Graded Rings and Hilbert Functions

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zur Erlangung des Grades eines

Doktors der Naturwissenschaften

Vorgelegt von

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Overview

The present cumulative thesis is based on two articles in the area of graded commutative algebra.

The topic of paper [A], entitled *A Note on the Dimension Theory of $\mathbb{Z}^n$–graded Rings*, is a generalization of an important elementary result on polynomial rings to arbitrary $\mathbb{Z}^n$–graded rings. The article appeared in volume 37 of the journal `Communications in Algebra`.

The paper [B], entitled *Remarks on Hilbert series of graded modules over polynomial rings* is about a general result on certain formal Laurent series and some applications of this result to Hilbert series of modules over standard–$\mathbb{Z}$–graded polynomial rings. The article appeared in volume 132 of the journal `manuscripta mathematica`.

The thesis has two parts: In the second part preprint versions of the articles [A] and [B] are reproduced. The first part consists of three supplementary sections: The initial section outlines the relevant concepts and terminology; it is mainly addressed to readers less familiar with graded commutative algebra. The following sections give a detailed summary, some additional explanantion, and also a brief outlook on possible objects of further research for each of the articles.

Part I

Summary of the Articles

Review of Basic Facts

All rings considered in the sequel are commutative and have an identity element.

Graded Rings

A $\mathbb{Z}^n$–graded ring is a ring R together with a decomposition $R = \bigoplus_{a\in\mathbb{Z}^n} R_a$ of the underlying additive group such that $R_a \cdot R_b \subseteq R_{a+b}$ holds for any $a, b \in \mathbb{Z}^n$. The elements of R_a are called homogeneous of degree a. The zero element is the only element that is homogeneous of any degree. It is easy to show that the identity element of R has to be homogeneous of degree 0, hence R_0 is a proper subring of R, and any R_a is an R_0–module. For any $x \in R$ there is a unique presentation $x = \sum_{a\in\mathbb{Z}^n} x_a$; the summands ($\neq 0$) are called the homogeneous components of x. This term is also used to denote the subgroups R_a, the homogeneous components of R.

A graded ideal is an ideal $I \subseteq R$ with the property that for any $x \in I$ all homogeneous components of x belong to I as well; this is equivalent with I being generated by homogeneous elements. The quotient ring R/I is then also graded in a natural way. For an arbitrary ideal I the graded core I^* is defined as the ideal generated by all homogeneous elements of I. It is the largest graded ideal contained in I.

For a graded ideal I to be prime it is enough that the implication $xy \in I \Rightarrow x \in I \vee y \in I$ holds for homogeneous $x, y \in R$ – so there is no need to introduce the notion of a *prime ideal[1]. As a consequence the graded core of an arbitrary prime ideal in R is again a prime ideal.

A *maximal ideal is a graded ideal not contained in another proper graded ideal. Any graded ideal except R is contained in a *maximal ideal, as can be shown by the usual proof involving Zorn's Lemma. The quotient ring of a $\mathbb{Z}^n$–graded by a *maximal ideal is a *simple ring, i. e. a graded ring not containing other graded ideals than (0) and R. By a result of Goto and Watanabe, see [7, Thm. 1.1.4], such a ring is either a field or isomorphic to a Laurent polynomial ring $\mathbb{F}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]$ over a field $\mathbb{F}$ in $m \leq n$ indeterminates. The latter is impossible if R is $\mathbb{N}^n$–graded, i. e. if $R_a = 0$ for all $a \in \mathbb{Z}^n \setminus \mathbb{N}^n$. Hence, in this case (or, more generally, if $R_a \neq 0$ implies $R_{-a} = 0$ for all $a \neq 0$) the *maximal ideals are also maximal in the usual sense, while in general a *maximal ideal only needs to be prime.

The graded version of localization is defined the natural way: The set S of homogeneous elements not contained in the prime ideal $\mathfrak{p}$ is multiplicatively closed. The ring $S^{-1}R$, denoted by $R_{(\mathfrak{p})}$, is called the homogeneous localization at $\mathfrak{p}$. It is $\mathbb{Z}^n$–graded by setting $\deg(r/s) = \deg(r) - \deg(s)$ for any homogeneous $r \in R$, and it is *local, i. e. it has a unique *maximal ideal – every proper graded ideal of $R_{(\mathfrak{p})}$ is contained in $\mathfrak{p}^* R_{(\mathfrak{p})}$.

Given a group homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^k$, one may view a $\mathbb{Z}^n$–graded ring R as $\mathbb{Z}^k$–graded by defining $\bigoplus_{a\in\varphi^{-1}(b)} R_a$ to be the homogeneous component of degree $b \in \mathbb{Z}^k$ in this new grading. This is called the φ–induced grading of R. The graded core of an

[1] This is a feature of gradings by torsion–free groups, while in general prime and *prime ideals are different concepts.

ideal $I \subseteq R$ with respect to the φ–induced grading will be denoted by $I^{*\varphi}$. The most important case is $\varphi = p_i$, the projection onto the ith factor of $\mathbb{Z}^n$. Let $U_1 \oplus U_2$ be a direct–sum decomposition of $\mathbb{Z}^n$ with projections $\pi_i : \mathbb{Z}^n \to U_i$, then an element of R is $\mathbb{Z}^n$–graded if and only if it is p_1– and p_2–graded, and hence $(I^{*\pi_1})^{*\pi_2} = I^* = (I^{*\pi_2})^{*\pi_1}$ for any ideal $I \subseteq R$. In particular an ideal of R is $\mathbb{Z}^n$–graded if and only if it is graded with respect to all factors of $\mathbb{Z}^n$.

Graded Modules

A $\mathbb{Z}^n$–graded module over a $\mathbb{Z}^n$–graded ring R is an R–module M together with a decomposition $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$ of its additive group such that $R_a \cdot M_b \subseteq M_{a+b}$ holds for any $a, b \in \mathbb{Z}^n$. The notion of a homogeneous element or component is used as for graded rings, and a graded submodule of a graded module M is a submodule N with the property that for any $x \in N$ all homogeneous components of x belong to N as well.

