these, isoperimetric inequalities can be formalized. They play an important role in the studies of large cells in random tessellations. Afterwards the notions of a polytope and its k-dimensional faces are introduced. The section of integral geometry deals with invariant measures and their construction, followed by a, not only in this thesis, frequently used theorem stating the affine Blaschke-Petkantschin formula.

From the area of probability theory we introduce the moment-generating, the (probability-)generating, and the cumulant-generating function. We will apply all of them in Chapter 4. Furthermore, prevalently used distributions are listed.

Section 2.3 is devoted to stochastic geometry and thus to point processes and the construction of random polytopes. We begin to define a random measure by locally finite Borel measures and a (simple) point process by a special subclass, the one of (simple) counting measures. The common notions of stationarity and isotropy are explained and the intensity measure of a random measure is defined. Concerning the intensity measure Campbells formula is stated. Then we give the definition of a Poisson point process, which is an underlying concept for the whole thesis. Ensuing the second repeatedly used formula is presented - the Slivnyak-Mecke formula. In order to be able to explain the construction of random tessellations later, we go on to introduce two special kinds of processes. The first ones are particle processes attended by the notions of the intensity, the grain distribution and a 'typical' grain. The second ones are processes of k-flats and in particular hyperplane processes. Here the intensity of such a process and the directional, respectively spherical directional distribution play a role.

The second part belonging to the basic facts of stochastic geometry deals with the construction of random polytopes. As mentioned above, two different models are explained - convex hulls of random points and intersections of random halfspaces. The former arise by taking the convex hull of n points chosen independently and according to some distribution function in $\mathbb{R}^d$. A natural choice for such a one is the uniform distribution, which requires a restriction to a bounded set. But also d-dimensional normal distributions are of special interest in that situation. The third considered model is the one of Poisson polytopes arising as the convex hull of random points generated by a Poisson point process in $\mathbb{R}^d$ and lying in the intersection of the process with a convex set in $\mathbb{R}^d$.

Intersections of random half spaces similarly construct random polytopes, but this point of view also leads us to the notion of random tessellations or mosaics. These locally finite partitions of $\mathbb{R}^d$ into compact convex polytopes, which do not overlap, are defined via particle processes in $\mathbb{R}^d$. The main intention lies on Poisson hyperplane tessellations which are build by Poisson hyperplane processes in $\mathbb{R}^d$. Certain polytopes out of such a tessellation, called the zero cell and the typical cell, are of special interest.

Basic facts from analysis that we will use in this thesis are the properties of a function of being concave or star-shaped. Furthermore, we list the definitions of the Beta, the Gamma and the upper incomplete Gamma function. Later on we will apply the Stirling numbers of the first kind and introduce them here. An important ingredient of some proofs in Chapter 3 is a theorem of Abelian type. We will state this theorem and in particular the special case of it that we will actually use.

Part I contains our results about random polytopes constructed as intersections of random half spaces and this means, to be more precisely, statements about the asymptotic shape of small cells in a special class of Poisson line tessellations. We begin with a short introduction mentioning the rise of the well studied contrary problem of the shape of large cells in random tessellations as a motivation to consider small cells.

Then we set the framework for this chapter. We consider what we call a rectangular Poisson line tessellation, which means that we construct a Poisson hyperplane process in $\mathbb{R}^2$ with a directional distribution concentrated on two orthogonal directions such that the emerging random tessellation only consists of two-dimensional rectangles. Thus the intensity measure of this process is concentrated on two families of lines parallel to the chosen directions. All of our results also hold (and will be proved) for the slightly more general case, where the two directions do not have to be orthogonal, which means that the cells are all parallelograms. After formalizing the idea of a typical cell, we extend our setting to the d-dimensional case getting a Poisson cuboid tessellation. For this we state and proof a proposition, which is essential for later considerations. It says that the edge lengths of a typical cell of such a tessellation are independent and exponentially distributed random variables.

In order to state results for the shape of small cells of the special Poisson line tessellation, we have to fix how to measure the shape and introduce our two deviation functionals $\sigma = 2\,\frac{\min\{X,Y\}}{X+Y}$ and $\tau = \max\{X, Y\}$, where X, Y denote the edge lengths of the typical cell. Using them, our main result then shows that the asymptotic shape of a typical cell of a Poisson line tessellation consisting of parallelograms tends to that of a line segment when we measure 'small' by the area. Moreover, it tells us that, in the limit, this line can not have positive length. Besides that, we also look at cells having small perimeter. There we get that the asymptotic shape is not uniquely determined since the conditional deviation functional σ, given small perimeter, follows a uniform distribution. To extend and underpin our theoretical results, we add a simulation study describing and reflecting the stated results, but also showing histograms for the 3-dimensional analog concerning small volume, surface area and total edge length.

Part II about random polytopes constructed as the convex hull of random points is devided into two chapters. Chapter 4 is devoted to distributional results for Poisson polytopes and is based on, respectively, goes beyond the Efron and Buchta identities. We start by giving a short overview how Bradley Efron linked the starting point of questions about random points, namely Sylvester's problem, and thus the expected number of vertices of a random polytope with respect to the uniform distribution (or some arbitrary probability measure μ) to the expected area, respectively volume. A long time after Efrons results, Christian Buchta was able to extend these to higher moments. The aim of Chapter 4 is to state analogs for Buchta's identities in the Poisson model and to link the generating functions of the μ-content and the number of vertices.

After the short introduction we set the framework for these considerations. We explain the Poisson model and fix all notations for the Poisson polytope Π_t and the treated functionals - the number of vertices $N(\Pi_t)$, the number of inner points $I(\Pi_t)$ and the μ-content of the complement of the polytope $\Delta(\Pi_t) = \mu(\mathbb{R}^d \setminus \Pi_t)$, that means of the missed set (recall that the Poisson polytope lies inside a convex body in $\mathbb{R}^d$).

Section 4.3 contains our results concerning the number of inner points. We begin with a theorem displaying the quite direct relation between the k-th factorial moment of the number of inner points and the k-th moment of the μ-content of the Poisson polytope. This identity enables to deduce a connection between the generating function of $I(\Pi_t)$ and the moment-generating function of the μ-measure of the Poisson polytope $\mu(\Pi_t)$. Furthermore, we were able to find a formula linking the cumulants of these two functions, where the Stirling numbers of the first kind play an important role.

Since we could state results for the number of inner points, we also tried to find similar formulas for the number of vertices. But the corresponding relations appeared to be more complex. We could also ascertain a connection between the generating function of the number of vertices and the moment-generating function of the μ-measure of the missed set, though this relation is not that immediate as in the case of the inner points. Hence, the relation between the factorial moments of $N(\Pi_t)$ and the moments of the missed set are more complicated.

In Section 4.5 we point out some applications for our theorems. In the last 30 years many papers had a focus on asymptotic distributions of the mentioned quantities, in many cases for the Poisson model and under the assumption that μ is the uniform distribution in a smooth convex set or a polytope, or for the d-dimensional Gaussian measure. But most of these results can be converted to the binomial model by some de-Poissonization arguments. We can apply our theorems to these results by deducing the asymptotic behavior of the variance of one quantity from that of another. The Appendix provides a theorem to refine the inference that we state in Corollary 4.5.1, namely the deduction of the variance of the, in this case, Lebesgue measure of the Poisson polytope from the variance of the number of vertices.

Chapter 5 deals with the question: 'What happens to the number of facets of a random polytope constructed as the convex hull if the number of generating points increases? Does it increase monotonically?' There are only a few papers devoted to that problem until now. For these considerations we chosed four different settings, meaning four different kinds of random polytopes. The first one is the above explained Gaussian polytope. The second one is the Poisson polytope, where the random points again are distributed according to the d-dimensional Gaussian measure. The third one arises as the convex hull of n random points chosen according to the uniform distribution from a d-dimensional ball. And the last one is the Poissonized case of the third one.

The main part of this chapter is divided into four sections, one for each kind of random polytope. For all of these cases we can show that the expected number of facets increases if the number of generating points, respectively, the intensity of the generating Poisson process (in the cases of the Poisson models), increases. An essential tool for the proofs of the theorems from this chapter are integral estimates using concave functions. That means that we substitute inside of integral expressions concave functions by linear functions to get easier terms where the monotonicities can be deduced.

2. Background Material

In this chapter we want to set basic notations and review some facts, which we will use in Chapters 3, 4 and 5. We start by recalling notions from convex and integral geometry to provide all necessary tools we need for the part of stochastic geometry. The last part of this section will be about certain functions and their properties, which will be applied later on.

Basic Notation

We will use the following notation throughout this work:

- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are the natural numbers including zero
- $\mathbb{R}^d, d \in \mathbb{N}$ is the d-dimensional Euclidean space
- $\lambda_d, d \in \mathbb{N}$ denotes the Lebesgue measure
- $B^d, d \in \mathbb{N}$ is the d-dimensional ball with radius 1
- $S^{d-1}, d \in \mathbb{N}$ denotes the $(d-1)$-dimensional unit sphere
- $\kappa_j, j \in \mathbb{N}$ is the volume of B^j
- $\omega_j, j \in \mathbb{N}$ is the surface area of S^{j-1}.

2.1. Background Material from Convex and Integral Geometry

To construct random geometric objects and to be able to analyse them some fundamentals from convex and integral geometry are required. Therefore, these subjects are a key tool for applications in stochastic geometry. This section is mainly based on [39].

2.1.1. Convex Geometry

A subset K of $\mathbb{R}^d$ is **convex** if for every pair x and y of points in K, every point on the straight line segment $[xy]$ that joins them is also within K. By a **convex body** we understand a compact, convex subset of $\mathbb{R}^d$ with non-empty interior. The set of all convex bodies will be denoted by $\mathcal{K}$. We say that a convex body is k**-dimensional**, $0 \leq k \leq d$, if its affine hull is a k-dimensional subspace in $\mathbb{R}^d$.

We want to introduce the intrinsic volumes of a convex body $K \subset \mathbb{R}^d$. For $x \in \mathbb{R}^d$ we denote by $p(K, x)$ the unique point in K nearest to x and thus by $d(x, K) = \| x - p(K, x) \|$ the distance of x from K. Moreover, for $K \in \mathcal{K}$ and $\epsilon > 0$ the parallel body of K at distance ϵ is the set

$$K_\epsilon := K + \epsilon B^d = \{x \in \mathbb{R}^d : d(x, K) \leq \epsilon\},$$

where + is the Minkowski sum. Then the **Steiner formula**

$$V_d(K_\epsilon) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} V_j(K) = \sum_{i=0}^{d} \epsilon^i \binom{d}{i} W_i(K)$$

says that the volume of K_ϵ is a polynomial in ϵ. This defines the **intrinsic volumes** $V_0(K), \ldots, V_d(K)$ of K and its **Minkowski quermassintegrals** $W_0(K)(= V_d(K))$, $W_1(K), \ldots, W_d(K)$. These two sets of functionals only differ by their normalization. We want to mention some meanings for certain intrinsic volumes. $V_0(K)$ is called the **Euler characteristic** of K. It is identically 1 for every convex body K. $V_1(K)$ is proportional to the mean width $b(K)$ of K

$$\frac{d\kappa_d}{2} b(K) = \kappa_{d-1} V_1(K).$$

$V_{d-1}(K)$ is half of the **surface area** $S(K)$ of K and $V_d(K)$ is simply the volume $V(K)$ of K. The intrinsic volume of the d-dimensional unit ball is

$$V_j(B^d) = \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}}, \quad j \in \{0, \ldots, d\}.$$

The intrinsic volumes have the following properties. They are

- additive, i.e. $V_j(K \cup L) = V_j(K) + V_j(L) - V_j(K \cap L)$, where $K, L \in \mathcal{K}$.
- motion invariant, i.e. $V_j(gK) = V_j(K)$ for any rigid motion g.

- nonnegative, i.e. $V_j(K) \geq 0$.
- continuous with respect to the Hausdorff metric.

We want to recall the geometric meaning and set the notation for the intrinsic volumes in dimensions 2 and 3:

$d = 2$	$V_2(K)$: area $A(K)$
	$2V_1(K)$: boundary length (perimeter) $L(K)$
	$V_0(K)$: Euler characteristic $\chi(K)$
$d = 3$	$V_3(K)$: volume $V(K)$
	$2V_2(K)$: surface area $S(K)$
	$\frac{1}{2}V_1(K)$: mean width $b(K)$
	$V_0(K)$: Euler characteristic $\chi(K)$

We want to mention one important inequality which is a key tool when extremum problems of random tessellations are concerned. This is the so called **isoperimetric inequality** for convex bodies. It makes a statement about the ratio of surface area and volume

$$\left(\frac{S(K)}{d\kappa_d}\right)^d \geq \left(\frac{V(K)}{\kappa_d}\right)^{d-1},$$

where equality holds only for d-dimensional balls, cf. [37, Eq. 7.20]. There is also an analogue for mean width instead of surface area

$$\left(\frac{b(K)}{2}\right)^d \geq \frac{V(K)}{\kappa_d},$$

again with equality for d-dimensional balls, cf. [37, Eq. 7.21].

An important class of convex bodies is the one of polytopes. We can understand a convex **polytope** $P \subset \mathbb{R}^d$ either as the intersection of a finite number of closed half spaces of $\mathbb{R}^d$, given that this intersection is bounded, or as the convex hull of a finite number of points in $\mathbb{R}^d$. A **face** of a convex polytope P is either P itself or a subset $P \cap H$ of P, where H is a hyperplane with P fully contained in one of the closed half spaces H^+ or H^- determined by H. A face is called k**-face** if it is of dimension $0 \leq k \leq d$. The 0-faces are the **vertices** of P, the 1-faces are the **edges** and the $(d-1)$-faces are the **facets**.

2.1.2. Integral Geometry

We will need some background information from integral geometry, which can be understood as the theory of invariant measures on a geometrical space. Here the geometrical space will be $\mathbb{R}^d$ and the measures are invariant under Euclidean motions. We will use this theory to introduce the affine Blaschke-Petkantschin formula, which will be an important tool in the following chapters. We start by considering three groups of bijective affine maps of $\mathbb{R}^d$ onto itself:

- the **translation group** T_d
- the **rotation group** SO_d
- the **rigid motion group** G_d

In addition to these topological groups we have to cover some homogeneous spaces in Euclidean integral geometry. These are

- the **linear k-dimensional Grassmannian** G_k^d, $k \in \{0, \dots, d\}$: the set of all k-dimensional linear subspaces of $\mathbb{R}^d$
- the **affine k-dimensional Grassmannian** A_k^d, $k \in \{0, \dots, d\}$: the set of all k-dimensional affine subspaces of $\mathbb{R}^d$.

The rotation group SO_d acts on the space G_k^d and the rigid motion group G_d acts on A_k^d. By starting from the **Lebesgue measure** λ_d we can now construct invariant measures on the mentioned groups and homogeneous spaces. The Lebesgue measure is the only translation invariant measure on $\mathbb{R}^d$ with $\lambda_d([0,1]^d) = 1$. It is the Haar measure on the homogeneous G_d-space $\mathbb{R}^d$ because it is rigid motion invariant, and it is normalized such that

$$\kappa_d := \lambda_d(B^d) = \frac{\pi^{\frac{d}{2}}}{\Gamma(1 + \frac{d}{2})},$$

where Γ denotes the Gamma function; see 2.6. The unit sphere S^{d-1} is a homogeneous SO_d-space. The Haar measure on the unit sphere can be derived from the Lebesgue measure as follows. For a Borel set of S^{d-1}, $A \in \mathcal{B}(S^{d-1})$, we set $\hat{A} := \{\alpha x \in \mathbb{R}^d : x \in A, 0 \leq \alpha \leq 1\}$. Since $\hat{A} \in \mathcal{B}(\mathbb{R}^d)$ we can define $\sigma_d(A) := \mathrm{d}\lambda_d(\hat{A})$. Therefore σ_d is a finite measure on S^{d-1}, where it holds that

$$\sigma_d(S^{d-1}) =: \omega_d = d\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

Since λ_d is rotation invariant, σ_d is also. Up to a constant, σ_d is the only rotation invariant measure on S^{d-1} and is called the **spherical Lebesgue measure**.

We can define a new measure $\bar{\nu}$ on SO_d as the image measure of $\sigma_d^d = \sigma_d \otimes \ldots \otimes \sigma_d$ under the measurable mapping $\psi : (S^{d-1})^d \to SO_d$, i.e. $\bar{\nu} = \psi(\sigma_d^d)$. Then $\bar{\nu}$ is a finite measure and the rotation invariance of the spherical Lebesgue measure implies the invariance of $\bar{\nu}$. Thus, $\nu = \frac{\bar{\nu}}{\bar{\nu}(SO_d)}$ is invariant and normalized so that ν is the unique Haar measure on the rotation group SO_d with $\nu(SO_d) = 1$.

To construct the Haar measure, denoted by μ, on the rigid motion group G_d we use the invariant measures λ_d and ν. μ is the image of the product measure $\lambda_d \otimes \nu$ under the homeomorphism $\gamma : \mathbb{R}^d \times SO_d \to G_d$. It is normalized such that $\mu(\gamma([0,1]^d \times SO_d)) = 1$. Up to a constant μ is the only left Haar measure on G_d.

To introduce invariant measures on the linear and affine Grassmannians we remark that some of the mentioned transformation groups operate continuously on G_k^d and A_k^d, respectivley. But only the operations of G_d and of SO_d on A_k^d and G_k^d respectively, are transitive. Thus, an invariant measure on A_k^d has to be rigid motion invariant and on G_k^d rotation invariant.

We fix a subspace $L_k \in G_k^d$ and denote the orthogonal complement by $L_k^\perp$. Then the invariant measure on G_k^d, denoted by ν_k, is the image of the invariant measure ν under the mapping $\beta_k : SO_d \to G_k^d$. It is normalized by $\nu_k(G_k^d) = 1$. The invariant measure on A_k^d, denoted by μ_k, is the image measure of the product measure $\lambda_{L_k^\perp} \otimes \nu$ under the mapping $\gamma_k : L_k^\perp \times SO_d \to A_k^d$, where $\lambda_{L_k^\perp}$ denotes the $(d-k)$-dimensional Lebesgue measure on $L_k^\perp$. It is normalized by $\mu_k(\{E \in A_k^d : E \cap B^d \neq \emptyset\}) = \kappa_{d-k}$ and satisfies (cf. [39, Thm.13.2.12])

$$\int_{A_k^d} f \, \mathrm{d}\mu_k = \int_{G_k^d} \int_{L^\perp} f(L+y) \lambda_{d-k}(\mathrm{d}y) \nu_k(\mathrm{d}L)$$

for every measurable function $f \geq 0$ on A_k^d.

2.1.3. The Affine Blaschke-Petkantschin Formula

An important tool emerging from integral geometry are geometric transformation formulas. A formula of **Blascke-Petkantschin type** deals with an integration over a product of homogeneous spaces of geometric objects.

These objects can be points, lines or hyperplanes. In most of the cases, the integration variable, which is a tuple of geometric objects, determines a new geometric object - for example, by span or intersection. This new object is called the **pivot**. The integration is then decomposed into an outer and an inner integration. The outer integration space is the space of all possible pivots and for a given pivot the inner integration space consists of the tuples of the initial integration space which determine exactly this pivot.

Assume we have to integrate a function of k-tupels of points in $\mathbb{R}^d$, where $k \in \{0, \ldots, d-1\}$, with respect to the product measure λ_d^k. In this situation it might be easier to integrate first over the k-tuples of points in a fixed k-dimensional linear subspace L with respect to the product measure λ_L^k, with a suitable Jacobian, and then over all linear subspaces L with respect to the invariant measure ν_k on G_k^d.

Such a formula exists also for affine subspaces, where instead of the linear Grassmannian, the affine Grassmannian is used. Here the initial integration is over $(\mathbb{R}^d)^{k+1}$, and the pivot is the k-dimensional subspace affinely spanned by the integration variable $(x_0, \ldots, x_k) \in (\mathbb{R}^d)^{k+1}$. The outer integration space is the affine Grassmannian A_k^d. The inner integration space is the product E^{k+1} for $E \in A_k^d$. The occurring Jacobian is $\Delta_k(x_0, \ldots, x_k)$, which denotes the k-dimensional volume of the simplex with vertices $x_0, \ldots, x_k$.

Theorem 2.1.1. *([39, Thm. 7.2.7])*
If $k \in \{1, \ldots, d\}$ and $f : (\mathbb{R}^d)^{k+1} \to \mathbb{R}$ is a nonnegative measurable function, then

$$\int\limits_{(\mathbb{R}^d)^{k+1}} f \, \mathrm{d}\lambda_d^{k+1} = \frac{\omega_{d-k+1} \cdots \omega_d}{\omega_1 \cdots \omega_k} (k!)^{d-k} \int\limits_{A_k^d} \int\limits_{E^{k+1}} f \Delta_k^{d-k} \, \mathrm{d}\lambda_E^{k+1} \mu_k(\mathrm{d}E).$$

2.2. Background Material from Probability Theory

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the underlying probability space and $\mathbb{E}X$ stands for the expectation of the random variable X over $(\Omega, \mathcal{A}, \mathbb{P})$. Then $L^p(\mathbb{P}), 0 < p < \infty$ denotes the set of all random variables X with

$$\mathbb{E}|X|^p = \int\limits_{\omega} |X|^p \, \mathrm{d}\mathbb{P} < \infty.$$

A common and significant way to describe the behavior of a random variable X is to use moments and cumulants. If $X \in L^p(\mathbb{P})$, the so called **p-th**

moment of X, denoted by $\mathbb{E}X^p$, exists.