The annihilator of a graded module M is a graded ideal, and so are the associated prime ideals, which turn out to be annihilators of homogeneous elements of M. Further a prime ideal $\mathfrak{p}$ is contained in $\operatorname{Supp}(M)$ if and only if $\mathfrak{p}^* \in \operatorname{Supp}(M)$.

The grading of a module can be changed by the technique of shifting: For any $\mathbb{Z}^n$–graded R–module M and any $u \in \mathbb{Z}^n$ the shifted module $M(u)$ is defined as the graded R–module with component $\big(M(u)\big)_a = M_{a+u}$ for all $a \in \mathbb{Z}^n$. Since M and $M(u)$ are isomorphic as ungraded objects, properties of M not depending on the grading are not affected by the shifting.

The depth of a finitely generated graded module M over a Noetherian *local $\mathbb{Z}$–graded ring R with *maximal ideal $\mathfrak{m}$ is defined as the maximal length of an M–regular sequence in $\mathfrak{m}$, i. e. the grade of $\mathfrak{m}$ on M, and denoted by $\operatorname{depth}(M)$ rather than $\operatorname{grade}(M, \mathfrak{m})$. This deviation from the standard terminology, where "depth" is used exclusively in the context of true local rings, may be justified by the fact that $\operatorname{grade}(M, \mathfrak{m})$ agrees with $\operatorname{depth}(M_{\mathfrak{m}})$, see [3, Prop. 1.5.15]. If R is positively graded, then the M–regular sequence can be composed of homogeneous elements. These can be chosen of degree 1 provided that furthermore the local ring R_0 has an infinite residue field and $\mathfrak{m}$ is generated by elements of degree 1.

Formal Power Series and Integer Laurent Functions

The ring of formal power series $\mathbb{Z}[[t]]$ is an overring of $\mathbb{Z}[t]$ and can be introduced formally in the same way – instead of the direct sum one considers the direct product (of abelian groups) $\prod_{n=0}^{\infty} \mathbb{Z}$ equipped with a multiplication defined by

$$(a_n)_{n \in \mathbb{N}} \cdot (b_n)_{n \in \mathbb{N}} := \left(\sum_{i=0}^{n} a_i b_{n-i} \right)_{n \in \mathbb{N}} .$$

As with the polynomial ring, one writes $\sum_{n=0}^{\infty} a_n t^n$ rather than $(a_n)_{n \in \mathbb{N}}$. The adjective *formal* emphasizes that the elements of $\mathbb{Z}[[t]]$ are *not* power series in the sense of complex analysis: The indeterminate t is not meant to be replaced by a ring element, and so no questions of convergence arise. In particular an equation as

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$$

is just a way to express the fact that $(1-t)\cdot\sum_{n\geq 0} t^n = 1$ in $\mathbb{Z}[[t]]$ – there is no "radius of convergence" to be regarded.

The localization of $\mathbb{Z}[[t]]$ at the multiplicative set $\{t^n \mid n \in \mathbb{N}\}$ is called the ring of formal Laurent series. Its elements can be written as formal series $\sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n = 0$ for $n \ll 0$, so this ring is denoted by $\mathbb{Z}[[t]][t^{-1}]$.

An integer Laurent function is a function $f : \mathbb{Z} \to \mathbb{Z}$ with $f(n) = 0$ for $n \ll 0$. The formal Laurent series associated to f is the series $H_f(t) := \sum_{n\in\mathbb{Z}} f(n)t^n$.

The (first) difference function Δf of an integer Laurent function f is defined by

$$\Delta f(n) := f(n) - f(n-1) \quad \text{for all} \quad n \in \mathbb{Z},$$

it is obviously an integer Laurent function, too. The series associated to Δf can be computed by

$$H_{\Delta f}(t) = \sum_{n\in\mathbb{Z}} \Delta f(n)t^n = \sum_{n\in\mathbb{Z}} \big(f(n) - f(n-1)\big)t^n = (1-t)\sum_{n\in\mathbb{Z}} f(n)t^n = (1-t)H_f(t).$$

A function $f : \mathbb{Z} \to \mathbb{Z}$ is of polynomial type d if there exists a polynomial $P_f \in \mathbb{Q}[t]$ of degree d such that $f(n) = P_f(n)$ for $n \gg 0$ – the polynomial P_f is uniquely determined. An integer Laurent function $f \neq 0$ is of polynomial type d if and only if Δf is of polynomial type $d-1$.

1 Paper [A] – Dimension Theory of $\mathbb{Z}^n$–graded Rings

Dimension theory is a main part of commutative algebra, and a standard topic in this area is the dimension of a polynomial ring $R[X_1, \ldots, X_n]$ and its relationship with the dimension of the ground ring. If R is Noetherian, a classical theorem of Krull, see [11, Satz 13], gives the equality

$$\dim\left(R[X_1, \ldots, X_n]\right) = \dim(R) + n. \tag{1}$$

The same holds if R is a Prüfer domain, but in the general case we just have the estimate

$$\dim(R) + n \leq \dim\left(R[X_1, \ldots, X_n]\right) \leq (n+1) \cdot \dim(R) + n. \tag{2}$$

Seidenberg showed in [17] that in the special cases of $n = 1$ or $\dim(R) = 1$ the dimension of the polynomial ring may in fact take any value inside this range. For illustration we recall a vintage example due to Krull.

Example 1.1 *(cf. [12, p. 670], [16, Thm. 4])*
Let k be an algebraically closed field, and let $\mathcal{R}$ denote the subring of $\mathsf{k}(X,Y)$ consisting of those $r \in \mathsf{k}(X,Y)$ that may be written in the form f/g with $f, g \in \mathsf{k}[X,Y]$ where g is not divisible by X, and satisfying $r(0,Y) = c$ for some $c \in \mathsf{k}$. Then $\dim(\mathcal{R}) = 1$ and $\dim(\mathcal{R}[\mathbf{T}]) = 3$

Proof: The ideal $\mathfrak{m} := \mathcal{R}X$ is maximal, since $\mathcal{R}/\mathfrak{m} \cong \mathsf{k}(Y)$, and the elements outside of $\mathfrak{m}$ are units – let $f/g \notin \mathfrak{m}$ be reduced to lowest terms, then f is not divisible by X and hence $g/f \in \mathcal{R}$ as well. Any non–zero $r \in \mathfrak{m}$ may be written in the form $r = X^n f/g$ with some $f, g \in \mathsf{k}[X,Y]$ not divisible by X. Since f/g is a unit in $\mathcal{R}$, the nth power of any $s \in \mathfrak{m}$ is divisible by r. This implies that any non–zero proper ideal of $\mathcal{R}$ has radical $\mathfrak{m}$; hence there are no prime ideals except (0) and $\mathfrak{m}$. But the extension of $\mathfrak{m}$ to the polynomial ring $\mathcal{R}[\mathbf{T}]$ has height 2, since the ideal $\mathfrak{p} := \{h \in \mathcal{R}[\mathbf{T}] \mid h(Y) = 0\}$ appears: It is obviously prime, and it is distinct from (0) and $\mathfrak{m}[\mathbf{T}]$ because of $X\mathbf{T} - XY \in \mathfrak{p}$ and $X\mathbf{T} \in \mathfrak{m}[\mathbf{T}] \setminus \mathfrak{p}$. □

Important objects for the dimension theory of $R[X_1, \ldots, X_n]$ are those prime ideals which are extensions of prime ideals $\mathfrak{q} \subset R$ to the polynomial ring. For any prime ideal $\mathfrak{p}$ in $R[X_1, \ldots, X_n]$ its re–extended contraction $\mathfrak{p}^{ce} := (\mathfrak{p} \cap R)R[X_1, \ldots, X_n]$ is the largest of these extended prime ideals contained in $\mathfrak{p}$, cf. [1, Prop. 1.17].

An elementary result in this context was remarkably not stated before 1972, when Brewer, Montgomery, Rutter and Heinzer published their article [2].

Theorem 1.2 *(Theorem 1 in [2])*
Let $\mathfrak{p}$ be a prime ideal in $R[X_1, \ldots, X_n]$, then

$$\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}^{ce}) + \operatorname{ht}\left(\mathfrak{p}/\mathfrak{p}^{ce}\right). \tag{3}$$

This statement allows to simplify the proofs of some further results in the dimension theory of polynomial rings:

For example it yields the fundamental fact

$$\dim\left(\mathbb{F}[X_1,\ldots,X_n]\right) = n$$

for any field $\mathbb{F}$ without invoking more advanced tools as Krull's Principal Ideal Theorem.

Polynomial rings $R[X_1,\ldots,X_n]$ are the prototypes of $\mathbb{Z}^n$–graded rings, and therefore it is not surprising that several statements on polynomial rings have analogues for general $\mathbb{Z}^n$–graded rings. Such an analogue of [2, Thm. 1] is the main result of paper [A], *A Note on the Dimension Theory of $\mathbb{Z}^n$–graded Rings*.

1.1 Motivation and Main Result

In the dimension theory of a general $\mathbb{Z}^n$–graded ring R the graded prime ideals play the same distinguished rôle as the extended prime ideals in the polynomial case. There is also a counterpart of the re–extended contraction: For any prime ideal $\mathfrak{p} \subset R$ the *graded core* $\mathfrak{p}^*$ is also prime, and so it is the unique largest graded prime ideal contained in $\mathfrak{p}$. Furthermore the inequality $\mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^*) \leq n$ follows in the same way as $\mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^{ce}) \leq n$ in the polynomial case – consider the homogeneous localization of $R/\mathfrak{p}^*$ at the ideal $\mathfrak{p}/\mathfrak{p}^*$.

Therefore the following theorem, the above–mentioned main result of paper [A], may in fact be considered as the graded–rings analogue of Theorem 1.2.

Theorem 1.3 *(Theorem 2.1 in [A])*
Let $\mathfrak{p}$ be a prime ideal in a $\mathbb{Z}^n$–graded ring. Then

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + \mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^*)\,. \tag{4}$$

We point out that in one special case (4) is just equation (3) rewritten: Consider the polynomial ring $R[X_1,\ldots,X_n]$ equipped with the standard $\mathbb{Z}^n$–grading. Here any graded prime ideal is of the form $\mathfrak{q}[X_1,\ldots,X_n]+I$, where $\mathfrak{q}$ is a prime ideal of R, and I is generated by a subset of $\{X_1,\ldots,X_n\}$. For an arbitrary prime ideal $\mathfrak{p} \subset R[X_1,\ldots,X_n]$ this implies

$$\mathfrak{p}^* = \mathfrak{p}^{ce} + \sum_{X_i \in \mathfrak{p}} (X_i)\,,$$

so if $\mathfrak{p}$ does not contain an indeterminate, then $\mathfrak{p}^*$ is exactly $\mathfrak{p}^{ce}$.

In the main case, that is $\mathfrak{p}$ being a non–graded prime ideal of finite height, the proof of Theorem 1.3 follows the same pattern as the proof of Theorem 1.2 given in [2], using two–fold induction: In the first step the assertion is shown for $\mathbb{Z}$–graded rings (as previously for polynomial rings in one variable) by induction on the height of the prime ideal. The general case is then treated by induction on n, regarding $\mathbb{Z}^{n+1}$ as $\mathbb{Z}^n \oplus \mathbb{Z}$ (as before by regarding $R[X_1,\ldots,X_n,X_{n+1}]$ as $R[X_1,\ldots,X_n][X_{n+1}]$). In both cases the proof depends mainly on one crucial argument: In the polynomial case this is the fact that in $R[X]$ any prime ideal is either the extension of a prime ideal of R or lies immediately above such a prime ideal, see [13, Satz 11], and for prime ideals in an arbitrary $\mathbb{Z}$–graded ring the same statement holds with "the extension of a prime ideal of R" replaced by "graded", cf. [3, Thm. 1.5.8].