The **moment-generating function** of a random variable X is

$$h_X(z) = \mathbb{E}e^{zX}$$

for $z \in \mathbb{R}$. The **(probability-)generating function** of a random variable X is

$$g_X(z) = \mathbb{E}z^X$$

for $z \in \mathbb{C}$. We set $n_{(k)} = \frac{n!}{(n-k)!}$, $n, k \in \mathbb{N}$. If $g_X(z)$ is an entire function, the k-th derivatives of $g_X(z)$ at $z = 1$ are the **k-th factorial moments** of X.

$$g_X^{(k)}(1) = \mathbb{E}X(X-1)\cdots(X-k+1)z^{X-k} \mid_{z=1} = \mathbb{E}X_{(k)}$$

Concerning the cumulants we want to introduce the **cumulant-generating function** of a random variable X. It is given by the logarithm of the moment-generating function

$$\ln h_X(z) = \ln \mathbb{E}e^{zX} = \sum_{k=1}^{\infty} \kappa_k(X) \frac{z^k}{k!},$$

where $\kappa_k(X)$ is the **cumulant of X of order k**.

In this thesis we will use different probability distributions and we want to recall some of them and set their notation here. The first one is the **d-dimensional normal distribution** or d-dimensional Gaussian distribution. Assume that C is a (strictly) positive definite, symmetric, real $d \times d$-matrix and $\mu \in \mathbb{R}^d$. A random vector $X = (X_1, \ldots, X_d)^T$ has a d-dimensional normal distribution with mean μ and covariance matrix C if it has the density

$$\phi^d_{\mu,C}(x) = \frac{1}{\sqrt{(2\pi)^d \det(C)}} e^{-\frac{1}{2}\langle x-\mu, C^{-1}(x-\mu)\rangle}$$

for $x \in \mathbb{R}^d$. X has a d-dimensional standard normal distribution if the mean is equal to zero, i.e. $\mu = 0$, and the covariance matrix is the $d \times d$-unit matrix, i.e. $C = I_d$. In this case we will denote the density function by ϕ_d and in case of $d = 1$ by ϕ. The distribution function of the d-dimensional standard normal distribution will be denoted by Φ^d and in case of $d = 1$ by Φ. Moreover, we will consider **exponentially distributed** random variables. A real valued random variable X is exponentially distributed with parameter λ if it has the density

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Related to the Exponential distribution is a special case of the **Gamma distribution**, the so called **Erlang distribution**. Here in difference to the Gamma distribution the second parameter is a natural number. This distribution has two parameters - one for shape ϱ and one for the rate k, which is a positive integer. The density function of the Erlang distribution reads as

$$f_{k,\varrho}(x) = \frac{\varrho^k x^{k-1} e^{-\varrho x}}{(k-1)!}$$

for $x, \varrho \geq 0, k \in \mathbb{N}$. The most frequently applied distribution in this work will be the **Poisson distribution**. It is a discrete probability distribution, which plays a prominent role in stochastic geometry. It determines, for example, the so called Poisson point process, which will be defined in the next section. An integer valued random variable X is Poisson distributed with parameter $t \in [0, \infty)$ if

$$\mathbb{P}(X = k) = \frac{t^k}{k!} e^{-t}$$

for $k \in \mathbb{N}_0$.

2.3. Background Material from Stochastic Geometry

The aim of this part is to provide all basic facts to describe constructions of random polytopes. In this sense, we want to explain the notion of a point process, which we will understand as a random counting measure. Therefore, we start by introducing random measures in general. Since it plays an outstanding role in the following, we will introduce the Poisson process in particular. To define random polytopes as parts of random tessellations we will explain processes of flats. The material of this section is mainly taken from [39].

2.3.1. Point Processes

We assume to have a locally compact space E with a countable base. The corresponding Borel σ-algebra is denoted by $\mathcal{B}(E)$. Then $\mathbf{M}(E)$ are all Borel measures η which are locally finite. $\mathbf{M}(E)$ is supplied with the σ-algebra $\mathcal{M}$, which is generated by the evaluation maps

$$f_A : \mathbf{M}(E) \to \mathbb{R} \cup \{\infty\}, \quad \eta \mapsto \eta(A)$$

for $A \in \mathcal{B}(E)$. The class $\mathbf{N}(E)$ of counting measures build a special class of measures on E. This is the class of measures $\eta \in \mathbf{M}(E)$ with $\eta(A) \in \mathbb{N}_0 \cup \{\infty\}$. We denote by $\mathcal{N}$ the trace-σ-algebra of $\mathcal{M}$ on $\mathbf{N}(E)$. Every counting measure is a finite or countable sum of Dirac measures δ_x, which are defined by

$$\delta_x(A) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

for $A \in \mathcal{B}(E)$. A counting measure is **simple** if $\eta(\{x\}) \leq 1$ for all $x \in E$. We denote by $\mathbf{N}_s(E)$ the subclass of $\mathbf{N}(E)$ of simple counting measures on E.

A **random measure** on E is a random variable with values in $(\mathbf{M}(E), \mathcal{M})$ and a **point process** on E is a random variable with values in $(\mathbf{N}(E), \mathcal{N})$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A point process η is simple if $\eta \in \mathbf{N}_s$ with probability one. If η and η' are random measures on E, respectively point processes on E, then $\eta + \eta'$ is a random measure on E, respectively a point process on E. Also the restriction $\eta \lfloor A$ of a random measure, respectively point process, η to A is again a random measure, respectively a point process.

Now we want to explain the notions of stationarity and isotropy. If a topological group G acts measurably on E, then G operates in a canonical way on $\mathbf{M}(E)$. Thus, for a random measure η on E, respectively a point process, and for $g \in G$ also $g\eta$ is a random measure, respectively a point process, on E. As an example, it holds for $E = \mathbb{R}^d$ or $E = \mathcal{F}'(\mathbb{R}^d)$, the system of nonempty and closed subsets of $\mathbb{R}^d$, where G is the group G_d of rigid motions of $\mathbb{R}^d$. In the case if $E = \mathbb{R}^d$ or $E = \mathcal{F}'(\mathbb{R}^d)$ and t_x the translation by a vector x, the image measures are denoted by $\eta + x$. Now a random measure, respectively a point process, is **stationary** if $\eta \overset{(\mathcal{D})}{=} \eta + x$ for all $x \in \mathbb{R}^d$ and **isotropic** if $\eta \overset{(\mathcal{D})}{=} \vartheta\eta$ for all $\vartheta \in SO_d$ ($\overset{(\mathcal{D})}{=}$ means equality in distribution).

The **intensity measure** of a random measure η on E is defined by

$$\Theta(A) := \mathbb{E}\eta(A) \qquad \text{for } A \in \mathcal{B}(E).$$

If η is a simple point process, $\Theta(A)$ gives the mean number of points of η lying in A. If η is a stationary random measure on $\mathbb{R}^d$, its intensity measure Θ is invariant under translations. The Lebesgue measure λ_d is, up to a constant, the only translation invariant, locally finite measure on $\mathbb{R}^d$. Thus, if Θ is locally finite, it is $\Theta = t\lambda_d$ with a constant $t \in [0, \infty)$. The number t is called the **intensity** of the random measure η. The following theorem, called **Campbell's formula**, is used frequently in the theory of random measures.

Theorem 2.3.1. *([39, Thm. 3.1.2]) Let η be a random measure on E with intensity measure Θ, and let $f : E \to \mathbb{R}$ be a nonnegative, measurable function. Then $\int_E f \mathrm{d}\eta$ is measurable, and*

$$\mathbb{E} \int_E f \, \mathrm{d}\eta = \int_E f \, \mathrm{d}\Theta.$$

If $\eta \in \mathbf{N}_s(E)$, Campbell's formula can be written as

$$\mathbb{E} \sum_{x \in \eta} f(x) = \int_E f \, \mathrm{d}\Theta.$$

The intensity measure is also called the **first moment measure**. The m**-th moment measure** $\Theta^{(m)}$ of η, $m \in \mathbb{N}$, on E^m is defined by

$$\Theta^{(m)}(A_1 \times \cdots \times A_m) = \mathbb{E}\eta^m(A_1 \times \cdots \times A_m) = \mathbb{E}\eta(A_1) \cdots \eta(A_m)$$

for $A_1, \ldots, A_m \in \mathcal{B}(E)$. For $m \in \mathbb{N}$ we set

$$E^m_{\neq} := \{(x_1, \ldots, x_m) \in E^m : x_i \text{ pairwise distinct}\}.$$

Then, the m**-th factorial moment measure** on E^m is defined by

$$\Theta_{(m)}(A_1 \times \cdots \times A_m) := \mathbb{E}\eta^m(A_1 \times \ldots \times A_m \cap E^m_{\neq})$$

for $A_1, \ldots, A_m \in \mathcal{B}(E)$. If η is a simple point process and $A \in \mathcal{B}(E)$, the m-th factorial moment measure of $\eta(A)$ is

$$\begin{aligned}\Theta_{(m)}(A^m) &= \mathbb{E} \sum_{x_1 \in \eta \cap A} \sum_{\substack{x_2 \in \eta \cap A \\ x_1 \neq x_2}} \cdots \sum_{\substack{x_m \in \eta \cap A \\ x_m \neq x_1, \ldots, x_{m-1}}} 1 \\ &= \mathbb{E}\eta(A)(\eta(A) - 1) \cdots (\eta(A) - m + 1).\end{aligned}$$

Now we want to introduce an important and commonly occuring type of point processes - the **Poisson point process**.

Definition 2.3.2. *([39, cf. Def. 3.2.1]) A Poisson process in E with intensity measure Θ is a simple point process in E with*

- ***Poisson counting variables**, i.e. for each $A \in \mathcal{B}(E)$ the random variable $\eta(A)$ is Poisson distributed with parameter $\Theta(A)$, and*
- ***independent increments**, i.e. for pairwise disjoint Borel sets $A_1, \dots, A_m$ in E, $m \in \mathbb{N}$, the random variables $\eta(A_1), \dots, \eta(A_m)$ are independent.*

It is possible, that $\Theta(A) = \infty$ appears in the first condition. Then $\eta(A)$ is infinite almost surely. A stationary Poisson point process in $\mathbb{R}^d$ is automatically isotropic. A very useful tool in this context is the **Slivnyak-Mecke formula**.

Theorem 2.3.3. *([39, Corol. 3.2.3]) Let η be a Poisson point process in E with intensity measure Θ, let $m \in \mathbb{N}$, and let $f : \mathbf{N}(E) \times E^m \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\mathbb{E} \sum_{(x_1,\dots,x_m)\in\eta^m_{\neq}} f(\eta; x_1, \dots, x_m)$$

$$= \int_E \dots \int_E \mathbb{E} f(\eta + \sum_{i=1}^m \delta_{x_i}; x_1, \dots, x_m)\ \Theta(\mathrm{d}x_1) \dots \Theta(\mathrm{d}x_m).$$

If we choose $f(\eta, x_1, \dots, x_m) := \mathbb{1}_{A_1\times\dots\times A_m}(x_1, \dots, x_m)$ with $A_1, \dots, A_m \in \mathcal{B}(E)$, then

$$\mathbb{E} \sum_{(x_1,\dots,x_m)\in\eta^m_{\neq}} f(\eta; x_1, \dots, x_m) = \mathbb{E}\eta^m(A_1 \times \dots \times A_m \cap E^m_{\neq})$$

$$= \Theta_{(m)}(A_1 \times \dots \times A_m).$$

This means that for a Poisson process η in E with intensity measure Θ and for $m \in \mathbb{N}$, the m-th factorial moment measure satisfies $\Theta_{(m)} = \Theta^m$ (cf. [39, Corol.3.2.4]).

In the following section we want to speak about random tessellations. Such a one will be described as a particle process in $\mathbb{R}^d$. By a **particle process** in $\mathbb{R}^d$ we understand a random point process on the space $\mathcal{F}'$ of nonempty closed subsets of $\mathbb{R}^d$ that is concentrated on the subset $\mathcal{C}'$ of nonempty compact sets. A particle process in the subset $\mathcal{K}' = \mathcal{K} \setminus \emptyset$ can be defined as a particle process concentrated on $\mathcal{K}'$ and is called a **process of convex particles**. We assume that the intensity measure Θ of a particle process is always locally finite, which means that $\Theta(\mathcal{F}_C) < \infty$ for all $C \in \mathcal{C}$. If for a particle process η it holds $\eta \overset{\mathcal{D}}{=} \eta + x$ for all $x \in \mathbb{R}^d$, then the particle process is called stationary. Analogous, the property of isotropy can be defined. To decompose the intensity measure Θ of a particle process, we need a **center function** $c : \mathcal{C}' \to \mathbb{R}^d$. An example for such a function is the circumcenter of C. We set $\mathcal{C}_0 = \{C \in \mathcal{C}' : c(C) = 0\}$ and define the mapping

$$\Psi : \mathbb{R}^d \times \mathcal{C}_0 \to \mathcal{C}', \quad (x, C) \mapsto x + C.$$

If η is a stationary particle process in $\mathbb{R}^d$ with intensity measure $\Theta \neq 0$, there exists a number $t \in (0, \infty)$ and a probability measure $\mathbb{G}$ on $\mathcal{C}_0$ such that

$$\Theta = t\Psi(\lambda_d \otimes \mathbb{G}).$$

t and $\mathbb{G}$ are uniquely determined and we call t the **intensity** and $\mathbb{G}$ the **grain distribution**. A random set with distribution $\mathbb{G}$ is called the **typical grain**. t can be interpreted as the expected number of particles per unit volume.

Since we want to explain the construction of a random polytope as intersections of random half spaces in the next section, we now introduce processes of flats. A point process in the space A_k^d of k-flats, $k \in \{1, \ldots, d-1\}$, in $\mathbb{R}^d$, meaning a point process in the space $\mathcal{F}'$ of nonempty closed subsets of $\mathbb{R}^d$ with intensity measure concentrated on A_k^d is called **process of k-flats** or k-flat process in $\mathbb{R}^d$. For $k = 1$ we get a line process and for $k = d-1$ it is a **hyperplane process**. If the k-flat process is stationary, it is possible to decompose its intensity measure. Since, for a locally finite, translation invariant measure Θ on A_k^d there exists a uniquely determined measure Θ_0 on G_k^d such that

$$\Theta(A) = \int_{G_k^d} \int_{L^\perp} \mathbb{1}_A(L + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \Theta_0(\mathrm{d}L)$$

for every Borel set $A \in \mathcal{B}(A_k^d)$. Applying this to the intensity measure $\Theta \neq 0$ of a stationary k-flat process in $\mathbb{R}^d$ we get

$$\int_{A_k^d} f \, \mathrm{d}\Theta = t \int_{G_k^d} \int_{L\perp} f(L + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}L) \tag{2.1}$$

for all nonnegative measurable functions f on A_k^d, where $t \in (0, \infty)$ and the probability measure $\mathbb{Q}$ on G_k^d are uniquely determined by Θ. Then t is called the **intensity** and $\mathbb{Q}$ the **directional distribution**. If the k-flat process is isotropic, $\mathbb{Q}$ is rotation invariant. As we have stated in 2.1.2, there is only one normalized rotation invariant measure on G_k^d, that is ν_k. This formulation gives us a direct interpretation of t and $\mathbb{Q}$, if we use the mapping

$$\pi_0 : \bigcup_{k=1}^{d-1} A_k^d \to \bigcup_{k=1}^{d-1} G_k^d,$$

which associates with every plane its translate through 0. Then, denoting by $\mathcal{F}$ the system of closed subsets of $\mathbb{R}^d$, we can write for a stationary k-flat process η and $A \in \mathcal{B}(G_k^d)$

$$t\mathbb{Q}(A) = \frac{1}{\kappa_{d-k}} \mathbb{E}\eta(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A)),$$

in particular

$$t = \frac{1}{\kappa_{d-k}} \mathbb{E}\eta(\mathcal{F}_{B^d})$$

and

$$\mathbb{Q} = \frac{\mathbb{E}\eta(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A))}{\mathbb{E}\eta(\mathcal{F}_{B^d})}.$$

It is convenient to replace the directional distribution $\mathbb{Q}$ by the **spherical directional distribution** φ in the cases $k = 1$ and $k = d-1$. It is a measure on the unit sphere S^{d-1} and defined by

$$\varphi(A) := \frac{1}{2}\mathbb{Q}(\{L(u) : u \in A\}) \qquad \text{if } k = 1,$$

for $L(u) := \text{lin}(u)$ and

$$\varphi(A) := \frac{1}{2}\mathbb{Q}(\{u^\perp : u \in A\}) \qquad \text{if } k = d-1,$$

for a set $A \in \mathcal{B}(S^{d-1})$. φ is an even probability measure on S^{d-1}. If η is a stationary hyperplane process in $\mathbb{R}^d$ with intensity $t \neq 0$, we can use the spherical directional distribution φ to decompose the intensity measure Θ. For that, we represent a hyperplane in the form

$$H(u, \tau) := \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\}$$

with a unit vector $u \in S^{d-1}$ and a number $\tau \in \mathbb{R}$. Every hyperplane $H \in A^d_{d-1}$ has two such representations. Then the decompostion of the intensity measure Θ given by (2.1) can be written as

$$\int\limits_{A^d_{d-1}} f \,\mathrm{d}\Theta = t \int\limits_{S^{d-1}} \int\limits_{\mathbb{R}} f(H(u,\tau))\,\mathrm{d}\tau\varphi(\mathrm{d}u)$$

for all nonnegative functions f on A^d_{d-1} and $t \in (0,\infty)$. We call two linear subspaces L, L' of $\mathbb{R}^d$ in **general position** if

$$\mathrm{lin}\,(L \cup L') = \mathbb{R}^d \quad \text{or} \quad \dim\,(L \cap L') = 0.$$

Thus, two k-planes E, E' are said to be in general position if their direction spaces $\pi_0(E), \pi_0(E')$ are in general position. Again there is a Poisson version of a k-flat process, which we will use in the following.

2.3.2. Construction of Random Polytopes

There are many different models or ways of constructing random polytopes. We want to consider two of them. Random polytopes as

- convex hulls of random points and
- intersections of random half spaces.

Random polytopes constructed as the convex hull can be generated by different kinds of point processes which live in $\mathbb{R}^d$ or in a convex body $K \subset \mathbb{R}^d$. The latter leads us to the subject of random hyperplane tessellations. This gives us a system of convex random polytopes which cover the whole space. Here we want to introduce in particular Poisson hyperplane tessellations and have a look at special polytopes out of it - the so called zero cell and the typical cell. For further information see the survey articles [38], [34] and [21].

Convex Hulls of Random Points

The first construction of random polytopes which we introduce here is a quite direct one. Since every polytope is the convex hull of its vertices, it is natural to generate a random polytope as the convex hull of finitely many random points. Thus, choose n points $X_1, \ldots, X_n$ independently and according to some distribution function in $\mathbb{R}^d$. The **convex hull** of these points, denoted by

$$P_n = [X_1, \dots, X_n],$$

is a random polytope. A natural choice is to choose **uniformly distributed** random points. This requires a restriction to a bounded set K in $\mathbb{R}^d$. Since we consider the convex hull of these points, it is straightforward to assume K to be convex.

But there are also other distributions which appeared to be interesting in this framework - as for instance distributions concentrated on the boundary of a convex body or, which will also play a role in this work, normal distributions. In this case we consider a Gaussian sample $X_1, \dots, X_n$ in $\mathbb{R}^d$. This means that we choose n independent random points according to the d-dimensional standard normal distribution. The convex hull of these points is called a **Gaussian polytope**.

Another situation, which will be studied here, is the Poisson model. There we assume that η is a Poisson point process in $\mathbb{R}^d$ with intensity t. The intersection of a convex set $K \subset \mathbb{R}^d$ with η consists of uniformly distributed random points $X_1, \dots, X_M$, where M is a random variable. Then the convex hull of these points is called a **Poisson polytope**. This is equivalent to consider M uniformly distributed random points in K, where M itself is a Poisson distributed random variable.

Intersections of Random Half Spaces

A dual model, to the explained one above, involves random hyperplanes since a convex polytope can also be understood as the intersection of a finite number of closed halfspaces. One issue is to look at random polytopes as parts of random tessellations generated by hyperplane processes.

Hyperplane processes in $\mathbb{R}^d$, as defined above, divide the whole space in a system of d-dimensional closed sets which have pairwise no common interior points. The arising object is called a random hyperplane tessellation. Therefore, we want to start by giving the definition and main ideas of a - not necessarily random - tessellation; cf. [39, Ch. 10]. A **tessellation** or also called mosaic in $\mathbb{R}^d$ is a countable system m of subsets satisfying the following conditions:

(a) m is a locally finite system of nonempty closed sets.

(b) The sets $K \in m$ are compact, convex and have interior points.

(c) The sets of m cover the whole space: $\cup_{K \in m} K = \mathbb{R}^d$.

(d) If $K, K' \in m$ and $K \neq K'$, then K and K' have no interior points in common: $\text{int } K \cap \text{int } K' = \emptyset$.

The d-dimensional closed sets of the tessellation are called **cells**. They are **convex polytopes**. The **faces** of a polytope P are the intersections of P with its supporting hyperplanes, where a k-dimensional face is said to be a k-face, $k \in \{0, \ldots, d-1\}$. More particularly, the 0-faces are the **vertices**, the 1-faces are the **edges**, and the $(d-1)$-faces are the **facets**. A tessellation is called **face-to-face** if the intersection of two different cells is a face of both cells again or the empty set.

By a **random tessellation** we understand a particle process in $\mathbb{R}^d$, where the emerging tessellation is face-to-face. This means that a random tessellation can be understood as a point process of convex polytopes, which are pairwise not overlapping, covering the whole space and satisfy the face-to-face condition.