An immediate consequence of Theorem 1.3 is a result on the dimension of modules:

Corollary 1.4 *(Cor. 2.2 in [A])*
Let M be a finitely generated $\mathbb{Z}^n$–graded module over a $\mathbb{Z}^n$–graded ring R, and let $\mathfrak{p}$ be a prime ideal in $\operatorname{Supp} M$. *Then*

$$\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + \operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) .$$

For a Noetherian R this was already proven by Goto and Watanabe, see [7, Prop. 1.2.2], in a different and more complicated way.

1.2 Graded Dimension

The graded dimension or *dimension of a $\mathbb{Z}^n$–graded ring R is defined by rephrasing the basic definitions of dimension theory in terms of graded prime ideals: The *height of a graded prime ideal $\mathfrak{p}$, denoted by $^*\mathrm{ht}(\mathfrak{p})$, is the supremum of the lengths of descending chains

$$\mathfrak{p} = \mathfrak{p}_r \supset \mathfrak{p}_{r-1} \supset \ldots \supset \mathfrak{p}_1 \supset \mathfrak{p}_0$$

of graded prime ideals, and the *dimension of R, denoted by $^*\dim(R)$, is the supremum of the *heights of all graded prime ideals of R.

The general estimate $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) \leq n$ implies

$$^*\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}) \leq (n+1)\,^*\mathrm{ht}(\mathfrak{p}).$$

for any graded prime ideal (Lemma 1.2 and Cor. 1.3 in [A]). Combining these results with Theorem 1.3 yields

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + \mathrm{ht}(\mathfrak{p}/\mathfrak{p}^*) \leq \mathrm{ht}(\mathfrak{p}^*) + n \leq (n+1)\,^*\mathrm{ht}(\mathfrak{p}^*) + n \leq (n+1)\,^*\dim(R) + n \quad (5)$$

for any prime ideal $\mathfrak{p}$ of R, and hence we get bounds for the graded dimension of R in terms of $\dim(R)$.

Corollary 1.5 *(Cor. 2.3 in [A])*
Let R be a $\mathbb{Z}^n$–graded ring, then

$$^*\dim(R) \leq \dim(R) \leq (n+1)\,^*\dim(R) + n. \quad (6)$$

Note the difference between this result and the corresponding inequality (2) of the polynomial case: The *dimension of R may equal $\dim(R)$, as the simple example of a standard–graded polynomial ring over a field shows. To illustrate the opposite extreme we reconsider Krull's example: Let $\mathcal{R}[\mathbf{T}]$ be equipped with the usual $\mathbb{Z}$–grading, i. e. $\deg(\mathbf{T}) := 1$. There are just three graded prime ideals $(0) \subset \mathfrak{m}[\mathbf{T}] \subset \mathfrak{m}[\mathbf{T}] + (\mathbf{T})$, so $^*\dim(\mathcal{R}[\mathbf{T}]) = 2$. Inverting of $\mathbf{T}$ shortens the chain of graded prime ideals by the last link, therefore the Laurent polynomial ring $\mathcal{R}[\mathbf{T}, \mathbf{T}^{-1}]$ has *dimension 1, while $\dim(\mathcal{R}[\mathbf{T}, \mathbf{T}^{-1}]) = \dim(\mathcal{R}[\mathbf{T}]) = 3$.

Similar to the polynomial case, the inequality (6) can be improved for certain rings. If the *height equals the usual height for any graded prime ideal in R, then the factor $(n+1)$ in (5) may be dropped, leading to a stronger estimate:

Corollary 1.6 *(Cor. 2.4 in [A])*
Let R be a $\mathbb{Z}^n$–graded ring such that ${}^*\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ *holds for any graded prime ideal* $\mathfrak{p} \subset R$. *Then*

$$ {}^*\dim(R) \leq \dim(R) \leq {}^*\dim(R) + n. $$

Perhaps not unexpectedly, this additional prerequisite is fulfilled for the same classes of rings already mentioned above as distinguished in the polynomial case:

Lemma 1.7
Let R be a $\mathbb{Z}^n$–graded ring. If R is Noetherian or Prüfer, then ${}^*\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ *holds for any graded prime ideal* $\mathfrak{p} \subset R$.

The proof is similar to that of the corresponding condition

$$ \mathrm{ht}_{R[X_1,\ldots,X_n]}\left(\mathfrak{p}[X_1,\ldots,X_n]\right) = \mathrm{ht}_R(\mathfrak{p}) $$

in the polynomial case; for a Noetherian ring Krull's Principal Ideal Theorem provides the essential argument, while in a Prüfer ring the invertibility of every graded ideal forces a prime ideal contained in a graded prime ideal to be graded itself.

Even in this special case the dimension may exceed the *dimension, since there might be no graded ideal of height $\dim(R)$ – the easiest example is a Laurent polynomial ring $\mathbb{F}[X, X^{-1}]$ over a field. Another additional condition on R assures the existence of such an ideal, allowing to improve inequality (6) in a different way[2]:

Corollary 1.8
Let R be a $\mathbb{Z}^n$–graded ring such that every *maximal ideal of R is maximal in the usual sense. Then

$$ {}^*\dim(R) \leq \dim(R) \leq (n+1)\,{}^*\dim(R). $$

PROOF: If $\dim(R) = \infty$, then ${}^*\dim(R)$ is also infinite. Let therefore $\mathfrak{m}$ be a maximal ideal with $\mathrm{ht}(\mathfrak{m}) = \dim(R) < \infty$. We show that there exists a graded ideal with the same height as $\mathfrak{m}$. By induction it is enough to treat the case $n = 1$: If $\mathfrak{m}$ is not graded, then $\mathfrak{m}^* \subset \mathfrak{m}$ is not *maximal by the assumption on R. Therefore it is contained in a *maximal ideal $\mathfrak{n} \supset \mathfrak{m}^*$. By Theorem 1.3 we have

$$ \mathrm{ht}(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m}^*) + 1 \leq \mathrm{ht}(\mathfrak{n}) \leq \dim(R) = \mathrm{ht}(\mathfrak{m}), $$

so the graded ideal $\mathfrak{n}$ also has height $\dim(R)$. □

This condition on R is satisfied in particular for $\mathbb{N}^n$–graded rings such as polynomial rings equipped with the usual $\mathbb{Z}^n$–grading. If R is also Noetherian or Prüfer, then we get

$$ \dim(R) = {}^*\dim(R), $$

the analogue of equation (1).

[2] This result was not mentioned in [A], since it depends just on the simplest case of Theorem 1.3.

1.3 Jaffard's Special Chain Theorem

The notion of a *special chain* of prime ideals was introduced by Jaffard in his monograph [14] on dimension theory of polynomial rings. A chain $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in $R[X_1, \ldots, X_n]$ is called a special chain, if for every $i = 0, \ldots, r$ the re–extended contraction $(\mathfrak{p}_i \cap R)R[X_1, \ldots, X_n]$ also appears in this chain. The main result in this context, known as Jaffard's Special Chain Theorem, see [14], Chapitre II, Théorème 3, says that in a finite–dimensional polynomial ring the dimension can be realized as the length of a special chain. This follows easily from Theorem 1.2, while the original proof given in [14] is rather complicated.

Again using the graded core as a substitute for the re–extended contraction, the notion of a special chain may be adapted to the setting of general $\mathbb{Z}^n$–graded rings:

Definition 1.9
A chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in a $\mathbb{Z}^n$–graded ring is called a special chain*, if for any $i = 0, \ldots, r$ there is a $j \leq i$ with $\mathfrak{p}_i^* = \mathfrak{p}_j$; i. e. the graded cores of all members also appear in this chain.*

With this definition Jaffard's Special Chain Theorem in the version [2, Corollary 3] can be adopted verbatim:

Theorem 1.10 *(Graded analogue of JSCT; Theorem 2.7 in [A])*
Let $\mathfrak{p}$ be a prime ideal of height d in a $\mathbb{Z}^n$–graded ring R. Then there is a special chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$$

of length d in R. In particular, if R is finite–dimensional, then $\dim R$ *can be realized as the length of a special chain of prime ideals of R.*

This follows easily from Theorem 1.3 by induction on d, but, as one checks in an instant, the reverse implication also holds. Hence these results are equivalent.

Theorem 1.3 also leads to a related result on the height of a prime ideal over its graded core, which may be viewed as a relative version of Jaffard's Theorem. This result involves the $\mathbb{Z}$–gradings of R that can be obtained from the original grading just by regarding only a single factor of $\mathbb{Z}^n$. The graded core with respect to the grading by the ith factor is denoted by ^{*p_i}. Using this terminology we introduce a notion for another specific sort of prime ideal chains.

Definition 1.11
A chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in a $\mathbb{Z}^n$–graded ring is called a ladder*, if for any $i = 1, \ldots, r$ there is a $j_i \in \{1, \ldots, n\}$ with $\mathfrak{p}_i^{*p_{j_i}} = \mathfrak{p}_{i-1}$.*

Induction on $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*)$ leads to the announced result:

Theorem 1.12 *(Theorem 3.3 in [A])*
Let R be a $\mathbb{Z}^n$–graded ring and $\mathfrak{p} \subset R$ a prime ideal. Then there exists a ladder

$$\mathfrak{p}^* = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$$

of length d if and only if $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) = d$.

Note that the essential content of Theorem 1.12 is *not* the mere existence of a ladder between $\mathfrak{p}$ and $\mathfrak{p}^*$, but the fact that any such ladder is of the maximal achievable length $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*)$.

Theorem 1.12 can be useful for extending assertions related to a prime ideal and its graded core from the $\mathbb{Z}$–graded to the $\mathbb{Z}^n$–graded case by means of induction. An important theorem in the range of homological invariants of graded modules provides a good example for this method:

Let R be an arbitrary Noetherian ring, M an R–module, and $\mathfrak{p} \subset R$ a prime ideal. For any integer $i \geq 0$ the ith Bass number of M at $\mathfrak{p}$ is defined as

$$\mu_i(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}} \left(k(\mathfrak{p}), M_\mathfrak{p}\right),$$

here $k(\mathfrak{p})$ denotes the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

Theorem 1.13 *([7, Thm. 1.2.3], Theorem 3.3 in [A])*
Let R be a Noetherian $\mathbb{Z}^n$–graded ring, M a $\mathbb{Z}^n$–graded R–module, and $\mathfrak{p}$ a non–graded prime ideal of R with $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) = d$. Then $\mu_{i+d}(\mathfrak{p}, M) = \mu_i(\mathfrak{p}^*, M)$ for every integer $i \geq 0$, and $\mu_i(\mathfrak{p}, M) = 0$ for $0 \leq i < d$.

This follows easily by repeated application of the well–known $\mathbb{Z}$–graded version of this theorem, [6, Thm. 1.12], to the ideals in a ladder between $\mathfrak{p}$ and $\mathfrak{p}^*$. In combination with Corollary 1.4 we get the $\mathbb{Z}^n$–version of [3, Thm. 1.5.9].

Corollary 1.14 *(Cor. 3.6 in [A])*
Let R be a Noetherian $\mathbb{Z}^n$–graded ring, M a finitely generated $\mathbb{Z}^n$–graded R–module, and $\mathfrak{p} \subset R$ a prime ideal. Then $M_\mathfrak{p}$ is Cohen–Macaulay if and only if $M_{\mathfrak{p}^*}$ is.