Let $\mathcal{H}$ be a locally finite system of hyperplanes in $\mathbb{R}^d$. Then the connected components of the complement of the union $\cup_{H \in \mathcal{H}} H$ are open polyhedral sets. Their closures are the **cells** of the arising tessellation. Such a tessellation in $\mathbb{R}^d$ is said to be a **hyperplane tessellation**. It is called in **general position** if the system $\mathcal{H}$ is in general position, which means that every k-dimensional plane of $\mathbb{R}^d$ is contained in at most $d-k$ hyperplanes of the system, $k = 0, \ldots, d-1$.

We consider now the situation where the system of hyperplanes $\mathcal{H}$ is induced by a hyperplane process η in $\mathbb{R}^d$. Then the emerging tessellation is a **random hyperplane tessellation** (we will skip 'random' in the following). It is said to be in general position if η is almost surely in general position and it is stationary if and only if η is stationary. Hyperplane tessellations are face-to-face.

Every hyperplane H of a hyperplane process in $\mathbb{R}^d$ divides the space into two half spaces H^- and H^+. Now fix a point $p \in \mathbb{R}^d$ which determines that all half spaces emerging from the hyperplane process which contain this point p are labeled by H^+ and all others by H^-. The intersection of all half spaces H^+ then is a convex polytope containing p. In this way every cell of a hyperplane tessellation is build by the intersection of random half spaces.

We want to focus on a special class, namely the **Poisson hyperplane tessellations**. They are generated by Poisson hyperplane processes and have interesting properties. Let η be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity $t > 0$ and spherical direction distribution φ. We assume that φ is not concentrated on a great subsphere, which implies that the process is nondegenerate. This means that there does not exist a line to which all hyperplanes of η are parallel. If η is in addition stationary, the system of its induced cells is a random tessellation in general position.

Later on in this thesis we want to consider two certain kinds of cells in a random tessellation - the zero cell and the typical cell. The cell which contains the zero point is almost surely uniquely determined and is called the **zero cell** or in case of a stationary Poisson hyperplane tessellation sometimes also **Crofton cell**. The **typical cell** can be interpreted as a properly shifted cell, randomly chosen from a very large window out of the tessellation while every cell had the same chance to be chosen. Since we understand a random tessellation as a particle process in $\mathbb{R}^d$, we say that a random polytope Z with grain distribution $\mathbb{G}$ is a typical cell.

There are some relations between the zero cell, usually denoted by Z_0, and the typical Z in a stationary random tessellation. Let $f : \mathcal{K}' \to \mathbb{R}$ be a translation invariant, nonnegative, measurable function, where $\mathcal{K}'$ denotes the family of nonempty, compact convex subsets of $\mathbb{R}^d$. Then

$$\mathbb{E}f(Z_0) = t^d\, \mathbb{E}(f(Z)V_d(Z)). \tag{2.2}$$

Thus, the distribution of the zero cell can be understood as the volume-weighted distribution of the typical cell up to translations. An implication of this is the fact that the zero cell has stochastically larger volume than the typical cell. To formulate this, let F_0 be the distribution function of $V_d(Z_0)$ and F be the distribution function of $V_d(Z)$. Then by using (2.2) and the fact that the mean volume of the typical cell of a stationary random mosaic is the reciprocal intensity of the particle process generating it (cf. [39, Eq. 10.4]) we get

$$F_0(x) \leq F(x) \qquad \text{for } 0 \leq x < \infty$$

and as a consequence

$$\mathbb{E}V_d^k(Z_0) \geq \mathbb{E}V_d^k(Z) \qquad \text{for } k \in \mathbb{N}.$$

2.4. Background Material from Analysis

In the courses of different proofs later on we will need some special functions or certain properties of functions. One of these properties is concavity. We say a real valued function f is **concave** on an interval $[a, b], a < b, a, b \in \mathbb{R}$ if for any $x, y \in [a, b]$ and for any $t \in [0, 1]$

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y). \tag{2.3}$$

We call a function f **star-shaped with respect to** 0 if for any $y \in [0, 1]$ it holds

$$f((1-t)y) \geq (1-t)f(y). \tag{2.4}$$

This means, if the definition for concavity holds for a function f on $[0, 1]$, we get for $x = 0$ and in case of $f(0) = 0$ the property of beeing star-shaped. Equally, we will use the following definition. A differentiable, real valued function f is **concave** on an interval $[a, b], a < b, a, b \in \mathbb{R}$ if its derivative function f', provided that it exists, is monotonically decreasing.

Now we want to mention two well-known functions, which we will apply in Chapter 5. The **Beta function** and the **Gamma function** are defined by

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x, y \geq 0 \tag{2.5}$$

and

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t, \quad x \geq 0. \tag{2.6}$$

We will make use of the well known property of the Beta function

$$\mathrm{B}(x, y) = \frac{(x-1)!(y-1)!}{(x+y-1)!} \quad \text{for } x, y \in \mathbb{N}. \tag{2.7}$$

Just as for the Gamma function it is known that

$$\Gamma(x+1) = x\Gamma(x) \quad \text{with } \Gamma(1) = 1 \quad \text{for } x \in \mathbb{N}. \tag{2.8}$$

Furthermore, we will need the **upper incomplete Gamma function**

$$\Gamma(x, s) = \int_s^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad x \geq 0. \tag{2.9}$$

In Chapter 4 the **Stirling numbers of the first kind** appear. They are defined by the expansion of the function $z_{(n)} = z(z-1)\dots(z-n+1)$ for $n \in \mathbb{N}$ into a power series in z,

$$z_{(n)} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k.$$

The Stirling numbers of the first kind satisfy (or can equivalently be defined by)

$$\frac{\ln^j(z+1)}{j!} = \sum_{k=j}^{\infty} \begin{bmatrix} k \\ j \end{bmatrix} \frac{z^k}{k!}. \tag{2.10}$$

A useful tool to determine the behavior of integrals that have the form of an integral transformation

$$f(s) = \int_a^b K(s,t)F(t)\,\mathrm{d}t,$$

where $K(s,t)$ is called the kernel function and $F(t)$ is an appropriate objective function, are **theorems of Abelian type**. These theorems imply statements about the asymptotical behavior of the transformation from the behavior of the objective function. A well-known transformation is the **Laplace transformation** $f(s) = \mathcal{L}\{F(t)\}$, which is of the form

$$f(s) = \int_0^\infty e^{-st}F(t)\,\mathrm{d}t,$$

where $F(t)$ is an appropriate function for which the integral exists. The following theorem of Abelian type allows to deduce the behavior of $f(s)$ as s is going to a finite number a knowing the behavior of $F(t)$ as t is going to ∞. To formulate the theorem, we first have to recall the notion of a so called J-function. A reel or complex valued function $F(t)$, $t \in \mathbb{R}$, is said to be a J-function, if (cf. [16, p.30])

(a) $F(t)$ is defined in $0 \le t < \infty$ or in $-\infty < t < \infty$ (it's value can be equal to ∞).

(b) The function $F(t)$ is summable in each finite interval $T_1 \le t \le T_2$ in the sense of Lebesgue. This implies that besides $\int_{T_1}^{T_2} F(t)\,\mathrm{d}t$ always $\int_{T_1}^{T_2} |F(t)|\,\mathrm{d}t$ exists.

Theorem 2.4.1. *[17, ch.4 §2 Thm.1] Let the J-function $F(t)$ have the asymptotical expansion*

$$F(t) \approx \sum_{\nu=0}^{\infty} \left[a_\nu^{(0)} t^{\alpha_\nu^{(0)}} + \ldots + a_\nu^{(k_\nu)} t^{\alpha_\nu^{(k_\nu)}} \right] e^{s_\nu t} \qquad \text{as } t \to \infty, \tag{2.11}$$

where the coefficients a as well as the exponents α and s_ν are arbitrary complex with $\mathcal{R}s_0 > \mathcal{R}s_1 > \ldots \to -\infty$, such that

$$F(t) = \sum_{\nu=0}^{n} \left[a_\nu^{(0)} t^{\alpha_\nu^{(0)}} + \ldots \right] e^{s_\nu t} + O(e^{(\mathcal{R}s_n - \epsilon_n)t}) \quad \textbf{\textit{with}} \ \epsilon_n > 0 \qquad \textbf{\textit{as}} \ t \to \infty. \tag{2.12}$$

(It is allowed that at a time finitely many s_ν have the same real part, where the associated terms always have to be included in the sum $\sum_{\nu=0}^{n}$ in the same time.) Then the function $f(s) = \mathcal{L}(F)$ exists at first in the half plane $\mathcal{R}s > \mathcal{R}s_0$, but can be extended analytically to the whole plane with except for the points s_ν, where it has singularities with the main term

$$a_\nu^{(0)} \frac{\Gamma(\alpha_\nu^{(0)} + 1)}{(s - s_\nu)^{\alpha_\nu^{(0)}+1}} + \ldots + a_\nu^{(k_\nu)} \frac{\Gamma(\alpha_\nu^{(k_\nu)} + 1)}{(s - s_\nu)^{\alpha_\nu^{(k_\nu)}+1}} \tag{2.13}$$

if $\alpha_\nu^{(\mu)} \neq -1, -2, \ldots$. If $\alpha_\nu^{(\mu)} = -p$ $(p = 1, 2, \ldots)$, the corresponding term has to be replaced by

$$a_\nu^{(\mu)} \frac{(-1)^p}{(p-1)!} (s - s_\nu)^{p-1} \ln(s - s_\nu). \tag{2.14}$$

Notation: It is also allowed that $\mathcal{R}s_\nu$ tends to a finite limit ζ instead of $-\infty$. Then $f(s)$ is analytical in the half plane $\mathcal{R}s > \zeta$ except from the points s_ν. Analogously, this holds if the expansion (2.11) has only finitely many terms.

We will only use a special case of Theorem 2.4.1, where we have an expansion as in (2.12) with $n = 0$ and $s_0 = 0$.

Corollary 2.4.2. *Let the J-function $F(t)$ be such that*

$$F(t) = \left[a^{(0)} t^{\alpha^{(0)}} + a^{(1)} t^{\alpha^{(1)}} + \ldots \right] + O(e^{-\epsilon t}) \quad \textbf{\textit{with}} \ \epsilon > 0 \qquad \textbf{\textit{as}} \ t \to \infty. \tag{2.15}$$

where the coefficients a as well as the exponents α are arbitrary complex. Then the function $f(s) = \mathcal{L}(F)$ exists at first in the half plane $\mathcal{R}s > 0$, but *can be extended analytically to the whole plane with except for the point* 0, *where it has a singularity with the main term*

$$a^{(0)} \frac{\Gamma(\alpha^{(0)} + 1)}{s^{\alpha^{(0)}+1}} + \ldots + a^{(k)} \frac{\Gamma(\alpha^{(k)} + 1)}{s^{\alpha^{(k)}+1}} \tag{2.16}$$

if $\alpha^{(\mu)} \neq -1, -2, \ldots$. *If* $\alpha^{(\mu)} = -p$ $(p = 1, 2, \ldots)$, *the corresponding term has to be replaced by*

$$a^{(\mu)} \frac{(-1)^p}{(p-1)!} s^{p-1} \ln(s). \tag{2.17}$$

Part I.

Results for Intersections of Random Half Spaces

3. Asymptotic Shape of Small Cells

3.1. Introduction

In the preface of the book [40], D.G. Kendall re-phrased a conjecture about the shape of planar tessellation cells having large area. He considered a stationary and isotropic Poisson line tessellation in the plane and conjectured that the shape of the cell containing the origin is approximately circular if its area is large. First contributions to Kendall's conjecture are due to Goldman [20], Kovalenko [27] and Miles [31]. The first result for higher dimensions is by Mecke and Osburg [29] who considered what they call Poisson cuboid tessellations. In a series of papers, Calka [11], Calka and Schreiber [13], Hug, Reitzner and Schneider [23] and Hug and Schneider [24, 25] treated very general higher-dimensional versions and variants of Kendall's problem for quite general tessellation models (Poisson hyperplanes, Poisson-Voronoi and Poisson-Delaunay tessellations) and size functionals; see also the book chapter [12] for an overview. The respective results either rely on asymptotic theory for high-density Boolean models or on sharp inequalities of isoperimetric type.

In this part, we focus on the analysis of the shape of small tessellation cells. So far, we were not able to discover a general principle as the one mentioned above for the large cells behind the asymptotic geometry of small cells. For this reason, we restrict attention to the following simple model and its affine images. Take two independent stationary (homogeneous) Poisson point processes of unit intensity on the two coordinate axes in the plane and draw vertical lines through the points on the x-axis and horizontal lines through the points on the y-axis; see Figure 3.1 (left). The collection of these lines (without the two coordinate axes) decomposes the plane into a countable number of non-overlapping rectangles, the collection of which is called a *rectangular Poisson line tessellation*; see [19]. Of interest here is the shape of a typical rectangle of the tessellation (the precise definition follows below). Mecke and Osburg [29] have shown that a typical rectangle tends to be 'more and more cubical as the area tends to infinity'. Now we are interested in the

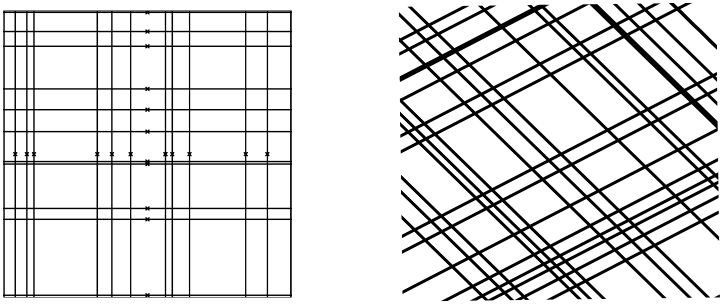


Figure 3.1.: A rectangular Poisson line tessellation (left) and a non-orthogonal Poisson parallelogram tessellation (right).

converse question and ask for the shape of a typical rectangle of small area. We will show that, in contrast to the large area case, the shape of typical rectangles with small area is asymptotically degenerate. Besides rectangles of small area, we also consider rectangles that have small perimeter. For such a situation we obtain a uniform distribution for our parameter measuring the shape of the rectangles (in fact not in general, but at least for the case described above). Again, this result is in contrast to the large perimeter case indicated in [29] (with proofs given in [32]). Concerning the mathematical analysis of small cells in random tessellations there only have appeared some conjectures together with heuristic arguments earlier in [31].

3.2. Framework

Denote by $\mathcal{L}$ the space of lines in $\mathbb{R}^2$ and by $\mathcal{L}_0$ the subspace containing only lines through the origin. We let L_1 and L_2 be two different lines in $\mathcal{L}_0$ and fix $q \in (0,1)$. On $\mathcal{L}_0$ we define the probability measure $\mathbb{Q}$ by $\mathbb{Q} = q\delta_{L_1} + (1-q)\delta_{L_2}$, where δ_{L_i} stands for the Dirac measure concentrated at L_i, $i = 1, 2$. This is to say, $\mathbb{Q}$ is concentrated on L_1 and L_2 with weights q and $1-q$, respectively. We also define the translation invariant measure Θ on $\mathcal{L}$ by the relation

$$\int_{\mathcal{L}} f(L)\Theta(\mathrm{d}L) = t \int_{\mathcal{L}_0} \int_{G^\perp} f(G+x)\, \lambda_{G^\perp}(\mathrm{d}x)\mathbb{Q}(\mathrm{d}G), \tag{3.1}$$

where $\lambda_{G^\perp}$ stands for the Lebesgue measure on $G^\perp$, $t \in (0, \infty)$ and $f : \mathcal{L} \to \mathbb{R}$ is a nonnegative measurable function. In other words, Θ is concentrated on two families of lines parallel to L_1 and L_2, whereas t is an intensity parameter.

Let now η be a Poisson point process on $\mathcal{L}$ with intensity measure Θ as at (3.1); cf. [39, 40] for definitions. Clearly, the lines of η decompose the plane into countably many parallelograms – *cells* – which have pairwise no interior points in common; see Figure 3.1 (right). The collection of all cells is denoted by $\mathcal{C} = \mathcal{C}(\eta)$ and the intersection point of the two diagonals of a parallelogram P is denoted by $c(P)$.

We want to formalize the idea of a *typical cell* of the tessellation. Recall that we can understand a random tessellation as a particle process in $\mathbb{R}^d$ and thus has an intensity t and a grain distribution $\mathbb{G}$; see Section 2.3.2. The grain distribution depends on the choice of the center function c. It is defined as a probability law $\mathbb{G}$, in this case, on the (measurable) space of parallelograms as follows:

$$t^{(d)}\mathbb{G}(A) = \mathbb{E}\sum_{P\in\mathcal{C}} \mathbb{1}_A\{P - c(P)\}\mathbb{1}_{\lfloor 1\rfloor}\{c(P)\},$$

where A is a measurable subset of parallelograms, $\lfloor 1\rfloor$, resp. (see below) $\lfloor n\rfloor$, stands for the centered square of area 1, resp. n, and $t^{(d)} = \kappa_d(\frac{\kappa_{d-1}t}{\omega_d})^d$. Heuristically, we get the typical cell, up to translations, if we choose randomly a cell of the tessellation within a large bounded part of space, giving equal weight to each of the cells. Therefore, we define

$$\mathbb{G}(A) = \lim_{n\to\infty} \frac{\mathbb{E}\sum\limits_{P\in\mathcal{C}} \mathbb{1}_A\{P - c(P)\}\,\mathbb{1}_{\lfloor n\rfloor}\{c(P)\}}{\mathbb{E}\sum\limits_{P\in\mathcal{C}} \mathbb{1}_{\lfloor n\rfloor}\{c(P)\}}. \tag{3.2}$$

Then a random parallelogram with distribution $\mathbb{G}$ is called a *typical cell* of the tessellation; cf. [39, 40] for further information. Note that by definition the typical cell Z is centered in the origin, i.e. $c(Z) = 0$. In other words, Z is the cell containing the origin 0 when the point process of centroids is conditioned on containing 0.

The following fact will turn out to be crucial for our further investigations. First we state it here for two dimensions, its proof, however, will be presented below for the analogous model in arbitrary space dimension. This extension is applied in Section 3.4.

Proposition 3.2.1. *The edge lengths of the typical cell of a Poisson parallelogram tessellation with intensity measure* Θ *given by* (3.1) *are independent* *and exponentially distributed random variables with parameters* $t(1-q)|\cos\alpha|$ *(for the edge parallel to* L_1*) and* $tq|\cos\alpha|$ *(for the edge parallel to* L_2*), where* $\alpha = \angle(L_1, L_2^\perp)$ *is the intersection angle between* L_1 *and* $L_2^\perp$*.*

As announced, we will formulate and prove a higher-dimensional version of Proposition 3.2.1. To state the result, we first need to introduce the higher-dimensional model. So, fix a space dimension $d \geq 2$, let $u_1, \dots, u_d$ be linearly independent unit vectors in $\mathbb{R}^d$ and fix weights $q_1, \dots, q_d \in [0,1]$ such that $q_1 + \dots + q_d = 1$. We define linear hyperplanes $H_1, \dots, H_d$ by

$$H_i = \operatorname{span}(\{u_1, \dots, u_d\} \setminus \{u_{d-i+1}\}), \qquad i \in \{1, \dots, d\},$$

and the probability measure $\mathbb{Q}$ on the space $\mathcal{H}_0$ of hyperplanes through the origin by putting $\mathbb{Q} = q_1\delta_{H_1} + \dots + q_d\delta_{H_d}$. The translation invariant measure Θ on the space $\mathcal{H}$ of (affine) hyperplanes in $\mathbb{R}^d$ induced by $\mathbb{Q}$ is given by

$$\int_{\mathcal{H}} f(H)\,\Theta(\mathrm{d}H) = t \int_{\mathcal{H}_0} \int_{H_0^\perp} f(H_0 + x)\,\lambda_{H_0^\perp}(\mathrm{d}x)\mathbb{Q}(\mathrm{d}H_0),$$

where $0 < t < \infty$ is a fixed constant and where $f : \mathcal{H} \to \mathbb{R}$ is nonnegative and measurable. Note that taking $d = 2$ and $q_1 = 1 - q_2 = q$ we get back the set-up for the planar case $d = 2$.

Now, let η be a Poisson point process on $\mathcal{H}$ with intensity measure Θ as defined above. The union of all hyperplanes in η decomposes $\mathbb{R}^d$ into a countable set of random parallelepipeds and the distribution of the typical cell (parallelepiped) of this tessellation is defined similarly as in (3.2). For $i \in \{1, \dots, d\}$ let $L_i = \operatorname{lin}(u_i)$ be the line spanned by u_i. Then the discussion around [30, Equation (6.3)] together with [39, Theorem 4.4.7] shows that $\eta \cap L_i$ is a homogeneous Poisson point process on L_i of intensity

$$t_{L_i} = t \sum_{j=1}^{d} q_j\, |\cos\angle(L_i, H_j^\perp)|,$$

where $\angle(L_i, H_j^\perp)$ is the angle between L_i and $H_j^\perp$ ($i, j \in \{1, \dots, d\}$). In the particular planar case $d = 2$ we have $t_{L_1} = t(1-q)|\cos\angle(L_1, L_2^\perp)| = t(1-q)|\cos\alpha|$ and $t_{L_2} = tq|\cos\angle(L_2, L_1^\perp)| = tq|\cos\alpha|$ with $\alpha = \angle(L_1, L_2^\perp)$. We can now state the higher-dimensional version of Proposition 3.2.1.