By the characterization of Gorenstein rings (see [3], Thm. 3.2.10) this yields another interesting result:

Corollary 1.15 *(Cor. 3.7 in [A])*
Let R be a Noetherian $\mathbb{Z}^n$–graded ring and $\mathfrak{p} \subset R$ a prime ideal. Then $R_\mathfrak{p}$ is Gorenstein if and only if $R_{\mathfrak{p}^*}$ is.

Addendum: Further Questions

One may ask whether there are analogues of Theorem 1.3 for rings graded by other abelian groups. The methods of [A] intrinsically used that the grading group is finitely generated and torsion–free, and any group with these properties is isomorphic to some group $\mathbb{Z}^n$ by the Fundamental Theorem on Finitely Generated Abelian Groups. Therefore one cannot expect such an extension of Theorem 1.3 to be straightforward. Retaining one of the essential conditions on the grading group G leads to the following situations.

i) The group G is torsion–free but not finitely generated, e. g. $G = \mathbb{Q}$. Here the graded core $\mathfrak{p}^*$ of any prime ideal $\mathfrak{p}$ is still prime, but there is no general upper bound for $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*)$ and, of course, no rank of the group available for an induction. In the case of a prime ideal of finite height and G being free abelian there might be a reduction to the $\mathbb{Z}^n$–case by considering a suitable subgroup of G, but if indeed a result could be obtained in the general case, it will probably require a completely new idea.

ii) If G is finitely generated but not torsion–free, then induction on the number of generators may come in handy, but there is a major obstacle: In general the graded core of a prime ideal $\mathfrak{p}$ is not a true prime ideal, but just a *prime ideal, i. e. the implication $xy \in \mathfrak{p}^* \Rightarrow x \in \mathfrak{p}^* \vee y \in \mathfrak{p}^*$ only holds for *homogeneous* elements x, y. Therefore an analogue of Theorem 1.3 could perhaps only be given in the setting of a *graded dimension theory*, i. e. a dimension theory in terms of *prime ideals. Such a theory was developed by Kamoi in [10] mainly for Noetherian (more precisely *Noetherian) G–graded rings.

Another, less concrete, question could be whether there are other general–graded–rings counterparts of familiar results on polynomial rings to be discovered, but a systematic investigation of this problem might be "Too Far Afield".

2 Paper [B] – Hilbert Series of Modules Over Polynomial Rings

The most important $\mathbb{Z}$–graded rings are the homogeneous graded affine $\mathbb{F}$–algebras: rings of the form $S = R/I$ with $\mathbb{F}$ a field and I a graded ideal of the polynomial ring $R = \mathbb{F}[X_1, \ldots, X_n]$ equipped with the standard $\mathbb{Z}$–grading, i. e., $\deg(X_i) = 1$ for $i = 1, \ldots, n$.

Let M be a $\mathbb{Z}$–graded module over such an algebra S, then the homogeneous components of M are $\mathbb{F}$–vectorspaces. In the most interesting case of a finitely generated module M these vectorspaces are finite–dimensional, and, since S is positively graded, $M_k = 0$ for $k \ll 0$. Hence we have an integer Laurent function

$$H(M, -) : \mathbb{Z} \to \mathbb{Z},\ k \mapsto \dim_{\mathbb{F}} (M_k)$$

with the associated formal Laurent series

$$H_M(t) = \sum_{k\in\mathbb{Z}} H(M,k)t^k = \sum_{k\in\mathbb{Z}} (\dim_{\mathbb{F}} M_k)\, t^k \in \mathbb{Z}[[t]][t^{-1}], \tag{7}$$

called the Hilbert function resp. the Hilbert series of M. By a fundamental result due to Hilbert, the function $H(M, -)$ is of *polynomial type*:

Theorem 2.1 *(Hilbert, [8])*
Let M be a finitely generated $\mathbb{Z}$–graded module of Krull dimension d over a homogeneous graded affine $\mathbb{F}$–algebra, then there exists a polynomial $P_M \in \mathbb{Q}[t]$ of degree $d - 1$ such that $H(M, k) = P_M(k)$ for $k \gg 0$.

Therefore the following well–known result in the theory of *generating functions* applies to the Hilbert series.

Theorem 2.2 *(cf. [3, Lemma 4.1.7], [9, Thm. 5.2.10])*
Let f be non–zero integer Laurent function with associated formal Laurent series H_f. The following conditions are equivalent:

i) The function f is of polynomial type d.

ii) There exists $Q \in \mathbb{Z}[t, t^{-1}]$ with $Q(1) \neq 0$ and $H_f(t) = \frac{Q(t)}{(1-t)^{d+1}}$.

The Hilbert series of $M \neq 0$ may therefore be written in the form

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d} \tag{8}$$

where $Q_M \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial with $Q(1) \neq 0$, and $d = \deg(P_M) + 1$ equals the Krull dimension of M. Following Stanley, see [19], the vector of coefficients of Q_M is denoted the h–vector of M.

The central result of article [B], *Remarks on Hilbert series of graded modules over polynomial rings*, is a kind of refinement of Theorem 2.2 for integer Laurent functions of polynomial type with non–negative values. This result has several consequences for Hilbert series of finitely generated graded modules over *polynomial* rings, namely a characterization of possible Hilbert functions and h–vectors and an arithmetic description of the maximal depth of a module with a specific Hilbert series.

2.1 Motivation and Main Result

The basic examples of homogeneous graded affine $\mathbb{F}$–algebras are the standard–graded polynomial rings themselves, and they provide the simplest Hilbert series: Let $R = \mathbb{F}[X_1, \ldots, X_n]$, then $H(R, k)$ is the number of monomials of total degree k in n variables; this number equals $\binom{n+k-1}{n-1}$, and therefore

$$H_R(t) = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} t^k = \frac{1}{(1-t)^n}.$$

This implies that any series $t^i/(1-t)^m$ with $m \leq n$ occurs as Hilbert series of a finitely generated graded R–module: let $M := R/\left(X_{m+1}, \ldots, X_n\right) \cong \mathbb{F}[X_1, \ldots, X_m]$, then the *shifted* module $M(-i)$ has Hilbert series

$$H_{M(-i)}(t) = t^i \cdot H_M(t) = \frac{t^i}{(1-t)^m}.$$

The Hilbert series is additive with respect to finite direct sums, hence any fraction of the form $Q(t)/(1-t)^m$, where $Q \in \mathbb{Z}[t, t^{-1}]$ has non-negative coefficients can be realized as the Hilbert series of an R–module. (In the sequel elements of $\mathbb{Z}[[t]][t^{-1}]$ without negative coefficients will be called non–negative for short.) Thus we have:

Lemma 2.3 *(Lemma 2.2 in [B])*
Let $H \neq 0$ be a formal Laurent series such that there is a decomposition

$$H(t) = \sum_{j=r}^{d} \frac{Q_j(t)}{(1-t)^j}$$

with non–negative numerators $Q_j \in \mathbb{Z}[t, t^{-1}]$ and $Q_r \neq 0$. Then there exists a finitely generated graded $\mathbb{F}[X_1, \ldots, X_n]$–module, $n \geq d$, with Hilbert series H and depth r.

This simple observation suggests to investigate which formal Laurent series H admit a decomposition of such a form. Obviously H must be non–negative, and, by Theorem 2.2, its coefficients have to be given by a function of polynomial type. In fact, these necessary conditions turn out to be sufficient as well – this is the announced main result.

Theorem 2.4 *(Theorem 2.1 in [B])*
Let f be an integer Laurent function of polynomial type $d-1$ which takes non–negative values. Then the associated Laurent series H_f admits a presentation

$$H_f(t) = \sum_{j=0}^{d} \frac{Q_j(t)}{(1-t)^j} \quad \text{with non–negative } Q_j \in \mathbb{Z}[t, t^{-1}]. \tag{9}$$

The proof uses induction on d, starting with the trivial case $d = 0$. For $d > 0$ there exists $K \in \mathbb{N}$ such that f is non–decreasing for $k \geq K$. So the series

$$(1-t) \sum_{k=K}^{\infty} f(k) t^k = f(K) t^K + \sum_{k=K+1}^{\infty} \big(f(k) - f(k-1)\big) t^k = f(K) t^K + \sum_{k=K+1}^{\infty} \Delta f(k) t^k$$

is non–negative, and the induction hypothesis applies to it, since the difference function Δf is of polynomial type $d-2$.

As an immediate consequence we get a characterization of Hilbert functions of finitely generated graded modules over standard–graded polynomial rings:

Corollary 2.5 *(Cor. 2.3 in [B])*
Let f be an integer Laurent function with associated Laurent series H_f. The following conditions are equivalent:

a) The function f is of polynomial type and takes non–negative values.

b) The series H_f admits a presentation as in Theorem 2.4.

c) The series H_f is the Hilbert series of a finitely generated graded module over some standard–graded polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$.

d) The function f is the Hilbert function of a finitely generated graded module over some standard–graded polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$.

Note that this result deals with arbitrary finitely generated graded $\mathbb{F}[X_1, \ldots, X_n]$–modules; for modules of the special form $\mathbb{F}[X_1, \ldots, X_n]/I$, i. e. for affine $\mathbb{F}$–algebras, there are far more restrictive conditions by a classical theorem of Macaulay, see [3], Theorem 4.2.10.

In the sequel a Laurent series associated to an integer Laurent function of polynomial type with non–negative values will be called viable for short.

2.2 Hilbert Depth and Positivity

By Hilbert's theorem, the Krull dimension of a finitely generated graded module (over some homogeneous graded $\mathbb{F}$–algebra) is encoded in its Hilbert series – all modules with Hilbert series H share the same Krull dimension; we will denote this number by $\dim(H)$. On the contrary, there is no similar statement on depth, that is, there may be modules over $R = \mathbb{F}[X_1, \ldots, X_n]$ with the same Hilbert series, but different depths. A very simple example is given by the $\mathbb{F}[X]$–modules $\mathbb{F}[X]$ and $\big(\mathbb{F}[X]/(X)\big) \oplus \big(\mathbb{F}[X](-1)\big)$.

The depth of a finitely generated module is bounded above by its dimension, hence for any viable Laurent series H there exists a maximal depth an R–module with Hilbert series H can have. This number

$$\operatorname{Hdep}(H) := \max \left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a f. g. gr. } R\text{–module } N \\ \text{with } H_N = H \text{ and } \operatorname{depth}(N) = r. \end{array} \right\},$$

will be called the Hilbert depth of H. For an R–module M the notation $\operatorname{Hdep}(M)$ is used instead of $\operatorname{Hdep}(H_M)$.

By its definition the Hilbert depth seems to be an opaque quantity, since a vast class of modules has to be taken into account, but Theorem 2.4 allows to prove the equality of $\operatorname{Hdep}(H)$ with an arithmetic invariant of H, the positivity, defined by

$$\operatorname{p}(H) := \max \left\{ r \in \mathbb{N} \mid (1-t)^r H(t) \text{ is non–negative} \right\}.$$

The positivity is well–defined, since $(1-t)^{\dim(H)+1}H(t)$ inevitably has negative coefficients, and it is an upper bound for $\mathrm{Hdep}(H)$ by standard facts about regular sequences. The reverse inequality can be deduced by applying Theorem 2.4 to the viable series $(1-t)^{\mathrm{p}(H)}H(t)$ of dimension $\dim(H)-\mathrm{p}(H)$ and invoking of Lemma 2.3, and so we have the following.