Proposition 3.2.2. *The edge lengths of a typical cell of a Poisson cuboid tessellation induced by* η *are independent and exponentially distributed random variables with parameters* $t_{L_1}, \dots, t_{L_d}$*, respectively.*

Proof of 3.2.2. As a first step let us describe an alternative construction for the random set η, which in the planar case has already been considered in the introduction. Recall the definition of the lines L_i from above and let for each $i \in \{1, \dots, d\}$, ξ_i be a homogeneous Poisson point process on L_i with intensity t_{L_i}. We assume that $\xi_1, \dots, \xi_d$ are independent. Now, for each $i \in \{1, \dots, d\}$, place hyperplanes through the Poisson points on L_i orthogonal to L_i. The collection (or union) of all hyperplanes constructed this way has the same distribution as η.

As a next step we describe a construction of the typical cell. To carry this out, we denote by Z_0 the almost surely uniquely determined d-dimensional parallelepiped of the tessellation induced by η that contains the origin. Then Z_0 is divided by the hyperplanes $H_1, \dots, H_d$ into 2^d smaller parallelepipeds meeting at the origin. With probability one, exactly one of these parallelepipeds, Z^* say, has the property that for all its corners the first coordinate is nonnegative. Now, Theorem 10.4.7 in [39] implies that (up to translations) Z^* has distribution $\mathbb{G}$ defined by (the higher-dimensional analogue of) (3.2). This means that we consider the distribution $\mathbb{G}$ of the typical cell now for a different center function c which regards the highest vertex in a certain direction $u_i \in S^{d-1}$. By means of this distribution of the typical cell with respect to the highest vertex, it can be seen that the random polytope Z^* is stochastically equivalent to Z. In other words, Z^* has (again up to translations) the same distribution as the typical cell of the Poisson hyperplane tessellation induced by η; see also Section 4 in [28].

Note in particular that the random parallelepiped Z^* has one of its corners at the origin. In view of the construction of η described at the beginning of the proof, this implies that the edge lengths of Z^* are the distances from the origin of d independent and homogeneous Poisson point processes on $L_1, \dots, L_d$ with intensities $t_{L_1}, \dots, t_{L_d}$, respectively, to their next point on the left or right (depending on the position of Z^* within Z_0). Thus, standard properties of such point processes allow us to conclude that the edge-lengths of the typical parallelepiped are independent and exponentially distributed with parameters $t_{L_1}, \dots, t_{L_d}$. □

3.3. Results for Small Cells

We consider the typical cell of a Poisson line tessellation as described above. Its random edge lengths are denoted by X and Y and its area by $A = XY$. To measure the shape of the typical cell we introduce two *deviation functionals*. The first one is

$$\sigma = 2\,\frac{\min\{X,Y\}}{X+Y}, \tag{3.3}$$

which is a random variable taking values in $[0,1]$ (this was the reason for the choice of the factor 2). We notice that σ is scale invariant, i.e., σ does not change if the parallelogram is rescaled by some constant factor. Moreover, we have $\sigma = 0$ if exactly one of the edge lengths X or Y is zero, i.e., if the parallelogram degenerates in that it is a line segment of positive length. For single points, i.e. $X = Y = 0$, σ is not defined. As a second deviation functional we introduce

$$\tau = \max\{X,Y\}. \tag{3.4}$$

This is not a scale invariant quantity, but we notice that $\tau = 0$ if and only if the parallelogram is degenerated to a point.

We investigate first the asymptotic behavior of the deviation functionals σ and τ under the condition that the typical cell area A tends to zero. Our main result in this direction reads as follows.

Theorem 3.3.1. *Let* $0 < \varepsilon < \frac{1}{2}$. *It holds that*

$$\mathbb{P}(\sigma > \varepsilon | A < a) = \frac{c_1(\varepsilon)}{-\ln(a)}(1 + o(1)) \qquad \text{as} \quad a \to 0 \tag{3.5}$$

and consequently

$$\lim_{a\to 0} \mathbb{P}(\sigma > \varepsilon | A < a) = 0. \tag{3.6}$$

Moreover,

$$\mathbb{P}(\tau > \varepsilon | A < a) = \frac{c_2(\varepsilon)}{-\ln(a)}(1 + o(1)) \quad \text{as } a \to 0.$$

and thus

$$\lim_{a\to 0} \mathbb{P}(\tau > \varepsilon | A < a) = 0, \tag{3.7}$$

where $c_1(\varepsilon)$ *and* $c_2(\varepsilon)$ *are two constants depending on* ε.

Some comments are in order about the interpretation of Theorem 3.3.1. Firstly, (3.6) shows that the asymptotic shape of a typical cell of small area tends to that of a line segment. On the other hand, (3.7) shows that, in the limit, this line segment cannot have positive length. This phenomenon is well reflected in the simulation study presented in Section 3.4.

Besides cells of small area, also cells with small perimeter can be considered. In this case the picture is somewhat different from that presented for the small area case in Theorem 3.3.1. In what follows we denote by $L(Z) = L = X + Y$ half of the perimeter length of the typical cell (the factor 1/2 is chosen for simplicity as will become clear in the proof).

Theorem 3.3.2. *Let* $0 < \varepsilon < 1$. *If* $q = 1/2$, σ *is uniformly distributed on* $[0, 1]$ *given that* $L < p$, *i.e.,*

$$\mathbb{P}(\sigma > \varepsilon | L < p) = 1 - \varepsilon,$$

independently of p. *If otherwise* $q \neq 1/2$,

$$\mathbb{P}(\sigma > \varepsilon | \, L < p) = \frac{4t_1t_2\big((t_1 - t_2)\varepsilon + t_1 - t_2 - (\varepsilon(t_1 - t_2) - 2t_1)e^{-\frac{t_1+t_2}{2}p}\big)}{(t_1 + t_2)(t_1(1 - e^{-t_2p}) - t_2(1 - e^{-t_1p}))(\varepsilon(t_1 - t_2) - 2t_1)}$$
$$\frac{-4t_1t_2\big((t_1 + t_2)e^{-\frac{2t_1-\varepsilon(t_1-t_2)}{2}p}\big)}{(t_1 + t_2)(t_1(1 - e^{-t_2p}) - t_2(1 - e^{-t_1p}))(\varepsilon(t_1 - t_2) - 2t_1)},$$

where $t_1 = t(1 - q)|\cos\alpha|$ *and* $t_2 = tq|\cos\alpha|$ *with* $\alpha = \angle(L_1, L_2^\perp)$.

Theorem 3.3.2 shows that the conditional deviation functional σ, given $L < p$, follows a uniform distribution on its range $[0, 1]$ in the particular case $q = 1/2$. This means that not only in contrast to the case of large perimeter (see [29, 32]), but also in contrast to the case of small area considered in Theorem 3.3.1 above, the asymptotic shape of cells that have small perimeter is not uniquely determined. This phenomenon is well reflected by the simulation study in forthcoming Section 3.4. We also refer to a related short discussion at the beginning of Section 7 in [25] about the independence of the shape of the zero cell and its perimeter. Such an interpretation becomes less obvious whenever $q \neq 1/2$. However, it is easily seen from the precise formula stated in Theorem 3.3.2 that the limit relation

$$\lim_{p\to 0} \mathbb{P}(\sigma > \varepsilon | L < p) = 1 - \varepsilon$$

holds.

3.3.1. Proof of Theorem 3.3.1

Reduction

We claim that without loss of generality we can restrict the proof of Theorem 3.3.1 to the case $t = 2$, $q = 1/2$ and $\mathbb{Q} = \frac{1}{2}\big(\delta_{L_x} + \delta_{L_y}\big)$, where L_x and L_y

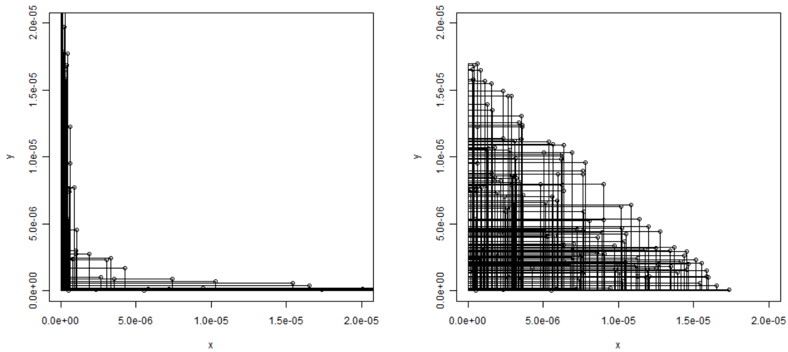


Figure 3.2.: (C. Redenbach) The 150 cells with smallest area (left) and smallest perimeter (right). Note that the axes on the left are cut off. The maximal edge length observed is 0.09.

are the two orthogonal coordinate axes. To show this, let us denote such a tessellation by PLT* and a tessellation with general parameters t, q and $\mathbb{Q}$ by PLT$(t, q, \mathbb{Q})$. We notice now that due to our assumptions on t and q there exists a non-degenerate linear transformation $f = f(t, q, \mathbb{Q}) : \mathbb{R}^2 \to \mathbb{R}^2$ such that PLT$(t, q, \mathbb{Q})$ after application of f has the same distribution as PLT*. We notice further that the images under f of a parallelogram, a line segment and a point are again a parallelogram, a line segment and a point, respectively. Thus, the statement of Theorem 3.3.1 is invariant under non-degenerate linear transformations of the underlying Poisson line tessellation. This implies that it is sufficient to establish the statement for one special choice of t, q and $\mathbb{Q}$, viz. PLT*.

Proof for PLT*

Proof of 3.5. To simplify the calculations we work from now on with $\widehat{\sigma} = \sigma/2$ and translate the result afterwards to the original deviation functional σ. To start with the calculation, we write the conditional probability as

$$\mathbb{P}(\widehat{\sigma} > \varepsilon | A < a) = \frac{\mathbb{P}(\widehat{\sigma} > \varepsilon, A < a)}{\mathbb{P}(A < a)}.$$

In what follows, we consider the numerator and the denominator separately. To deal with the numerator we have to consider the event $\hat{\sigma} = \frac{\min\{X,Y\}}{X+Y} > \varepsilon$ and $A = XY < a$. Without loss of generality we can assume that $\min\{X,Y\} = Y$. The condition $\frac{\min\{X,Y\}}{X+Y} > \varepsilon$ then becomes $\frac{Y}{X+Y} > \varepsilon$ and implies $\frac{\varepsilon}{1-\varepsilon}X < Y < X$. Therefore, $\varepsilon < \frac{1}{2}$ has to apply to retain Y as the minimum. Taking the second condition $A = XY < a$ into account, leads to $\min\left\{X, \dfrac{a}{X}\right\}$ as the upper bound for Y. Considering the lower bound for Y, that is $Y = \frac{\varepsilon}{1-\varepsilon}X$, we get the upper bound for X from the condition $XY < a$. As a consequence of this and because of Proposition 3.2.2, the numerator can be written as

$$
\begin{aligned}
\iint\limits_{\substack{\hat{\sigma}>\varepsilon \\ A<a}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x &= 2\int\limits_0^{\sqrt{\frac{a(1-\varepsilon)}{\varepsilon}}}\ \int\limits_{\frac{x\varepsilon}{1-\varepsilon}}^{\min\{x,\frac{a}{x}\}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x \\
&= 2\int\limits_0^{\sqrt{a}}\int\limits_{\frac{\varepsilon x}{1-\varepsilon}}^{x} e^{-x-y}\,\mathrm{d}y\mathrm{d}x + 2\int\limits_{\sqrt{a}}^{\sqrt{\frac{a(1-\varepsilon)}{\varepsilon}}}\ \int\limits_{\frac{\varepsilon x}{1-\varepsilon}}^{\frac{a}{x}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x \\
&\le 2\int\limits_0^{\sqrt{a}}\int\limits_{\frac{\varepsilon x}{1-\varepsilon}}^{x} \mathrm{d}y\mathrm{d}x + 2\int\limits_{\sqrt{a}}^{\sqrt{\frac{a(1-\varepsilon)}{\varepsilon}}}\ \int\limits_{\frac{\varepsilon x}{1-\varepsilon}}^{\frac{a}{x}} \mathrm{d}y\mathrm{d}x\,.
\end{aligned}
$$

The last expression can be determined by a straight forward integration procedure, which yields that asymptotically, as $a \to 0$, it behaves like a times a constant $c_1(\varepsilon)$ depending on ε (the two logarithmic terms cancel out).

Using the substitution $u = x\frac{1}{\sqrt{a}}$ and $v = y\frac{1}{\sqrt{a}}$ we can write the denominator as

$$
\iint\limits_{A<a} e^{-x-y}\,\mathrm{d}y\mathrm{d}x = \int\limits_0^{\infty}\int\limits_0^{\frac{a}{x}} e^{-(x+y)}\,\mathrm{d}y\mathrm{d}x = \int\limits_0^{\infty}\int\limits_0^{\frac{1}{u}} e^{-\sqrt{a}(u+v)}a\,\mathrm{d}v\mathrm{d}u\,.
$$

Applying the substitution $s = u + v$ and $t = u - v$ in the next step, we get

$$
\iint\limits_{A<a} e^{-x-y}\,\mathrm{d}y\mathrm{d}x = \frac{a}{2}\int\limits_0^{\infty}\int\limits_{-s}^{s} e^{-\sqrt{a}s}\,\mathbb{I}(s^2 - t^2 < 4)\,\mathrm{d}t\mathrm{d}s\,. \qquad (3.8)
$$

Now we split this double integral and calculate the resulting integrals directly as far as possible:

$$
\begin{aligned}
& \frac{a}{2}\int_0^\infty\int_{-s}^{s} e^{-\sqrt{a}s}\,\mathbb{1}(s^2-t^2<4)\,\mathrm{d}t\mathrm{d}s \\
= \;& \frac{a}{2}\int_0^\infty e^{-\sqrt{a}s}\left(\int_{-s}^{s}\mathbb{1}(s\le 2)\,\mathrm{d}t+\int_{-s}^{s}\mathbb{1}(|t|\ge\sqrt{s^2-4})\mathbb{1}(s>2)\,\mathrm{d}t\right)\mathrm{d}s \\
= \;& a\left(\int_0^2 se^{-\sqrt{a}s}\,\mathrm{d}s+\int_2^\infty e^{-\sqrt{a}s}(s-\sqrt{s^2-4})\,\mathrm{d}s\right) \\
= \;& 1-a\int_2^\infty e^{-\sqrt{a}s}\sqrt{s^2-4}\,\mathrm{d}s. \qquad (3.9)
\end{aligned}
$$

Even if the integral in (3.9) looks rather innocent, its asymptotic behavior as $a\to 0$ turns out to be not accessible with elementary methods as above. To overcome this difficulty we make use of a theorem of Abelian type ([17, ch.4 §2 Thm.1]) (see Theorem 2.4.1), more precisely of the special case in Corollary 2.4.2. So, let $F(s):=\sqrt{s^2+4s}$ and write

$$
\mathcal{I}(a):=a\int_2^\infty e^{-\sqrt{a}s}\sqrt{s^2-4}\,\mathrm{d}s=ae^{-2\sqrt{a}}\int_0^\infty e^{-\sqrt{a}s}F(s)\,\mathrm{d}s\,, \qquad (3.10)
$$

which arises by a shift $s\mapsto s+2$. We denote by

$$
f(t)=\mathcal{L}\{F(s)\}=\int_0^\infty e^{-ts}F(s)\,\mathrm{d}s
$$

the Laplace transformation of $F(s)$. The expansion of $F(s)$ fits to (2.15) and thus (2.16) and (2.17) are applicable. In terms and order of Corollary 2.4.2 we get

$$
\begin{aligned}
f(t) &= \frac{\Gamma[2]}{t^2}+2\frac{\Gamma[1]}{t}+(-2)\frac{(-1)}{0!}t^0\ln(t)+4\frac{(-1)^2}{1!}t^1\ln(t)+O(t^2\ln(t)) \\
&= \frac{1}{t^2}+\frac{2}{t}+2\ln(t)+4t\ln(t)+O(t^2\ln(t)) \quad \text{as } t\to 0.
\end{aligned}
$$

This means that we have for $s = \sqrt{a}$

$$f(\sqrt{a}) = \frac{1}{a} + \frac{2}{\sqrt{a}} + 2\ln(\sqrt{a}) + 4\sqrt{a}\ln(\sqrt{a}) + O(a\ln(\sqrt{a})) \quad \text{as } a \to 0.$$

It follows for the integral in (3.10)

$$\mathcal{I}(a) = ae^{-2\sqrt{a}}\left(\frac{1}{a} + \frac{2}{\sqrt{a}} + 2\ln(\sqrt{a}) + 4\sqrt{a}\ln(\sqrt{a}) + O(a\ln(\sqrt{a}))\right),$$

as $a \to 0$ and after a Taylor expansion of the exponential function this implies

$$\mathcal{I}(a) \quad = \quad 1 + 2a\ln(\sqrt{a}) - 2a + O(a^2\ln(\sqrt{a})) \quad \text{as } a \to 0.$$

Putting all together we can deduce

$$\begin{aligned}\iint\limits_{A<a} e^{-x-y}\,\mathrm{d}y\mathrm{d}x &= 1 - a\int\limits_2^\infty e^{-\sqrt{a}s}\sqrt{s^2-4}\,\mathrm{d}s \\ &= 1 - \mathcal{I}(a) = 1 - (1 + 2a\ln(\sqrt{a}) - 2a + O(a^2\ln(\sqrt{a}))) \\ &= -a\ln(a) + O(a) \quad \text{as } a \to 0.\end{aligned}$$

Combining this with the asymptotic behavior of the numerator implies (3.5) for $\widehat{\sigma}$ as well as for the original deviation functional σ. □

Proof of (3.7). We start by re-writing the conditional probability as

$$\mathbb{P}(\tau > \varepsilon | A < a) = \frac{\mathbb{P}(\max\{X,Y\} > \varepsilon, A < a)}{\mathbb{P}(A < a)}. \tag{3.11}$$

The denominator is the same as in the proof of (3.5) and behaves like $-a\ln(a)$ as $a \to 0$. Next, we consider the numerator and write

$$\begin{aligned}\mathbb{P}(\max\{X,Y\} > \varepsilon, A < a) &= \mathbb{P}(X > \varepsilon, A < a) + \mathbb{P}(Y > \varepsilon, A < a) \\ &\quad + \mathbb{P}(X > \varepsilon, Y > \varepsilon, A < a).\end{aligned}$$

The latter term equals 0, since for small a, the conditions $X > \varepsilon$, $Y > \varepsilon$ and $XY \leq a$ can not be fulfilled at the same time. Therefore, we look at

$$\begin{aligned}\mathbb{P}(\max\{X,Y\} > \varepsilon, A < a) &= \mathbb{P}(X > \varepsilon, A < a) + \mathbb{P}(Y > \varepsilon, A < a) \\ &= 2\mathbb{P}(X > \varepsilon, A < a) = 2\int_\varepsilon^\infty \int_0^{\frac{a}{x}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Since the numerator also tends to 0, we have to apply l'Hospital's rule to be able to calculate

$$\lim_{a\to 0} \frac{\int_\varepsilon^\infty \int_0^{\frac{a}{x}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x}{a} = \frac{\lim_{a\to 0} \frac{\mathrm{d}}{\mathrm{d}a} \int_\varepsilon^\infty \int_0^{\frac{a}{x}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x}{1}$$
$$= \lim_{a\to 0} \int_\varepsilon^\infty \frac{e^{-\frac{a}{x}-x}}{x}\,\mathrm{d}x = \Gamma(0,\varepsilon) < \infty,$$

where $\Gamma(\,\cdot\,,\,\cdot\,)$ stands for the upper incomplete Γ-function; see (2.9). We can follow that

$$a^{-1} \int_\varepsilon^\infty \int_0^{\frac{a}{x}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x = c_2(\varepsilon) + o(1) \quad \text{as } a \to 0$$

and thus

$$\int_\varepsilon^\infty \int_0^{\frac{a}{x}} e^{-x-y}\,\mathrm{d}y\mathrm{d}x = ac_2(\varepsilon) + o(a) \quad \text{as } a \to 0.$$

Combining the considerations for the numerator and the denominator we have

$$\mathbb{P}(\tau > \varepsilon | A < a) = \frac{ac_2(\varepsilon) + o(a)}{-a\ln(a) + O(a)} = \frac{c_2(\varepsilon) + o(a)}{-\ln(a) + O(a)} \quad \text{as } a \to 0.$$

This implies that

$$\lim_{a\to 0} \mathbb{P}(\tau > \varepsilon | A \le a) = 0.$$

□

3.3.2. Proof of Theorem 3.3.2

Applying the reduction step as in the proof of Theorem 3.3.1 is not possible here since the problem involving a small perimeter is not invariant under non-degenerate linear transformation. However, we can use Proposition 3.2.1 to give a direct proof. It implies that the random edge lengths X and Y of the typical cell (parallelogram) are independent and identically distributed according to an exponential distribution with mean $t_1 := t_{L_1}$ and $t_2 := t_{L_2}$. If $t_1 = t_2 = t$ it follows that the half perimeter length $L = X + Y$ is Erlang distributed with parameters t and 2. In fact this was the reason for the factor 1/2 in the definition of L. Thus,

$$\mathbb{P}(L < p) = 1 - (1 + tp)e^{-tp} \qquad \text{for any} \qquad p > 0. \tag{3.12}$$

If (without loss of generality) $t_1 > t_2$, $L = X + Y$ has the distribution function

$$\mathbb{P}(L < p) = 1 - \frac{t_1 e^{-t_2 p} - t_2 e^{-t_1 p}}{t_1 - t_2} \qquad \text{for any} \qquad p > 0. \tag{3.13}$$

Without loss of generality assume that $X > Y$ and have a closer look at the event $\{\sigma > \varepsilon,\, L < p\}$. If $X \in (0, p/2]$, then Y may range between $\frac{\varepsilon}{2-\varepsilon}X$ and X, and if $X \in [p/2, p(1-\varepsilon/2)]$ then Y ranges between $\frac{\varepsilon}{2-\varepsilon}X$ and $p - X$ (the remaining case $X > p(1-\varepsilon/2)$ contradicts $X + Y < p$ and $Y > \frac{\varepsilon}{2-\varepsilon}X$). Thus,

$$\begin{aligned}\mathbb{P}(\sigma > \varepsilon,\, L < p)&\\ &= 2\int_0^{p/2}\int_{\frac{\varepsilon}{2-\varepsilon}x}^{x} t_1 t_2 e^{-t_1 x - t_2 y}\,\mathrm{d}y\mathrm{d}x + 2\int_{p/2}^{p(1-\varepsilon/2)}\int_{\frac{\varepsilon}{2-\varepsilon}x}^{p-x} t_1 t_2 e^{-t_1 x - t_2 y}\,\mathrm{d}y\mathrm{d}x.\end{aligned}$$

If $t_1 = t_2 = t$, evaluation of these integrals yields

$$\mathbb{P}(\sigma > \varepsilon,\, L < p) = (1-\varepsilon)(1 - (1 + tp)e^{-tp})$$

and in view of (3.12) the exact distributional result $\mathbb{P}(\sigma > \varepsilon | L < p) = 1 - \varepsilon$. If otherwise $t_1 > t_2$, one shows that

$$\begin{aligned}\mathbb{P}(\sigma > \varepsilon|\, L < p) = &\frac{4t_1t_2\big((t_1 - t_2)\varepsilon + t_1 - t_2 - (\varepsilon(t_1 - t_2) - 2t_1)e^{-\frac{t_1+t_2}{2}p}\big)}{(t_1 + t_2)(t_1(1 - e^{-t_2 p}) - t_2(1 - e^{-t_1 p}))(\varepsilon(t_1 - t_2) - 2t_1)}\\ &\frac{-4t_1t_2\big((t_1 + t_2)e^{-\frac{2t_1 - \varepsilon(t_1 - t_2)}{2}p}\big)}{(t_1 + t_2)(t_1(1 - e^{-t_2 p}) - t_2(1 - e^{-t_1 p}))(\varepsilon(t_1 - t_2) - 2t_1)},\end{aligned}$$

which reduces to the separately treated uniform distribution if $t_1 = t_2$. It remains to notice that in view of Proposition 3.2.1, $t_1 = t_2$ if and only if $q = 1/2$. This completes the proof. □

Remark 1. We would like to point out that the first result of Theorem 3.3.2 has a well-known background. Namely, let X and Y be two independent and exponentially distributed random variables. Then X (or Y), given that $X + Y = s$ for some fixed $s > 0$, is uniformly distributed on $[0, s]$.

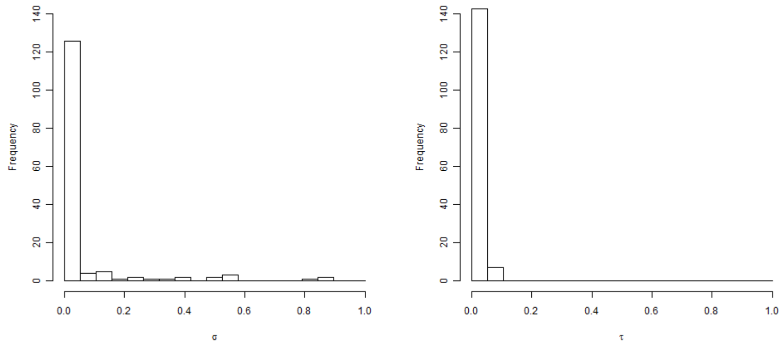


Figure 3.3.: (C. Redenbach) Histograms for the deviation functionals of the 150 cells with smallest area.

3.4. Simulation Results and Outlook to Higher Space Dimensions

To highlight and to underpin the theoretical results in the previous subsection we performed the following simulation study. We chose $t = 2$, $q = 1/2$ and $\mathbb{Q} = \frac{1}{2}\big(\delta_{L_x} + \delta_{L_y}\big)$, where L_x and L_y are the two orthogonal coordinate axes. In this case, the edge lengths of the typical cell of the rectangular Poisson line tessellation are independently exponentially distributed with mean 1; see Proposition 3.2.1. Hence, we simulated 10^{12} independent realizations of the random vector (X, Y) with X and Y i.i.d. Exponential(1). From the collection of cells obtained this way, we selected the 150 cells with smallest area. These are shown in Figure 3.2 (left). Histograms of the deviation functionals σ and τ are shown in Figure 3.3. Both the line segment shape of the cells with an accumulation around the origin and the peak at zero in the histograms for the deviation functionals are nicely visible.

As discussed above, the area is only one measure of size of a cell. We investigate now the shape of small cells in the sense that their perimeter tends to zero. Therefore, the 150 cells with smallest perimeter were extracted from the sample generated above. These cells together with their deviation functionals are shown in Figure 3.2 (right) and Figure 3.4, respectively. In this

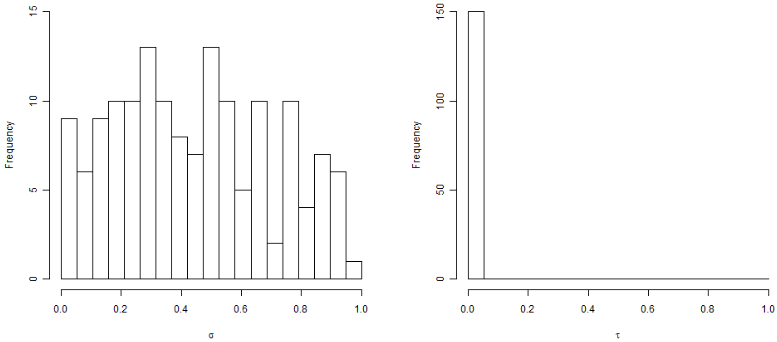


Figure 3.4.: (C. Redenbach) Histograms for the deviation functionals of the 150 cells with smallest perimeter.

situation, the maximum of the edge lengths also tends to zero, but Theorem 3.3.2 implies that σ follows a uniform distribution on the interval $[0, 1]$, which is also nicely visible in the histograms. The minimal and maximal sizes of area and perimeter of the cells included in the statistics described above are given in Table 3.1.

	minimum	maximum
area 2D	1.79e-14	8.46e-12
perimeter 2D	1.06e-6	3.52e-5

Table 3.1.: (C. Redenbach) Minimum and maximum of the size of the cells considered in the statistics (planar case).

The Poisson line tessellations considered here have natural analogues in higher space dimensions, the Poisson cuboid tessellations, for which we refer to [19] and to (3.2.2). Also for these tessellations the question about the shape of small cells can be asked. Natural candidates are the typical cell of small volume, small surface area or small total edge length. Unfortunately, we were not able to derive the higher-dimensional pendants to Theorem 3.3.1 or Theorem 3.3.2 in full generality. This is mainly due to technical complications that arise for space dimensions $d \geq 3$ and cause that the Abelian-type theorem used in the proof of Theorem 3.3.1 can no more be applied. For this reason we carried out a simulation study concerning small cells in $\mathbb{R}^3$. For this purpose, 10^{12} independent realizations of the random vector (X_1, X_2, X_3) with X_1, X_2, X_3 i.i.d. Exponential(1) were generated representing the random edge lengths of the typical cell (this is justified by Proposition 3.2.2). From the sample of cells the 150 cells with smallest volume, surface area and total edge length were extracted. Histograms for the deviation functionals $\sigma = 3\min(X_1, X_2, X_3)/(X_1 + X_2 + X_3)$ (again the factor 3 yields a value of 1 for a cube and the range $[0, 1]$) and $\tau = \max(X_1, X_2, X_3)$ for these cells are shown in Figure 3.5. The minimal and maximal sizes of the cells included in the statistics above are summarized in Table 3.2.

	minimum	maximum
volume 3D	3.20 e-15	4.97e-13
surface area 3D	1.74e-8	3.58e-7
total edge length 3D	6.80e-4	3.88e-3

Table 3.2.: (C. Redenbach) Minimum and maximum of the size of the cells considered in the statistics (spatial case).

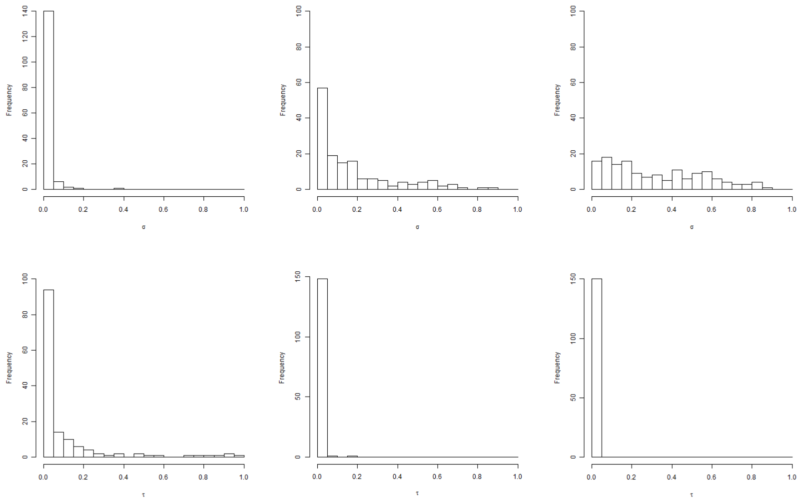


Figure 3.5.: (C. Redenbach) Histograms for the deviation functionals σ (first row) and τ (second row) of the 150 cells with smallest volume (left), surface area (middle) and total edge length (right) in $\mathbb{R}^3$.

Part II.

Results for Convex Hulls of Random Points

4. Beyond the Efron-Buchta Identities: Distributional Results for Poisson Polytopes

4.1. Introduction and Main Results

Let μ be some probability measure in $\mathbb{R}^d$ which is absolutely continuous with respect to Lebesgue measure. Choose m random points $X_1, \ldots, X_m$ in $\mathbb{R}^d$ independently according to the probability measure μ. The convex hull $P_m = [X_1, \ldots, X_m]$ of these points builds a random polytope. Numerous papers have been designated to the study of combinatorial and metric properties of such random polytopes, investigating, for example, the number of facets and the volume.

The problem to determine the expectation $\mathbb{E}N(P_m)$ of the number of vertices of such a random polytope in dimension $d = 2$ was first raised by Sylvester precisely 150 years ago in 1864 and so became known as Sylvester's problem. He suggested to choose the points according to Lebesgue measure λ_2, naturally restricted to some convex set K of finite area. In the following years, a large number of explicit results have been obtained. Most of them concerned the expected area $\mathbb{E}\lambda_2(P_m)$ of random polygons, where the random points are chosen uniformly in special convex bodies K, such as ellipses or polygons (see e.g. Buchta [5], [6], Buchta and Reitzner [9]). Yet, for $d \geq 3$ it appeared to be difficult to evaluate the expected volume for convex bodies different from the unit ball (see Buchta and Müller [8], Kingman [26], Affentranger [1], Buchta and Reitzner [10], and Zinani [41]). Thus, recent developments concentrate on asymptotic results as $m \to \infty$.

The question how to link Sylvester's original question asking for the expected number of vertices $\mathbb{E}N(P_m)$ to the expected area, respectively volume $\mathbb{E}\lambda_d(P_m)$ of the random polytope was answered by Efron [18], who proved for $d = 2, 3$

$$\frac{\mathbb{E}\lambda_d(P_m)}{\lambda_d(K)} = 1 - \frac{\mathbb{E}N(P_{m+1})}{m+1}.$$

More generally, one can replace Lebesgue measure by some arbitrary probability measure μ here and obtains

$$\mathbb{E}\mu(P_m) = 1 - \frac{\mathbb{E}N(P_{m+1})}{m+1}$$

for m random points chosen independently according to the probability measure μ.

For a long time, Efron's result – although frequently used – stood somehow isolated in the theory of random polytopes. Only recently, Buchta [7] was able to complement this equation by identities for higher moments. He proved that for $k \in \mathbb{N}$

$$\mathbb{E}\mu(P_m)^k = \mathbb{E}\prod_{i=1}^{k}\left(1 - \frac{N(P_{m+k})}{m+i}\right). \tag{4.1}$$

For the first time not only expectations but also higher moments of $\lambda_d(P_m)$, respectively $\mu(P_m)$, were linked to moments of $N(P_m)$. For example, Buchta's identities give rise to an identity for the variances of $\mu(P_m)$ and $N(P_m)$, thus correcting an error in previous results for the variances of these random variables; see [7].

It is desirable to go a step further by linking the generating functions of $\mu(P_m)$ and $N(P_m)$ and thus the distributions. But, to the best of our knowledge, Buchta's identity is still too complicated to lead to a simple identity between the generating functions of $\mu(P_n)$ and $N(P_n)$.

Yet switching from the binomial model described above to the Poisson model leads to surprisingly simple identities. It is the aim of this chapter to state analogs of Buchta's identities in the Poisson model, and then to link the generating functions of $\mu(\cdot)$ and $N(\cdot)$ by an extremely simple identity.

To describe the Poisson model, we assume that the number of random points itself is a Poisson distributed random variable M with parameter $t > 0$. Then the points $X_1, \ldots, X_M$ form a Poisson point process η in $\mathbb{R}^d$ of intensity measure $t\mu$; see Section 2.3.2. We denote by Π_t the convex hull of the points of η. Our main result concerns the number of inner points $I(\Pi_t) = M - N(\Pi_t)$ using the (probability-) generating function $g_{I(\Pi_t)}$ and the moment-generating function $h_{\mu(\Pi_t)}$ of $\mu(\Pi_t)$.

Theorem 4.1.1. *The generating function $g_{I(\Pi_t)}$ of the number of inner points and the moment-generating function $h_{\mu(\Pi_t)}$ of the μ-measure of Π_t are entire functions on $\mathbb{C}$ and satisfy*

$$g_{I(\Pi_t)}(z+1) = h_{\mu(\Pi_t)}(tz).$$

This theorem is a consequence of an identity between the moments of $I(\Pi_t)$ and $\mu(\Pi_t)$ and leads to an identity between the cumulants of $I(\Pi_t)$ and $\mu(\Pi_t)$. It is accompanied by a theorem connecting the generating function of the number of vertices to the moment-generating function of the μ-measure of $\mathbb{R}^d \backslash \Pi_t$.

For further material on random polytopes we refer to the recent survey articles by Hug [21] and Reitzner [34].

4.2. Framework

Let μ be a probability measure which is absolutely continuous with respect to Lebesgue measure. Assume that η is a Poisson point process with intensity measure $t\mu$, $t > 0$. The most important examples are given if μ is either the suitably normalized Lebesgue measure on some convex set $K \subset \mathbb{R}^d$ or the d-dimensional Gaussian measure.

More precisely, by $\mathbf{N}_s$ we denote the set of all simple and finite counting measures $\nu = \sum \delta_{x_i}$ with $x_i \in \mathbb{R}^d$, where simplicity of a counting measure $\nu = \sum \delta_{x_i}$ means that $x_i \neq x_j$ for all $i \neq j$. Alternatively, one can think of $\mathbf{N}_s$ as the set of all finite point configurations of distinct points in $\mathbb{R}^d$. This can be achieved by identifying the random measure ν with its support $\{x_1, x_2, \ldots\}$. Consequently, for $\nu \in \mathbf{N}_s$ and a Borel set $A \subset \mathbb{R}^d$, $\nu(A)$ denotes both, the restricted point configuration $\{x_1, x_2, \ldots\} \cap A$ and the counting measure $\sum \delta_{x_i}(A)$.

By Π_t we denote the convex hull of the points of η, which is a random polytope. Π_t^o will stand for the interior of the random polytope. We will use $N(\Pi_t)$ for the number of vertices and $I(\Pi_t)$ for the number of inner points of Π_t, where it holds with probability one that

$$N(\Pi_t) = \sum_{\eta} \mathbb{1}(x \notin \Pi_t^o)$$

and

$$I(\Pi_t) = \eta(\mathbb{R}^d) - N(\Pi_t) = \sum_{\eta} \mathbb{1}(x \in \Pi_t^o).$$

Let us write

$$\Delta(\Pi_t) = \mu(\mathbb{R}^d \backslash \Pi_t) = 1 - \mu(\Pi_t)$$

for the μ-content of the complement of Π_t.

We make use of the Slivnyak-Mecke formula [39, Corol. 3.2.3]); see Theorem 2.3.3. In this setting it says that for $m \in \mathbb{N}$ and $f : \mathbf{N}_s \times (\mathbb{R}^d)^m \to \mathbb{R}$ a nonnegative measurable function it holds

$$\mathbb{E} \sum_{(x_1,\dots,x_m)\in\eta^m_{\neq}} f(\eta; x_1, \dots, x_m) \tag{4.2}$$

$$= \quad t^m \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{E} f(\eta + \sum_{i=1}^m \delta_{x_i}; x_1, \dots, x_m)\mu(\mathrm{d}x_1) \dots \mu(\mathrm{d}x_m).$$

Here $\eta^m_{\neq}$ stands for the set of all m-tuples of distinct points in η.

Furthermore, we need a relative to the inclusion-exclusion principle. Assume $A \subset \mathbb{R}^d$, $k \in \mathbb{N}$, and assume $x_1, \dots, x_k \in \mathbb{R}^d$ to be fixed distinct points. Then

$$\mathbb{1}(\bigcup_{j=1}^k \{x_j\} \cap A \neq \emptyset) = \sum_{r=1}^k (-1)^{r+1} \sum_{I\in\{1,\dots,k\}^r_{\neq}} \mathbb{1}(\bigcup_{j\in I} \{x_j\} \subset A). \tag{4.3}$$

Here again $\{1, \dots, k\}^r_{\neq}$ stands for the set of all r-tuples of distinct numbers in $\{1, \dots, k\}$. This formula is just the binomial formula, applied to $(1 - 1)^m$, where m is the cardinality of $\bigcup_{j=1}^k \{x_j\} \cap A$.

4.3. Results for the Number of Inner Points

The aim of this section is to obtain relations between the factorial moments of the number of inner points $I(\Pi_t)$ and the moments of the μ-content of the random polytope Π_t. From this statement we will deduce Theorem 4.1.1.

Theorem 4.3.1. *Let $I(\Pi_t)$ be the number of inner points and $\mu(\Pi_t)$ the μ-content of the random polytope Π_t. Then for $k \in \mathbb{N}$*

$$\mathbb{E} I(\Pi_t)_{(k)} = t^k \mathbb{E}\mu(\Pi_t)^k.$$

We make this explicit in the particular cases $k = 1, 2$. For $k = 1$, Theorem 4.3.1 yields for the expectations of these random variables

$$\mathbb{E} I(\Pi_t) = t\mathbb{E}\mu(\Pi_t). \tag{4.4}$$

For $k = 2$ we obtain an identity for the variances,

$$\mathbb{V}I(\Pi_t) = t^2\mathbb{V}\mu(\Pi_t) + t\mathbb{E}\mu(\Pi_t). \tag{4.5}$$

Proof of Theorem 4.3.1. Consider the number of inner points $I(\Pi_t)$,

$$I(\Pi_t) = \sum_{x\in\eta} \mathbb{1}(x \in \Pi_t^o).$$

The number of (ordered) k-tuples of pairwise distinct inner points of Π_t is given by $I(\Pi_t)_{(k)}$. To calculate the expected value of this, we use for a point set ξ the notation $[\xi]$ for the convex hull of the points in ξ and apply the Slivnyak-Mecke formula (4.2).

$$\begin{aligned}
\mathbb{E}I(\Pi_t)_{(k)} &= \mathbb{E} \sum_{(x_1,\dots,x_k)\in\eta^k_{\neq}} \prod_{j=1}^{k} \mathbb{1}(x_j \in [\eta]^o) \\
&= t^k\mathbb{E} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{j=1}^{k} \mathbb{1}(x_j \in [\eta, x_1, \dots x_k]^o)\, \mathrm{d}\mu(x_1)\dots \mathrm{d}\mu(x_k) \\
&= t^k\mathbb{E} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{j=1}^{k} \mathbb{1}(x_j \in [\eta]^o)\, \mathrm{d}\mu(x_1)\dots \mathrm{d}\mu(x_k) \\
&= t^k\mathbb{E}\mu(\Pi_t^o)^k
\end{aligned}$$

Since μ is absolutely continuous, Theorem 4.3.1 follows. □

This identity leads to the relation between the generating function of the number of inner points and the moment-generating function of the μ-content of the random polytope Π_t, as already stated in

Theorem 4.1.1. *The generating function $g_{I(\Pi_t)}$ of the number of inner points and the moment-generating function $h_{\mu(\Pi_t)}$ of the μ-measure of Π_t are entire functions on $\mathbb{C}$ and satisfy*

$$g_{I(\Pi_t)}(z+1) = h_{\mu(\Pi_t)}(tz).$$

Proof of Theorem 4.1.1. For definitions of $g_{I(\Pi_t)}(z)$ and $h_{\mu(\Pi_t)}(z)$ see Section 2.2. Recall that the generating function of the inner points is given by

$$g_{I(\Pi_t)}(z) = \mathbb{E}z^{I(\Pi_t)} = \sum_{k=0}^{\infty} z^k \mathbb{P}(I(\Pi_t) = k).$$

For $|z| < 1$ the generating function is always absolutely convergent. Since $I(\Pi_t) \leq \eta(\mathbb{R}^d)$, we also have for $|z| \geq 1$

$$|z^{I(\Pi_t)}| \leq |z^{\eta(\mathbb{R}^d)}|.$$

This implies

$$|\mathbb{E}z^{I(\Pi_t)}| \leq \mathbb{E}|z^{\eta(\mathbb{R}^d)}| = \sum_{k=0}^{\infty} |z|^k e^{-t} \frac{t^k}{k!} = e^{t(|z|-1)} < \infty$$

because $\eta(\mathbb{R}^d)$ is Poisson distributed with parameter t. Hence, $g_{I(\Pi_t)}$ is an entire function on $\mathbb{C}$.

It is well known that if $g_{I(\Pi_t)}$ is an entire function, the k-th derivatives of $g_{I(\Pi_t)}$ at the point $z = 1$ are the k-th factorial moments of $I(\Pi_t)$.

$$g^{(k)}_{I(\Pi_t)}(1) = \mathbb{E}I(\Pi_t)(I(\Pi_t) - 1) \cdots (I(\Pi_t) - k + 1)z^{I(\Pi_t)-k}|_{z=1} = \mathbb{E}I(\Pi_t)_{(k)}$$

We evaluate the analytic function $g_{I(\Pi_t)}(z + 1)$ at $z = 0$ and deduce

$$g_{I(\Pi_t)}(z + 1) = \sum_{k=0}^{\infty} g^{(k)}_{I(\Pi_t)}(1) \frac{z^k}{k!} = \sum_{k=0}^{\infty} \mathbb{E}I(\Pi_t)_{(k)} \frac{z^k}{k!}. \tag{4.6}$$

Since the random variable $\mu(\Pi_t)$ is bounded by $\mu(\mathbb{R}^d) = 1$, the moment-generating function of $\mu(\Pi_t)$ is also an entire function. Its derivatives at $z = 0$ are given by the moments of $\mu(\Pi_t)$.

$$h^{(k)}_{\mu(\Pi_t)}(0) = \mathbb{E}\mu(\Pi_t)^k e^{z\mu(\Pi_t)}|_{z=0} = \mathbb{E}\mu(\Pi_t)^k$$

Because $h_{\mu(\Pi_t)}(z)$ is an entire function and thus analytic, we can write

$$h_{\mu(\Pi_t)}(z) = \sum_{k=0}^{\infty} h^{(k)}_{\mu(\Pi_t)}(0) \frac{z^k}{k!} = \sum_{k=0}^{\infty} \mathbb{E}\mu(\Pi_t)^k \frac{z^k}{k!}. \tag{4.7}$$

Combining (4.6) and (4.7) with Theorem 4.3.1 proves Theorem 4.1.1. □

In the next step we use this relation between the moment-generating function of $\mu(\Pi_t)$ and the generating function of $I(\Pi_t)$ to prove a relation between their cumulants. First recall that the cumulant-generating function of a random variable X is given by

$$\ln h_X(z) = \ln \mathbb{E}e^{zX} = \sum_{k=1}^{\infty} \kappa_k(X) \frac{z^k}{k!},$$

where $\kappa_k(X)$ is the cumulant of X of order k. Due to Theorem 4.1.1 we have

$$\ln h_{\mu(\Pi_t)}(tz) = \ln g_{I(\Pi_t)}(z+1) = \ln h_{I(\Pi_t)}(\ln(z+1)). \tag{4.8}$$

Essential for the relation between the cumulants of the moment-generating function of $\mu(\Pi_t)$ and the generating function of $I(\Pi_t)$ are the Stirling numbers of the first kind, denoted by $\begin{bmatrix} k \\ j \end{bmatrix}$ for $k, j \in \mathbb{N}$; see Section 2.4.

Theorem 4.3.2. *Let $\kappa_k(\mu(\Pi_t))$, resp. $\kappa_k(I(\Pi_t))$ be the cumulants of the μ-measure $\mu(\Pi_t)$, resp. of the number of inner points $I(\Pi_t)$. Then*

$$t^k \kappa_k(\mu(\Pi_t)) = \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \kappa_j(I(\Pi_t)).$$

Proof of Theorem 4.3.2. By definition of the cumulants and because of (4.8) we have

$$\begin{aligned} \sum_{k=1}^{\infty} t^k \kappa_k(\mu(\Pi_t)) \frac{z^k}{k!} &= \ln h_{\mu(\Pi_t)}(tz) \\ &= \ln h_{I(\Pi_t)}(\ln(z+1)). \end{aligned} \tag{4.9}$$

We expand the last expression in a series in $\ln(z+1)$ with coefficients given by the cumulants of $I(\Pi_t)$.

$$\ln \mathbb{E} e^{\ln(z+1) I(\Pi_t)} = \sum_{j=1}^{\infty} \kappa_j(I(\Pi_t)) \frac{\ln^j(z+1)}{j!}$$

Using property (2.10) for the logarithmic term gives

$$\begin{aligned} \sum_{j=1}^{\infty} \kappa_j(I(\Pi_t)) \frac{\ln^j(z+1)}{j!} &= \sum_{j=1}^{\infty} \kappa_j(I(\Pi_t)) \sum_{k=j}^{\infty} \begin{bmatrix} k \\ j \end{bmatrix} \frac{z^k}{k!} \\ &= \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \kappa_j(I(\Pi_t)) \Big) \frac{z^k}{k!}. \end{aligned} \tag{4.10}$$

Comparing coefficients of $\frac{z^k}{k!}$ in (4.9) and (4.10) proves our theorem. □

4.4. Results for the Number of Vertices

Analogous to Theorem 4.1.1, we want to state a theorem connecting the measure of the missed set $\Delta(\Pi_t) = \mu(\mathbb{R}^d \setminus \Pi_t)$ and the number of vertices $N(\Pi_t)$. Moreover, we find a relation between higher moments of these two variables. However, the relation in this case is not that immediate as the identity in the case of the inner points of Π_t.

Theorem 4.4.1. *The generating function $g_{N(\Pi_t)}$ of the number of vertices and the moment-generating function $h_{\Delta(\Pi_t)}$ of the μ-measure of $\mathbb{R}^d \setminus \Pi_t$ satisfy for $x \in [0,1]$*

$$g_{N(\Pi_t)}(x) = h_{\Delta(\Pi_{xt})}(t(x-1)).$$

Before giving the proof of this theorem, we compare Theorems 4.1.1 and 4.4.1. Substituting z by $z-1$ in the first mentioned theorem, the statements of these theorems read as

$$\begin{aligned} g_{I(\Pi_t)}(z) &= h_{\mu(\Pi_t)}(t(z-1)), \qquad (4.11) \\ g_{N(\Pi_t)}(x) &= h_{\Delta(\Pi_{xt})}(t(x-1)). \end{aligned}$$

The main difference is the occurrence of x in the random variable $\Delta(\Pi_{xt})$ in the second line, which makes sense only if x is in $\mathbb{R}_+$ and makes it impossible to extend the right-hand side to a holomorphic function. It would be of interest to deduce one of these identities from the other, but we have been unable to find a connection.

It should be remarked that it is possible to prove the identity (4.11) for $z \in [0,1]$ using the method applied in the proof of Theorem 4.4.1. By the identity theorem for holomorphic functions, we could deduce that equality holds for all $z \in \mathbb{C}$ because $g_{I(\Pi_t)}$ and $h_{\mu(\Pi_t)}$ are entire functions. It is straightforward to prove that also $g_{N(\Pi_t)}(z)$ and $h_{\Delta(\Pi_t)}(z)$ are both entire functions, but we make no use of this fact in our investigations.

Proof of Theorem 4.4.1. Suppose η_{xt} and $\bar{\eta}_{yt}$ are two independent Poisson point processes on $\mathbb{R}^d$ with intensity measure $xt\mu$, respectively $yt\mu$ with $x, y \geq 0,\ x+y=1$. It is well known that

$$\eta \overset{(\mathcal{D})}{=} \eta_{xt} + \bar{\eta}_{yt}.$$

Conversely, if we split η into two point sets by deciding for each point of η independently if it belongs to η_1 with probability x or to η_2 with probability $y = 1-x$, then η_1, resp. η_2, equals η_{xt}, resp. $\bar{\eta}_{yt}$, in distribution.

Denote by $\mathcal{F}_N(\Pi_t)$ the set of vertices of Π_t. As described above, we split η into η_{xt} and $\bar{\eta}_{yt}$ and consider the event that all vertices of Π_t emerge from η_{xt}. This event occurs if no point of $\bar{\eta}_{yt}$ is contained in $\mathbb{R}^d \setminus \Pi_{xt}$, where Π_{xt} is the convex hull of the points of η_{xt}. Because these point processes are independent, we have

$$\mathbb{P}(\mathcal{F}_N(\Pi_t) \subset \eta_{xt}) = \mathbb{P}(\bar{\eta}_{yt}(\mathbb{R}^d \setminus \Pi_{xt}) = 0) = \mathbb{E}(e^{-yt\Delta(\Pi_{xt})}). \tag{4.12}$$

Moreover, to compute $\mathbb{P}(\mathcal{F}_N(\Pi_t) \subset \eta_{xt})$ we first condition on the number of vertices

$$\mathbb{P}(\mathcal{F}_N(\Pi_t) \subset \eta_{xt} | N(\Pi_t) = k) = x^k,$$

which follows from the splitting argument stated above. Taking expectation and thus removing the condition, we get

$$\sum_{k=0}^{\infty} x^k \mathbb{P}(N(\Pi_t) = k) = \mathbb{E}x^{N(\Pi_t)}. \tag{4.13}$$

Combining (4.12) and (4.13) yields our theorem. □

Theorem 4.4.1 states the relation between the factorial moment-generating function of the number of vertices and the moment-generating function of the μ-content of the missed set $\mathbb{R}^d \setminus \Pi_{xt}$. Due to the occurrence of x in the random variable $\Delta(\Pi_{xt})$, it seems impossible to state a simple identity between factorial moments of $N(\Pi_t)$ and $\Delta(\Pi_t)$. As can be seen in the next theorem, there is a much more complicated relation for the moments of these two random variables. Again we use the notation $[\xi]$ for the convex hull of points of a point set ξ.

Theorem 4.4.2. *Let $N(\Pi_t)$ be the number of vertices and $\Delta(\Pi_t)$ the μ-content of the complement of Π_t. Then for $k \in \mathbb{N}$*

$$\begin{aligned}\mathbb{E}N(\Pi_t)_{(k)} &= t^k \,\mathbb{E}\Delta(\Pi_t)^k - t^k \sum_{r=1}^{k-1} (-1)^{r+1} \binom{k}{r} \\ &\quad \times \mathbb{E} \int\limits_{\mathbb{R}^d\setminus\Pi_t} \dots \int\limits_{\mathbb{R}^d\setminus\Pi_t} \mu([\eta, x_1, \dots, x_{k-r}] \setminus [\eta])^r \,\mathrm{d}\mu(x_1) \dots \mathrm{d}\mu(x_{k-r}).\end{aligned}$$

The particular case $k = 1$ gives a simple identity for the expected values

$$\mathbb{E}N(\Pi_t) = t\mathbb{E}\Delta(\Pi_t). \tag{4.14}$$

And for $k = 2$ we obtain the more complicated expression

$$\mathbb{E}N(\Pi_t)_{(2)} = t^2 \mathbb{E}\Delta(\Pi_t)^2 - 2t^2 \mathbb{E} \int\limits_{\mathbb{R}^d \setminus \Pi_t} \mu([\eta, x] \setminus [\eta]) \, \mathrm{d}\mu(x). \tag{4.15}$$

Formulas (4.14) and (4.15) can be used to deduce, for the variances, the relation

$$\mathbb{V}N(\Pi_t) = t^2 \mathbb{V}\Delta(\Pi_t) + t\mathbb{E}\Delta(\Pi_t) - 2t^2 \mathbb{E} \int\limits_{\mathbb{R}^d \setminus \Pi_t} \mu([\eta, x] \setminus [\eta]) \, \mathrm{d}\mu(x). \tag{4.16}$$

Proof of Theorem 4.4.2. We are interested in the factorial moments of the number of vertices $N(\Pi_t) = \sum \mathbb{1}(x \notin \Pi_t^o)$ of the random polytope Π_t. We apply the Slivnyak-Mecke formula (4.2) to obtain

$$\begin{aligned}
\mathbb{E}N(\Pi_t)_{(k)} &= \mathbb{E} \sum_{(x_1, \ldots, x_k) \in \eta^k_{\neq}} \mathbb{1}(x_1 \notin \Pi_t^o) \cdots \mathbb{1}(x_k \notin \Pi_t^o) \\
&= t^k \; \mathbb{E} \int\limits_{\mathbb{R}^d} \ldots \int\limits_{\mathbb{R}^d} \prod_{j=1}^k \mathbb{1}(x_j \notin [\eta, x_1, \ldots, x_k]^o) \, \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_k).
\end{aligned}$$

To go further we have to evaluate the occurring product. For this we make use of formula (4.3) with

$$A = [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o,$$

that is

$$\begin{aligned}
&\mathbb{1}\Big(\bigcup_{j=1}^k \{x_j\} \cap [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o \neq \emptyset\Big) \\
&\qquad = \sum_{r=1}^{k-1} (-1)^{r+1} \sum_{I \in \{1, \ldots, k\}^r_{\neq}} \mathbb{1}\Big(\bigcup_{j \in I} \{x_j\} \subset [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o\Big).
\end{aligned}$$

Because it is impossible that all points $\{x_1, \ldots, x_k\}$ are in $[\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o$, the term for $r = k$ is missing. If we multiply both sides by $\prod_{j=1}^k \mathbb{1}(x_j \notin [\eta]^o)$, we obtain

$$\prod_{j=1}^{k} \mathbb{1}(x_j \notin [\eta, x_1, \ldots, x_k]^o)$$

$$= \prod_{j=1}^{k} \mathbb{1}(x_j \notin [\eta]^o)\mathbb{1}(x_j \notin [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o)$$

$$= \prod_{j=1}^{k} \mathbb{1}(x_j \notin [\eta]^o)\Big(1 - \mathbb{1}(\bigcup_{j=1}^{k}\{x_j\} \cap [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o \neq \emptyset)\Big)$$

$$= \prod_{j=1}^{k} \mathbb{1}(x_j \notin [\eta]^o) - \sum_{r=1}^{k-1}(-1)^{r+1}$$

$$\times \sum_{I \in \{1,\ldots,k\}^r_{\neq}} \prod_{j \in I} \mathbb{1}(x_j \in [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o) \prod_{j=1}^{k} \mathbb{1}(x_j \notin [\eta]^o).$$

In the next step we have to integrate over all $x_1, \ldots, x_k$ in $\mathbb{R}^d$ or, more precisely, over $\mathbb{R}^d \setminus \Pi_t$ because of the indicator functions $\mathbb{1}(x_j \notin [\eta]^o)$. The integral of the first term on the right-hand side thus equals $\Delta(\Pi_t)^k$. We obtain

$$\mathbb{E}N(\Pi_t)_{(k)} = t^k\ \mathbb{E}\Delta(\Pi_t)^k - t^k\ \sum_{r=1}^{k-1}(-1)^{r+1} \sum_{I \in \{1,\ldots,k\}^r_{\neq}} \mathbb{E} \int_{\mathbb{R}^d\setminus\Pi_t} \cdots \int_{\mathbb{R}^d\setminus\Pi_t}$$

$$\times \prod_{j \in I} \mathbb{1}(x_j \in [\eta, x_1, \ldots, x_k]^o \setminus [\eta]^o) \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_k)$$

$$= t^k\ \mathbb{E}\Delta(\Pi_t)^k - t^k\ \sum_{r=1}^{k-1}(-1)^{r+1}\binom{k}{r}$$

$$\times\, \mathbb{E} \int_{\mathbb{R}^d\setminus\Pi_t} \ldots \int_{\mathbb{R}^d\setminus\Pi_t} \mu([\eta, x_1, \ldots, x_{k-r}] \setminus [\eta])^r \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_{k-r}).$$

□

4.5. Applications

In the last 30 years, many papers have been devoted to compute the asymptotic distribution of the quantities mentioned above, in many cases for the Poisson model and under the assumption that μ is the uniform distribution on a smooth convex set or a polytope, or for the d-dimensional Gaussian measure. Most of these results carry over to the binomial model by some de-Poissonization arguments; see for example the papers by Calka and Yukich [14] and Bárány and Reitzner [4, 3].

In this section, we want to contribute to these results giving an example of how our results can be applied. Assume that $K \in \mathcal{K}_+^k$, i.e., it has k-times continuously differentiable boundary of positive Gaussian curvature and volume one. Let $\mu(\cdot) = \lambda_d(K \cap \cdot)$ be Lebesgue measure restricted to the convex body K, and hence Π_t is a Poisson polytope inscribed in K. After planar results going back to Renyi and Sulanke ([35] and [36]) it was shown by Bárány [2] that for any d-dimensional smooth convex body $K \in \mathcal{K}_+^3$

$$\begin{aligned} \mathbb{E}N(\Pi_t) &= c_1\Omega(K)t^{\frac{d-1}{d+1}} + o(t^{\frac{d-1}{d+1}}), \\ \lambda_d(K) - \mathbb{E}\lambda_d(\Pi_t) &= c_1\Omega(K)t^{-\frac{2}{d+1}} + o(t^{\frac{d-1}{d+1}}) \end{aligned} \tag{4.17}$$

as $t \to \infty$, where $\Omega(K)$ denotes the affine surface area of the boundary of K. In fact, these results have been obtained for the binomial model, but it is easy to see that results for the binomial model immediately carry over to the Poisson model. For a long time, it was out of reach to compute the asymptotic behavior of the variance or even precise estimates. Only recently it was proved by Reitzner [33], using the Efron-Stein jackknife inequality, that for $K \in \mathcal{K}_+^2$ there are constants $c_2(K), c_3(K) > 0$ such that

$$\begin{aligned} c_2(K)t^{-\frac{d+3}{d+1}} &\le \mathbb{V}\lambda_d(\Pi_t) \le & c_3(K)t^{-\frac{d+3}{d+1}}, \\ c_2(K)t^{\frac{d-1}{d+1}} &\le \mathbb{V}N(\Pi_t) \le & c_3(K)t^{\frac{d-1}{d+1}}. \end{aligned}$$

A very recent breakthrough was achieved by Calka and Yukich, who calculated in [14] the precise asymptotics for the variances of the number of vertices and the volume of the random polytope Π_t. We have for $K \in \mathcal{K}_+^3$

$$\mathbb{V}N(\Pi_t) = c_4\Omega(K)t^{\frac{d-1}{d+1}} + o(t^{\frac{d-1}{d+1}}) \tag{4.18}$$

and for $K \in \mathcal{K}_+^6$

$$\mathbb{V}\lambda_d(\Pi_t) = c_5\Omega(K)t^{-\frac{d+3}{d+1}} + o(t^{-\frac{d+3}{d+1}}) \tag{4.19}$$

as $t \to \infty$. We can apply our identities to deduce one from the other. Because

$\mathbb{V}\Delta(\Pi_t) = \mathbb{V}\lambda_d(\Pi_t)$, Eq. (4.16) implies

$$t^2\mathbb{V}\lambda_d(\Pi_t) = \mathbb{V}N(\Pi_t) - \mathbb{E}N(\Pi_t) + 2t^2\mathbb{E}\int_{K\setminus\Pi_t} \lambda_d([\eta, x] \setminus [\eta])\,\mathrm{d}x\,.$$

By (4.18) and (4.17), it follows for $K \in \mathcal{K}_+^3$

$$\mathbb{V}\lambda_d(\Pi_t) = c_6\Omega(K)t^{-\frac{d+3}{d-1}} + o(t^{-\frac{d+3}{d-1}}) - 2\mathbb{E}\int_{K\setminus\Pi_t} \lambda_d([\eta, x] \setminus [\eta])\,\mathrm{d}x$$

as $t \to \infty$. We will prove in the appendix that $D_t = \int_{K\setminus\Pi_t} \lambda_d([\eta, x] \setminus [\eta])\,dx$ satisfies

$$\mathbb{E}D_t = c_7\Omega(K)t^{-\frac{d+3}{d+1}} + o(t^{-\frac{d+3}{d+1}})$$

for $K \in \mathcal{K}_+^2$ as $t \to \infty$. Combining these estimates proves the following corollary.

Corollary 4.5.1. *For $K \in \mathcal{K}_+^3$ we have*

$$\mathbb{V}\lambda_d(\Pi_t) = c_8\Omega(K)t^{-\frac{d+3}{d-1}} + o(t^{-\frac{d+3}{d-1}})$$

as $t \to \infty$.

This is the result of Calka and Yukich [14] for a slightly bigger class of convex bodies. As in their paper this could be transferred to a formula giving the asymptotic variance for the binomial model.

Furthermore, we can use this corollary, (4.17) and (4.5) to obtain asymptotically the variance of the number of inner points $I(\Pi_t)$ for $K \in \mathcal{K}_+^3$.

$$\begin{aligned}\mathbb{V}I(\Pi_t) &= t^2\mathbb{V}\lambda_d(\Pi_t) + t\mathbb{E}\lambda_d(\Pi_t)\\ &= t + c_9\Omega(K)t^{\frac{d-1}{d+1}} + o(t^{\frac{d-1}{d+1}})\end{aligned}$$

as $t \to \infty$. Observe that it follows immediately from $N(\Pi_t) + I(\Pi_t) = \eta(K)$ that

$$\mathbb{E}I(\Pi_t) = t - c_{10}\Omega(K)t^{\frac{d-1}{d+1}} + o(t^{\frac{d-1}{d+1}})$$

as $t \to \infty$.

Similarly, one could apply our identities in the case when the intensity measure of the Poisson point process is a multiple of the uniform measure on a polytope K, or a multiple of the Gaussian distribution. We refer to [3] and [22].

4.6. Appendix

Theorem 4.6.1. *Assume that $K \in \mathcal{K}^2_+$ with $\lambda_d(K) = 1$, and let $\Pi_t = [\eta]$ be the Poisson polytope chosen according to the intensity measure $t\lambda_d(K \cap \cdot)$. Define*

$$D_t = \int\limits_{K\setminus\Pi_t} \lambda_d([\eta, x] \setminus [\eta])\, dx.$$

Then there is a positive constant C_d depending on the dimension such that

$$\lim_{t\to\infty} \mathbb{E} D_t t^{1+\frac{2}{d+1}} = \mathrm{C}_d \Omega(K).$$

For $x \in K$ denote by $\mathcal{F}(\eta, x)$ the set of facets of Π_t which can be seen from x, i.e., which are facets of Π_t but not of $[\Pi_t, x] = [\eta, x]$. Note that this set is empty if $x \in \Pi_t$. Using this notation, we have

$$\lambda_d([\eta, x] \setminus [\eta]) = \frac{1}{d!} \sum_{(x_1,\dots,x_d)\in\eta^d_{\neq}} \mathbb{1}([x_1, \dots, x_d] \in \mathcal{F}(\eta, x))\lambda_d[x_1, \dots, x_d, x].$$

The Slivnyak-Mecke formula (4.2) yields

$$\begin{aligned} \mathbb{E} D_t &= \frac{1}{d!} \int\limits_K \mathbb{E} \sum_{(x_1,\dots,x_d)\in\eta^d_{\neq}} \mathbb{1}([x_1, \dots, x_d] \in \mathcal{F}(\eta, x))\lambda_d[x_1, \dots, x_d, x]\, \mathrm{d}x \\ &= \frac{1}{d!} t^d \int\limits_K \dots \int\limits_K \mathbb{E}\mathbb{1}(F \in \mathcal{F}(\eta + \sum \delta_{x_i}, x))\lambda_d[F, x]\, \mathrm{d}x_1 \dots \mathrm{d}x_d \mathrm{d}x, \end{aligned}$$

where $F = [x_1, \dots, x_d]$. The affine hull of F is a hyperplane which cuts K into two parts. Denote by $K_+(F)$ that part of K which contains x. The indicator function equals one if the affine hull of F separates x from η, i.e., if $\eta(K_+) = 0$. This happens with probability $e^{-t\lambda_d(K_+(F))}$. The volume of the simplex $[F, x]$ equals $1/d$ times the base $\lambda_{d-1}(F)$ times the height, which is the distance $d_{\mathrm{aff}\,F}(x)$ of x to the affine hull of F.

$$\mathbb{E} D_t = \frac{1}{d\, d!} t^d \int\limits_K \dots \int\limits_K e^{-t\lambda_d(K_+(F))} \lambda_{d-1}(F) \mathrm{d}_{\mathrm{aff}\,F}(x)\, \mathrm{d}x_1 \dots \mathrm{d}x_d \mathrm{d}x$$

The next lemma gives the asymptotic behavior of this integral and thus proves our theorem.

Lemma 4.6.2. *Assume that $K \in \mathcal{K}^2_+$ with $\lambda_d(K) = 1$. Then*

$$\int\limits_K \cdots \int\limits_K e^{-t\lambda_d(K_+(F))} \lambda_{d-1}(F) \mathrm{d}_{\mathrm{aff} F}(x)\, \mathrm{d}x_1 \ldots \mathrm{d}x_d \mathrm{d}x$$

$$= \quad c_d \Omega(K) t^{-(d+1)-\frac{2}{d+1}} + o\left(t^{-d-1-\frac{2}{d+1}}\right) \qquad (4.20)$$

as $t \to \infty$.

Principal ideas for the proof of this lemma are taken from [33], where the asymptotics of a similar integral was computed.

Proof of Lemma 4.6.2. In a first step, we transform the integral using the Blaschke–Petkantschin formula (cf., [39, Thm. 7.2.7]), see Sect. 2.1.3),

$$\int\limits_K \cdots \int\limits_K f(x_1, \ldots, x_d) \mathrm{d}x_1 \ldots \mathrm{d}x_d$$

$$= \quad (d-1)! \int\limits_{H \in \mathcal{H}^d_{d-1}} \int\limits_{K \cap H} \cdots \int\limits_{K \cap H} f(x_1, \ldots, x_d) \lambda_{d-1}(F) \mathrm{d}x_1 \ldots \mathrm{d}x_d \mathrm{d}H.$$

The differential dH corresponds to the suitably normalized rigid motion invariant Haar measure on the Grassmannian $\mathcal{H}^d_{d-1}$ of hyperplanes in $\mathbb{R}^d$. A hyperplane is given by its unit normal vector $u \in S^{d-1}$ and its signed distance h to the origin, $H(u,h) = \{y : \langle y, u \rangle = h\}$. Let $H_+ = \{y : \langle y, u \rangle \geq h\}$ be the corresponding halfspace. Denoting by du the element of surface area on S^{d-1}, we have $dH = \frac{1}{2} dh du$, $u \in S^{d-1}, h \in \mathbb{R}$. (Observe that $H(h,u) = H(-h,-u)$, which explains the factor $\frac{1}{2}$.)

Because of $\lambda_{d-1}(F)$ the integrand vanishes outside the interval

$$h \in [-h_K(-u), h_K(u)],$$

where $h_K(u)$ is the support function of K in direction u. Given $H = H(h,u)$, we assume that the additional point $x \in H_+$. Then $K_+(F) = K \cap H_+$ and $\lambda_+ = \lambda_d(K_+)$ only depends on H_+ but not on the relative position of the points $x_j \in H$. This yields

$$\int\limits_K \cdots \int\limits_K e^{-t\lambda_d(K_+(F))} \lambda_{d-1}(F) \mathrm{d}_{\mathrm{aff} F}(x)\, \mathrm{d}x_1 \ldots \mathrm{d}x_d \mathrm{d}x$$

$$= \quad \frac{(d-1)!}{2} \int\limits_{S^{d-1}} \int\limits_{-h_K(-u)}^{h_K(u)} e^{-t\lambda_+} \mathcal{I}_{K \cap H} \mathcal{J}_{K \cap H_+}\, \mathrm{d}h \mathrm{d}u \qquad (4.21)$$

with

$$\mathcal{I}_{K\cap H} = \int\limits_{K\cap H} \dots \int\limits_{K\cap H} \lambda\!\!\lambda_{d-1}(F)^2 \,\mathrm{d}x_1 \dots \mathrm{d}x_d, \quad \mathcal{J}_{K\cap H_+} = \int\limits_{K_+} \mathrm{d}_H(x)\,\mathrm{d}x.$$

Given some $\varepsilon > 0$, we split the integral in (4.21) with respect to h into two parts: $h \in [-h_K(-u), h_K(u) - \varepsilon]$ and $h \in [h_K(u) - \varepsilon, h_K(u)]$. Estimating the integral

$$\int\limits_{-h_K(-u)}^{h_K(u)-\varepsilon} e^{-t\lambda_+} \mathcal{I}_{K\cap H} \mathcal{J}_{K\cap H_+} \,\mathrm{d}h$$

is easy. The integrals $\mathcal{I}_{K\cap H}$ and $\mathcal{J}_{K\cap H_+}$ are always bounded by a constant γ_1 independent of h and u. There exists a constant $\gamma_2 = \gamma_2(\delta) > 0$ independent of u with $\lambda_+ = \lambda_+(h,u) \geq \gamma_2$. And $h_K(u) + h_K(-u)$ is bounded by some constant γ_3 independent of u. Thus for $h \leq h_K(u) - \varepsilon$ we have

$$0 \leq \int\limits_{-h_K(-u)}^{h_K(u)-\varepsilon} e^{-t\lambda_+} \mathcal{I}_{K\cap H} \mathcal{J}_{K\cap H_+} \,\mathrm{d}h \leq \gamma_1^2 \gamma_3 e^{-t\gamma_2}. \tag{4.22}$$

We estimate the second part of the integral. Let $u \in S^{d-1}$ be fixed. As K is of class $\mathcal{K}^2_+$, there is a unique point $p \in \partial K$ with outer normal vector u. Choose $\delta > 0$ sufficiently small. There exists a paraboloid $q^{(p)}(y)$ and a $\lambda = \lambda(\delta) > 0$ such that the λ-neighborhood of p in ∂K can be represented by a convex function $f^{(p)}(y)$ fulfilling

$$((1+\delta)^{-1} q^{(p)}(y) + p) \leq f^{(p)}(y) \leq ((1+\delta) q^{(p)}(y) + p). \tag{4.23}$$

Now we fix $\varepsilon > 0$ such that for each u the intersection $H(h_K(u) - \varepsilon, u) \cap \partial K$ is contained in this λ-neighborhood of the boundary point p.

Let $\mathbb{R}^d = \{(y,z) | y \in \mathbb{R}^{d-1}, z \in \mathbb{R}\}$. For the moment, identify the tangent hyperplane to ∂K at p with the plane $z = 0$ and p with the origin such that K is contained in the halfspace $z \geq 0$ and u coincides with $(0,-1)$. Hence, in this situation $h_K(u) = 0$. Define $H(z) = H(-h,u)$ to be the hyperplane parallel to $z = 0$ with distance z to the origin and, in accordance with the definition above, $H_+(z)$ to be the corresponding halfspace containing the new origin.

We introduce polar coordinates: let $\mathbb{R}^d = (\mathbb{R}^+ \times S^{d-2}) \times \mathbb{R}$ and denote by (rv, z) a point in $\mathbb{R}^d$, $r \in \mathbb{R}^+$, $v \in S^{d-2}$, $z \in \mathbb{R}$. Since $K \in \mathcal{K}^2_+$, by choosing a suitable Cartesian coordinate system in $\mathbb{R}^{d-1}$, the paraboloid can be parametrized by

$$b_2(rv) = \tfrac{1}{2}(k_1\langle rv, e_1\rangle^2 + \cdots + k_{d-1}\langle rv, e_{d-1}\rangle^2),$$

where $k_1, \ldots, k_{d-1}$ are the principal curvatures of K at p. The estimate (4.23) reads as

$$(1+\delta)^{-1} b_2(v) r^2 \leq z = f(rv) \leq (1+\delta) b_2(v) r^2,$$

which implies

$$(1+\delta)^{-\frac{1}{2}} b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}} \leq r = r(v,z) \leq (1+\delta)^{\frac{1}{2}} b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}}, \tag{4.24}$$

where r is the radial function of $K \cap H(z)$. From this we obtain estimates for the $(d-1)$-dimensional volume of $K \cap H(z)$

$$(1+\delta)^{-\frac{d-1}{2}} c_1 \kappa(u)^{-\frac{1}{2}} z^{\frac{d-1}{2}} \leq \lambda_{d-1}(K \cap H(z)) \leq (1+\delta)^{\frac{d-1}{2}} c_1 \kappa(u)^{-\frac{1}{2}} z^{\frac{d-1}{2}} \tag{4.25}$$

with a suitable constant $c_1 > 0$, where $\kappa(u) = \prod k_i$ is the Gaussian curvature of K at p. By definition

$$\lambda_+(z) = \int_0^z \lambda_{d-1}(K \cap H(t)) \mathrm{d}t, \tag{4.26}$$

which by (4.25) implies

$$(1+\delta)^{-\frac{d-1}{2}} \frac{2}{d+1} c_1 \kappa(u)^{-\frac{1}{2}} z^{\frac{d+1}{2}} \leq \lambda_+(z) \leq (1+\delta)^{\frac{d-1}{2}} \frac{2}{d+1} c_1 \kappa(u)^{-\frac{1}{2}} z^{\frac{d+1}{2}}. \tag{4.27}$$

For a given z, (4.24) shows that $K \cap H(z)$ contains an ellipsoid $\mathcal{E}_-$ defined by $(1+\delta)^{-1} b_2(v) r^2 = z$, resp., is contained in an ellipsoid $\mathcal{E}_+$ defined by $(1+\delta) b_2(v) r^2 = z$. We are interested in

$$\mathcal{I}_{K \cap H(z)} = \int_{K \cap H(z)} \cdots \int_{K \cap H(z)} \lambda_{d-1}(F)^2 \, dx_1 \ldots \mathrm{d}x_d.$$

Clearly, if the range of integration is increased, resp., decreased, $\mathcal{I}$ will increase, resp., decrease.

$$\mathcal{I}_{\mathcal{E}_-} \leq \mathcal{I}_{K \cap H(z)} \leq \mathcal{I}_{\mathcal{E}_+}$$

Note that these integrals are invariant under volume–preserving affinities. Thus, $\mathcal{I}_{\mathcal{E}_\pm}$ does not depend on the shape of the ellipsoids and is proportional to $\lambda_{d-1}(\mathcal{E}_\pm)^{d+2}$. Hence, there exists a suitable constant c_2 for which

$$(1+\delta)^{-\frac{(d-1)(d+2)}{2}} c_2 \kappa(u)^{-\frac{d+2}{2}} z^{\frac{(d-1)(d+2)}{2}} \leq \mathcal{I}_{K\cap H(z)} \leq (1+\delta)^{\frac{(d-1)(d+2)}{2}} c_2 \kappa(u)^{-\frac{d+2}{2}} z^{\frac{(d-1)(d+2)}{2}}.$$

In the last step, we estimate

$$\mathcal{J}_{K\cap H_+(z)} = \int_{K_+(z)} \mathrm{d}_H(x)\,\mathrm{d}x = \int_0^z \lambda_{d-1}(K\cap H(t))(z-t)\mathrm{d}t.$$

By the same monotonicity argument used above, we obtain

$$(1+\delta)^{-\frac{d-1}{2}} c_3 \kappa(u)^{-\frac{1}{2}} z^{\frac{d+3}{2}} \leq \mathcal{J}_{K\cap H_+(z)} \leq (1+\delta)^{\frac{d-1}{2}} c_3 \kappa(u)^{-\frac{1}{2}} z^{\frac{d+3}{2}}. \quad (4.28)$$

Now we are ready to estimate the integral

$$\int_{h_K(u)-\varepsilon}^{h_K(u)} e^{-t\lambda_+} \mathcal{I}_{K\cap H} \mathcal{J}_{K\cap H_+}\,\mathrm{d}h = \int_0^\varepsilon e^{-t\lambda_+(z)} \mathcal{I}_{K\cap H(z)} \mathcal{J}_{K\cap H_+(z)}\,\mathrm{d}z.$$

Note that (4.26) is equivalent to

$$\frac{d}{dz}\lambda_+(z) = -\lambda_{d-1}(K\cap H(z)),$$

and substituting $v = \lambda_+(z)$ implies

$$\int_0^\varepsilon e^{-t\lambda_+(z)} \mathcal{I}_{K\cap H(z)} \mathcal{J}_{K\cap H_+(z)}\,\mathrm{d}z = \int_0^{\lambda_+(\varepsilon)} e^{-tv} \mathcal{I}_{K\cap H(z(v))} \mathcal{J}_{K\cap H_+(z(v))} \lambda_{d-1}(K\cap H(z(v)))^{-1}\,\mathrm{d}v,$$

where $H(z(v))$ denotes the hyperplane parallel to $z=0$ cutting off from K a cap of volume v.

Combining this with (4.25) - (4.28) yields

$$c_4(1+\delta)^{-\frac{(d-1)(d^2+3d+3)}{(d+1)}} \kappa(u)^{-\frac{d}{d+1}} \int_0^{\lambda_+(\varepsilon)} e^{-tv} v^{\frac{d^2+d+2}{d+1}}\,\mathrm{d}v \leq \int_{h_K(u)-\varepsilon}^{h_K(u)} e^{-tv} \mathcal{I}_{K\cap H(z)} \mathcal{J}_{K\cap H_+(z)}\,\mathrm{d}h \leq c_4(1+\delta)^{\frac{(d-1)(d^2+3d+3)}{(d+1)}} \kappa(u)^{-\frac{d}{d+1}} \int_0^{\lambda_+(\varepsilon)} e^{-tv} v^{\frac{d^2+d+2}{d+1}}\,\mathrm{d}v$$

with a suitable constant c_4. Hence, we are interested in the asymptotic behavior of the Laplace transform

$$\int_0^{\lambda_+(\varepsilon)} e^{-tv} v^{\frac{d^2+d+2}{d+1}} \mathrm{d}v = \mathcal{L}\left(v^{\frac{d^2+d+2}{d+1}}\right)(t) + O\left((1-\gamma_2)^t\right)$$

as $t \to \infty$. (Recall that $\lambda_+(\varepsilon) \geq \gamma_2$.) By an Abelian theorem, cf., e.g., [17, chap. 3, § 1], we obtain

$$\mathcal{L}\left\{v^\alpha\right\}(t) = \Gamma(\alpha+1)\, t^{-\alpha-1} + O\left(t^{-\alpha-2}\right)$$

as $t \to \infty$. This implies the following bounds

$$\begin{aligned} & c_5(1+\delta)^{-\frac{(d-1)(d^2+3d+3)}{(d+1)}} \kappa(u)^{-\frac{d}{d+1}} t^{-(d+1)-\frac{2}{d+1}} \left(1 + O\left(t^{-1}\right)\right) \\ \leq & \int_{h_K(u)-\varepsilon}^{h_K(u)} e^{-tv} \mathcal{I}_{K\cap H(z)} \mathcal{J}_{K\cap H_+(z)}\, \mathrm{d}h \\ \leq & \; c_5(1+\delta)^{\frac{(d-1)(d^2+3d+3)}{(d+1)}} \kappa(u)^{-\frac{d}{d+1}} t^{-(d+1)-\frac{2}{d+1}} \left(1 + O\left(t^{-1}\right)\right) \end{aligned}$$

as $t \to \infty$, where the constants in $O(\cdot)$ and the constant c_5 are independent of p and u.

Concerning the remaining integration, note that the term

$$\int_{S^{d-1}} \kappa(u)^{-1+\frac{1}{d+1}} \mathrm{d}u = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \mathrm{d}x$$

is the affine surface area $\Omega(K)$. Since the terms in (4.22) are of smaller order, we finally obtain

$$\begin{aligned} & c_6(1+\delta)^{-\frac{(d-1)(d^2+3d+3)}{(d+1)}} \Omega(K) t^{-(d+1)-\frac{2}{d+1}} \left(1 + O\left(t^{-1}\right)\right) \\ \leq & \int_K \cdots \int_K e^{-t\lambda_d(K_+(F))} \lambda_{d-1}(F) d_{\mathrm{aff}\,F}(x)\, dx_1 \ldots dx_d dx \\ \leq & \; c_6(1+\delta)^{\frac{(d-1)(d^2+3d+3)}{(d+1)}} \Omega(K) t^{-(d+1)-\frac{2}{d+1}} \left(1 + O\left(t^{-1}\right)\right) \end{aligned}$$

as $t \to \infty$ with a suitable constant c_6. Since this holds for each $\delta > 0$, the proof is finished. □

5. Monotonicity of the Number of Facets of Random Polytopes

5.1. Introduction

A natural question about random polytopes, constructed as the convex hull P_n of n random points $X_1, \ldots, X_n$ chosen independently and according to a given density function in $\mathbb{R}^d$ or in a convex body K, is how the expected number of k-dimensional faces $\mathbb{E}f_k(P_n)$ behaves if the number of generating points increases.

The first studies concerning this issue have been done by Devillers, Glisse, Goaoc, Moroz and Reitzner ([15]). They considered convex hulls of uniformly distributed random points in a convex body K. It is proven that for planar convex sets the expected number of vertices $\mathbb{E}f_0(P_n)$ is increasing in n. Furthermore, they showed that, for $d \geq 3$, the number of facets $\mathbb{E}f_{d-1}(P_n)$ is increasing for n large enough if $\lim_{n\to\infty} \frac{\mathbb{E}f_{d-1}(P_n)}{An^c} = 1$ for some constants A and $c > 0$ and that $\mathbb{E}f_{d-1}(P_n)$ is asymptotically increasing in n for K being a smooth convex body. In this chapter we want to study the monotonicity of the expected number of facets $\mathbb{E}f_{d-1}(P_n)$ for four different settings of generating the random polytopes.

Let $X_1, \ldots, X_n$ be a Gaussian sample in $\mathbb{R}^d$, $d \in \mathbb{N}$, i.e., n random points chosen independently and according to the d-dimensional standard normal distribution with mean 0 and covariance matrix $\frac{1}{2}I_d$, where I_d is the $d \times d$-unit matrix. Then the convex hull of these points, denoted by $P_n = [X_1, \ldots, X_n]$, is a Gaussian polytope. The main result of this chapter concerns this kind of random polytopes. The question is: what happens if we choose $n + 1$ random points instead of n? Can we expect that the generated Gaussian polytope has more facets in average? The answer is given by the main result of this chapter:

Theorem 5.1.1. *Let $P_n = [X_1, \ldots, X_n]$ be the Gaussian polytope generated by the convex hull of n independent random points in $\mathbb{R}^d$ chosen according to* *the d-dimensional standard normal distribution. Then*

$$\mathbb{E}f_{d-1}(P_n) \text{ is monotonically increasing in } n.$$

In addition we want to show what happens concerning the monotonicity of the number of facets if the Gaussian polytope is generated by a random number of points. Therefore, we consider a Gaussian sample $X_1, \dots, X_N$ in $\mathbb{R}^d$ where N is Poisson-distributed with parameter $t > 0$. We will denote the convex hull of these random points by Π_t and call it a Gaussian Poisson polytope. It will be shown that the expected number of facets increases if the intensity t increases.

The third kind of random polytopes considered here emerges by the convex hull of n random points chosen independently and according to the uniform distribution from a d-dimensional ball B_d with volume 1. The random polytope which comes up in this way will be denoted by P_n^*. Here, too, we can adapt the poissonized variant as explained in the Gaussian case. This means that we have N independent uniformly distributed random points in B_d where again $N \sim Poisson(t)$.

5.2. Integral Estimates Using Concave Functions

In the proofs of this chapter will appear integrals of the form:

$$c\int_0^\alpha h(s)g(s)L(s)^{d-1}\mathrm{d}s \tag{5.1}$$

with $c > 0$, $\alpha \in \{0, \infty\}$, $h(s)$ nonnegative and $g(s)$ a linear function with negative slope and root denoted by s_N. $L(s)$ is positive on $[0,1]$ and in the case of $\alpha = \infty$ equal to zero on $(1, \infty)$. In each concrete situation we will see that $L(s)$ is a star-shaped function with respect to zero; see Eq.(2.4). One key ingredient in our proofs will be to estimate the integral (5.1) by substituting $L(s)$ by the linear function $f(s) = \frac{L(s_N)}{s_N}s,\ s \in [0, \alpha)$, such that we can deduce

$$c\int_0^\alpha h(s)g(s)L(s)^{d-1}\mathrm{d}s \geq c\int_0^\alpha h(s)g(s)f(s)^{d-1}\mathrm{d}s. \tag{5.2}$$

The star-shape property of $L(s)$ will be proved in any case by showing the concavity on the intervall $[0,1]$.

5.3. Gaussian Polytopes

In this section we want to study the monotonicity of the expected number of facets $\mathbb{E}f_{d-1}$ of the Gaussian polytope $P_n = [X_1, \ldots, X_n]$ regarding the number n of random points generating P_n. We can state

Theorem 5.3.1. *Let $P_n = [X_1, \ldots, X_n]$ be the Gaussian polytope generated by the convex hull of n independent random points in $\mathbb{R}^d$ chosen according to the d-dimensional standard normal distribution. Then*

$$\mathbb{E}f_{d-1}(P_n) \text{ is monotonically increasing in } n.$$

Proof of Theorem 5.3.1. At first, we have to set a formula describing $\mathbb{E}f_{d-1}$. Each $(d-1)$-dimensional face of P_n is the convex hull of exactly d random points with probability one. Since $X_1, \ldots, X_n$ are chosen independently and identically, it holds

$$\begin{aligned} \mathbb{E}f_{d-1}(P_n) &= \sum_{\{j_1,\ldots,j_d\}\subset\{1,\ldots,n\}} \mathbb{E}\ \mathbb{1}([x_{j_1}, \ldots, x_{j_d}] \text{ is a facet}) \\ &= \binom{n}{d} \mathbb{P}([x_1, \ldots, x_d] \text{ is a facet}). \end{aligned}$$

We denote by H the affine hull of the $(d-1)$-dimensional simplex $P_d = [x_1, \ldots, x_d]$, which devides $\mathbb{R}^d$ into two half spaces H^+ and H^-. P_d being a facet means that all other points $x_{d+1}, \ldots, x_n$ are either located in H^+ or in H^-. We denote by H^* the projection of H onto its orthogonal complement $H^\perp$ and conclude that P_d is a facet of P_n if the orthogonal projection of P_d is a vertex of the convex hull of the orthogonal projection of P_n onto $H^\perp$. Thus, we can use the onedimensional standard normal distribution Φ to calculate the probability that P_d is a facet of P_n.

$$\mathbb{E}f_{d-1}(P_n) = \binom{n}{d} \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \{(1-\Phi(H^*))^{n-d} + \Phi(H^*)^{n-d}\} \prod_{i=1}^{d} \phi_d(x_i)\ \mathrm{d}x_i. \tag{5.3}$$

Now we use the affine Blaschke-Petkantschin formula (see Section 2.1.3, for

$k = d - 1$) to conclude

$$\mathbb{E} f_{d-1}(P_n) = c_d \binom{n}{d} \int\limits_{\mathcal{H}_{d-1}^d} \int\limits_H \cdots \int\limits_H \left\{(1-\Phi(H^*))^{n-d} + \Phi(H^*)^{n-d}\right\} \times \Delta_{d-1}(x_1, \ldots, x_d) \prod_{i=1}^d \phi_d(x_i) \, \mathrm{d}_H x_i \, \mathrm{d}H,$$

where c_d is a positive constant only depending on the dimension d. This constant will vary in the following without getting different notations. The differential dH corresponds to the suitably normalized rigid motion invariant Haar measure on the Grassmannian $\mathcal{H}_{d-1}^d$ of hyperplanes in $\mathbb{R}^d$. Since every hyperplane is given by its unit normal vector $u \in S^{d-1}$ and its distance h to the origin, $H(u,h) = \{y : \langle y, u \rangle = h\}$, we can write

$$\begin{aligned} &\mathbb{E} f_{d-1}(P_n) \\ &= c_d \binom{n}{d} \int\limits_{S^{d-1}} \int\limits_0^\infty \left\{\Phi(h)^{n-d} + (1-\Phi(h))^{n-d}\right\} \\ &\quad \times \left(\int\limits_H \cdots \int\limits_H \Delta_{d-1}(x_1, \ldots, x_d) e^{-\left(\frac{\|x_1\|^2}{2} + \ldots + \frac{\|x_d\|^2}{2}\right)} \prod_{i=1}^d \mathrm{d}x_i \right) e^{-\frac{\|h\|^2 d}{2}} \mathrm{d}h \mathrm{d}u. \end{aligned}$$

Because the multiple integral in the last line is a constant regarding the integration variables h and u and because of the rotation invariance of the integrand, we simplify to get finally

$$\begin{aligned} &\mathbb{E} f_{d-1}(P_n) \\ &= c_d \binom{n}{d} \int\limits_{-\infty}^\infty (1-\Phi(h))^{n-d} \\ &\quad \times \left(\int\limits_H \cdots \int\limits_H \Delta_{d-1}(x_1, \ldots, x_d) e^{-\left(\frac{\|x_1\|^2}{2} + \ldots + \frac{\|x_d\|^2}{2}\right)} \prod_{i=1}^d \mathrm{d}x_i \right) e^{-\frac{\|h\|^2 d}{2}} \mathrm{d}h \\ &= c_d \binom{n}{d} \int_{\mathbb{R}} (1-\Phi(h))^{n-d} \phi(h)^d \mathrm{d}h. \end{aligned}$$

We substitute $s = \Phi(h)$ and set $\phi(\Phi^{-1}(s)) = L_1(s)$ such that it holds

$$\begin{aligned} Ef_{d-1}(\mathbb{P}_n) &= c_d \binom{n}{d} \int_0^1 (1-s)^{n-d} \phi(\Phi^{-1}(s))^{d-1} \mathrm{d}s \\ &= c_d \binom{n}{d} \int_0^1 (1-s)^{n-d} L_1(s)^{d-1} \mathrm{d}s. \end{aligned}$$

Now the aim is to proof that

$$Ef_{d-1}(\mathbb{P}_n) \geq Ef_{d-1}(\mathbb{P}_{n-1}),$$

which means that we have to show

$$\int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d} \right] (1-s)^{n-1-d} L_1(s)^{d-1} \mathrm{d}s \geq 0.$$

Now we use our tool (5.2). We want to substitute $L_1(s)$ by the linear function $f(s) = \frac{L_1(s_N)}{s_N} s,\ s \in [0,1]$, where s_N denotes the root of the function $g(s) = \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right]$, to have

$$\begin{aligned} &\int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d} \right] (1-s)^{n-1-d} L_1(s)^{d-1} \mathrm{d}s \\ &\qquad \geq \int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d} \right] (1-s)^{n-1-d} f(s)^{d-1} \mathrm{d}s. \end{aligned}$$

This is possible if $L_1(s)$ is concave. A continuously differentiable function is (strictly) concave if and only if its derivative is (strictly) monotonically decreasing. Since

$$\phi'(h) = -h\phi(h), \quad h \in (-\infty, \infty)$$

and

$$\left(\Phi^{-1}(s)\right)' = \frac{1}{\phi(\Phi^{-1}(s))}, \quad s \in (0,1),$$

it follows

$$L_1'(s) = \frac{d}{ds}\phi(\Phi^{-1}(s)) = -\Phi^{-1}(s)\phi(\Phi^{-1}(s))\frac{1}{\phi(\Phi^{-1}(s))} = -\Phi^{-1}(s).$$

$\Phi(s)$ is monotonically increasing in s and therefore $\Phi^{-1}(s)$ is too. This implies that $-\Phi^{-1}(s)$ is monotonically decreasing. So $L_1(s)$ is concave on $[0,1]$ and we can apply (5.2). We have

$$
\begin{aligned}
&Ef_{d-1}(\mathbb{P}_n) - Ef_{d-1}(\mathbb{P}_{n-1})\\
&\qquad \geq \int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right](1-s)^{n-1-d} f(s)^{d-1} \mathrm{d}s\\
&\qquad = \left(\frac{L(s_N)}{s_N}\right)^{d-1} \int_0^1 (1-s)^{n-1-d} s^{d-1} \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right] \mathrm{d}s\\
&\qquad = \left(\frac{L(s_N)}{s_N}\right)^{d-1} \binom{n}{d} \int_0^1 (1-s)^{n-1-d} s^{d-1} \left((1-s) - \frac{n-d}{n}\right) \mathrm{d}s.
\end{aligned}
$$

In the next step, we use the Beta function and the property (2.7) to complete the proof.

$$
\begin{aligned}
&\mathbb{E}f_{d-1}(P_n) - \mathbb{E}f_{d-1}(P_{n-1})\\
&\qquad \geq \left(\frac{L(s_N)}{s_N}\right)^{d-1} \binom{n}{d} \left(\mathrm{B}(d, n+1-d) - \frac{n-d}{n} B(d, n-d)\right) = 0
\end{aligned}
$$

□

5.4. Gaussian Poisson Polytopes

Now we consider a Gaussian polytope Π_t with a Poisson-distributed number of generating points. Concerning the monotonicity of the number $\mathbb{E}f_{d-1}(\Pi_t)$ of occuring facets we can set the following theorem.

Theorem 5.4.1. *Let $\Pi_t = [X_1, \ldots, X_N]$ be a Gaussian polytope generated by the convex hull of N independent random points chosen according to the d-dimensional standard normal distribution, where the number of points N is Poisson distributed with parameter t. Then*

$$\mathbb{E}f_{d-1}(\Pi_t) \text{ is monotonically increasing in } t.$$

We want to give two different proofs for Theorem 5.4.1. The first one follows the same idea as we used for proving Theorem 5.3.1. The second one applies Theorem 5.3.1.

Proof of Theorem 5.4.1. We start again by calculating the expected number of facets

$$\begin{aligned} Ef_{d-1}(\Pi_t) &= \mathbb{E} \sum_{(x_1,\dots,x_d)\in\eta^d_{\neq}} \mathbb{1}([x_1,\dots,x_d] \text{ is a facet}) \\ &= t^d \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{P}([x_1,\dots,x_d] \text{ is a facet }) \prod_{i=1}^{d} \phi_d(x_i)\, \mathrm{d}x_i \\ &= t^d \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{P}(\eta \cap H^+ = \emptyset) \prod_{i=1}^{d} \phi_d(x_i)\, \mathrm{d}x_i. \end{aligned}$$

Then we use the affine Blaschke-Petkantschin formula (2.1.3) to get

$$\begin{aligned} Ef_{d-1}(\Pi_t) &= c_d\, t^d \int_{\mathcal{H}^d_{d-1}} \int_H \dots \int_H e^{-t\Phi(H^*)} \triangle[x_i] \prod_{i=1}^{d} \phi_d(x_i)\, \mathrm{d}_H x_i\, \mathrm{d}H \\ &= c_d\, t^d \int_{\mathbb{R}} e^{-t\Phi(h)} \phi(h)^d \mathrm{d}h \qquad (5.4) \end{aligned}$$

with a positive constant c_d only depending on the dimension d. We substitute $s = \Phi(h)$ and set $\phi(\Phi^{-1}(s))\mathbb{1}_{[0,1]}(s) = L_2(s)$. Then we have

$$Ef_{d-1}(\Pi_t) = c_d \int_0^\infty e^{-ts} t^d L_2(s)^{d-1} \mathrm{d}s.$$

To proof that the expected number of facets is monotonically increasing with respect to t, we will show that the first derivative of $Ef_{d-1}(\Pi_t)$ with respect to t is nonnegative. This is (omitting the positive constant c_d)

$$\frac{\mathrm{d}\, Ef_{d-1}(\Pi_t)}{\mathrm{d}t} = \int_0^\infty e^{-st} t^{d-1} (d - st) L_2(s)^{d-1} \mathrm{d}s.$$

As in the the proof of Theorem 5.3.1 we now apply our tool (5.2). We substitute $L_2(s)$, in this case, by $f(s) = L_2\left(\frac{d}{t}\right) s\frac{t}{d}$, $s \in [0, \infty)$. This implies

$$\begin{aligned}
&\int_0^\infty e^{-st}t^{d-1}(d-st)L_2(s)^{d-1}\mathrm{d}s \\
&\qquad \geq \int_0^\infty e^{-st}t^{d-1}(d-st)f(s)^{d-1}\mathrm{d}s \\
&\qquad = t^{d-1}\left(\frac{t}{d}\right)^{d-1} L_2\left(\frac{d}{t}\right)^{d-1}\int_0^\infty e^{-st}(d-st)s^{d-1}\mathrm{d}s \\
&\qquad = c_{t,d}\int_0^\infty e^{-st}(d-st)s^{d-1}\mathrm{d}s
\end{aligned}$$

with a positive constant $c_{t,d}$. Using the Gamma function and the property (2.8) we complete the proof.

$$\begin{aligned}
\int_0^\infty e^{-st}t^{d-1}(d-st)f(s)^{d-1}\mathrm{d}s &= c_{t,d}\int_0^\infty e^{-u}(d-u)\left(\frac{u}{t}\right)^{d-1}t^{-1}\mathrm{d}u \\
&= c_{t,d}t^{d-2}\left(d\,\Gamma[d] - d\,\Gamma[d]\right) \\
&= 0.
\end{aligned}$$

□

Proof of Theorem 5.4.1. Another possibility to proof Theorem 5.4.1 is to apply Theorem 5.3.1. It holds

$$\begin{aligned}
\mathbb{E}f_{d-1}(\Pi_t) &= \sum_{n=0}^\infty \mathbb{E}f_{d-1}(P_n)\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^\infty \mathbb{E}f_{d-1}(P_n)\frac{e^{-t}t^n}{n!}.
\end{aligned}$$

We set $a_n = \mathbb{E}f_{d-1}(P_n), n \in \mathbb{N}$, and show that the first derivative of

$$f(t) = \sum_{n=0}^\infty a_n\frac{e^{-t}t^n}{n!}, \quad t \in (0, \infty)$$

is nonnegative with $a_n, n \in \mathbb{N}$, monotonically increasing in n (see Theorem 5.3.1).

$$\begin{aligned} f(t)' &= \sum_{n=0}^{\infty} a_n \frac{1}{n!}\left(-e^{-t}t^n + e^{-t}nt^{n-1}\right) \\ &= -\sum_{n=0}^{\infty} a_n \frac{1}{n!} e^{-t}t^n + \sum_{n=0}^{\infty} a_{n+1}\frac{1}{n!}e^{-t}t^n \\ &\geq 0. \end{aligned}$$

□

5.5. Random Polytopes in a Ball

In this section we look at the monotonicity of polytopes generated by random points chosen according to the uniform distribution from a d-dimensional ball with volume 1.

Theorem 5.5.1. *Let $P_n^* = [X_1, \dots, X_n]$ be a random polytope generated by the convex hull of n random points chosen according to the d-dimensional uniform distribution from a d-dimensional ball B_d with volume 1. Then*

$$Ef_{d-1}(P_n^*) \text{ is monotonically increasing in } n.$$

Proof of Theorem 5.5.1. Analogous to (5.3) we can describe the number of facets of P_n^* by (cf. [8])

$$\begin{aligned} \mathbb{E}f_{d-1}(P_n^*) &= \binom{n}{d}\mathbb{P}([x_1, \dots, x_d] \text{ is a facet}) \\ &= \binom{n}{d}\int_{B^d}\dots\int_{B^d}\left\{V^{n-d} + (1-V)^{n-d}\right\}\mathrm{d}x_1\dots\mathrm{d}x_d, \end{aligned}$$

where $V = \frac{V^*}{\rho_d}$, V^* denotes the volume of the bigger of the two parts of B_d cut off by the affine hull of $x_1, \dots, x_d$ and ρ_d is the abbreviation for the volume of B_d. After using the affine Blaschke-Petkantschin formula (see Section 2.1.3) this becomes

$$\mathbb{E}f_{d-1}(P_n^*) = c_d \binom{n}{d}(d-1)! \int\limits_{S^{d-1}} \int\limits_0^1 \{V^{n-d} + (1-V)^{n-d}\}$$
$$\times \left(\int\limits_H \dots \int\limits_H \Delta_{d-1}(x_1, \dots, x_d)\, \mathrm{d}x_1 \dots \mathrm{d}x_d \right) \mathrm{d}h\mathrm{d}u,$$

where $\Delta_{d-1}(x_1, \dots, x_d)$ is the $(d-1)$-dimensional volume of the convex hull of $x_1, \dots, x_d$. The volume V depends (only) on the signed distance h of the hyperplane representing the affine hull of $x_1, \dots, x_d$ cutting off the bigger of the two parts of B_d. We get

$$\mathbb{E}f_{d-1}(P_n^*) = c_d \binom{n}{d}(d-1)! \int\limits_{-1}^1 (1-V)^{n-d} v(h)^{d+1}\, \mathrm{d}h,$$

where $v(h) = \frac{\mathrm{d}V(h)}{\mathrm{d}h}$. We substitute $s = V(h)$ and end up with

$$\mathbb{E}f_{d-1}(P_n^*) = c_d \binom{n}{d}(d-1)! \int\limits_0^1 (1-s)^{n-d} v(V^{-1}(s))^d \mathrm{d}s. \tag{5.5}$$

To apply the same idea as in the proofs of Theorem 5.3.1 and 5.4.1, we have to use our established integral form (5.1). It means that we want to replace $L_3(s) = v(V^{-1}(s))^{\frac{d}{d-1}}$ by a linear function. This again is proper if $L_3(s)$ is concave on $[0,1]$. Thus, we want to show that the first derivative of $L_3(s)$ is monotonically decreasing. It is

$$\begin{aligned} L_3(s)' = (v(V^{-1}(s))^{\frac{d}{d-1}})' &= \frac{d}{d-1} v(V^{-1}(s))^{\frac{1}{d-1}} \frac{v(V^{-1}(s))'}{v(V^{-1}(s))} \\ &= c_d \frac{v'(V^{-1}(s))}{v(V^{-1}(s))^{\frac{d-2}{d-1}}}. \end{aligned}$$

Since $V^{-1}(s)$ is monotonically increasing in s, we set $v(V^{-1}(s)) = v(x)$. As a consequence of the general Brunn-Minkowski theorem (see [37]) it holds that $v(x)^{\frac{1}{d-1}}$ is a concave function on $[0,1]$. This implies that the first derivative of $v(x)$ is monotonically decreasing in x. According to that, we can say that

$$(v(x)^{\frac{1}{d-1}})' = \frac{1}{d-1} v(x)^{\frac{2-d}{d-1}} v'(x) = c_d \frac{v'(x)}{v(x)^{\frac{d-2}{d-1}}}$$

is monotonically decreasing. And for this reason,

$$L_3(s)' = c_d \frac{v'(V^{-1}(s))}{v(V^{-1}(s))^{\frac{d-2}{d-1}}}$$

is monotonically decreasing and $L_3(s)$ is concave. In this way we get the form

$$\mathbb{E}f_{d-1} = \binom{n}{d}(d-1)! \int_0^1 (1-s)^{n-d} L_3(s)^{d-1} \mathrm{d}s$$

with $L_3(s)$ concave on $[0,1]$ and can go on completing the proof as in the one of Theorem 5.3.1. □

5.6. Poisson Polytopes in a Ball

Our last issue is to modify the situation from Section 5.5, but with a Poisson-distributed number of generating points, as in Chapter 5.4.

Theorem 5.6.1. *Let $\Pi_t^* = [X_1, \ldots, X_N]$ be a random polytope generated by the convex hull of N random points chosen according to the d-dimensional uniform distribution from a d-dimensional ball B_d with volume 1, where the number of points N is Poisson distributed with parameter t. Then*

$$\mathbb{E}f_{d-1}(\Pi_t^*) \text{ is monotonically increasing in } t.$$

Proof of Theorem 5.6. The proof follows the same idea as the one from Theorem 5.5.1. We describe the number of facets by (see (5.4) and (5.5))

$$\begin{aligned}
\mathbb{E}f_{d-1}(\Pi_t^*) &= \mathbb{E} \sum_{(x_1,\ldots,x_d)\in\eta_{\neq}^d} \mathbb{1}([x_1,\ldots,x_d] \text{ is a facet}) \\
&= c_d\, t^d \int_{-1}^{1} e^{-tV} v(h)^{d+1} \mathrm{d}h \\
&= c_d\, t^d \int_0^t e^{-ts} v(V^{-1}(s))^d \mathrm{d}s,
\end{aligned}$$

where V and $v(h)$ are defined as before. Setting $L_4(s) = v(V^{-1}(s))^{\frac{d}{d-1}} \mathbb{1}_{[0,1]}(s)$, we obtain

$$\mathbb{E} f_{d-1}(\Pi_t^*) \quad = \quad c_d \, t^d \int\limits_0^1 e^{-ts} L_4(s)^{d-1} \mathrm{d}s$$

and go on completing the proof as explained above. □

In all four cases we could show that the expected number of facets increases if the number of generating points, respectively, the intensity of the generating Poisson process in Section 5.4 and the latter case, increases.

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Symbols

Keywords