Theorem 2.6 *(Theorem 3.2 in [B])*
Let H be a viable Laurent series, then $\mathrm{Hdep}(H) = \mathrm{p}(H)$.

Together with the additivity of the Hilbert series this result has a consequence for the behaviour of the Hilbert depth with respect to short exact sequences:

Corollary 2.7 *(Cor. 3.3 in [B])*
Let $0 \to U \to M \to N \to 0$ be an exact sequence of finitely generated graded modules over $\mathbb{F}[X_1,\ldots,X_n]$, then $\mathrm{Hdep}(M) \geq \min\{\mathrm{Hdep}(U), \mathrm{Hdep}(N)\}$.

Contrary to the corresponding result on depth, see [3, Prop. 1.2.9], there are no similar estimates for $\mathrm{Hdep}(U)$ and $\mathrm{Hdep}(N)$, as the example of the graded maximal ideal of R shows:

Example 2.8 *(Example 3.4 in [B])*
Let $\mathfrak{m} := (X_1,\ldots,X_n) \subset \mathbb{F}[X_1,\ldots,X_n]$, then $\mathrm{Hdep}(\mathfrak{m}) = \left\lceil \frac{n}{2} \right\rceil$.

For the proof of this statement we consider $(1-t)^r H_{\mathfrak{m}}(t)$ for $r \in \mathbb{N}$. An easy induction yields

$$(1-t)^r H_{\mathfrak{m}}(t) = nt + \sum_{k=2}^{r} \left[\binom{n+k-1-r}{k} + (-1)^{k-1}\binom{r}{k} \right] t^k + \sum_{k=r+1}^{\infty} \binom{n+k-1-r}{k} t^k.$$

The second coefficient is non–negative if and only if $r \leq \left\lceil \frac{n}{2} \right\rceil$, and it is easily shown that this condition is also sufficient for the entire series to be non–negative. The result can be extended to powers of the *maximal ideal: The estimate $\mathrm{Hdep}\left(\mathfrak{m}^s\right) \leq \left\lceil \frac{n}{s+1} \right\rceil$ follows again by a simple inspection of the $(s+1)$th coefficient in $(1-t)^r H_{\mathfrak{m}^s}(t)$, but the proof of the reverse inequality requires sophisticated combinatorial methods, see [5]. Even more difficult to determine is the Hilbert depth of the higher syzygy modules of $R/\mathfrak{m}$. Complicated asymptotic results are given in [4, Chapter 4], and a closed formula can probably not be found, due to the complex structure of the series in question, a so–called ${}_3F_2$*–series*.

2.3 The Numerator of a Hilbert Series

The characterization of Hilbert series of finitely generated graded modules over polynomial rings allows to develop also a characterization of the possible h–vectors, i. e. the Laurent polynomials that may appear as numerators in (8). Corollary 2.5 and Theorem 2.4 yield a first criterion:

Lemma 2.9 *(Lemma 4.1 in [B])*
Let $P \in \mathbb{Z}[t, t^{-1}]$. The following conditions are equivalent:

a) There exists $d \in \mathbb{N}$ such that the Laurent series $\frac{P(t)}{(1-t)^d}$ is non–negative.

b) There exists $d \in \mathbb{N}$ such that there is a finitely generated graded module over a polynomial ring $\mathbb{F}[X_1, \dots, X_n]$ with Hilbert series $\frac{P(t)}{(1-t)^d}$.

The fairly abstract condition a) turns out to be equivalent to a simple property of P regarded as a real valued function of a real variable: By Theorem 2.4 a Laurent polynomial $P \neq 0$ satisfying condition a) admits a presentation

$$P(t) = \sum_{j=0}^{d} Q_j(t) \cdot (1-t)^{d-j}$$

with non–negative $Q_j \in \mathbb{Z}[t, t^{-1}]$, and hence $P(x) > 0$ for all $x \in \,]0, 1[$. The proof of the converse is more elaborate. The basic idea is to factorize P, and therefore it is advisable to consider Laurent polynomials with arbitrary real coefficients as well. For brevity a non–zero $P \in \mathbb{R}[t, t^{-1}]$ satisfying condition a) of Lemma 2.9 will be called convenient.

We show that any $P \in \mathbb{R}[t, t^{-1}]$ taking positive values in $]0, 1[$ is convenient. The following ingredients of the proof are easy:

Lemma 2.10 *(Lemma 4.4 in [B])*
Let $P, Q \in \mathbb{R}[t, t^{-1}]$ be convenient. Then $P \cdot Q$ is also convenient.

Lemma 2.11 *(Lemma 4.5 in [B])*
Let $P = \alpha t + \beta \in \mathbb{R}[t]$ with $P(x) > 0$ for all $x \in \,]0, 1[$. Then P is convenient.

The critical step towards the general case are the quadratic polynomials:

Lemma 2.12 *(Lemma 4.6 in [B])*
Let $P(t) = \alpha t^2 + \beta t + \gamma \in \mathbb{R}[t]$ with $P(x) > 0$ for all $x \in \,]0, 1[$. Then P is convenient.

The case $\alpha < 0$ can be reduced to the linear case. In the main case $\alpha > 0$ a division by this positive number is harmless, so we may assume $\alpha = 1$. Now we distinguish three subcases by the position of the zeros of P. Among these the complicated one is that of P having no real zero. Here a somewhat lengthy calculation shows that for all $d \in \mathbb{N}$, $d \geq 3$ we have

$$\frac{t^2 + \beta t + \gamma}{(1-t)^d} = \gamma + (d\gamma + \beta)t + \sum_{k=2}^{\infty} T_d(k) \cdot \frac{\prod_{j=1}^{d-3}(k+j)}{(d-1)!} \cdot t^k,$$

where the quadratic polynomial T_d takes only positive values for $d \gg 0$.

After these preparatory steps the general case is easy:

Theorem 2.13 *(Theorem 4.7 in [B])*
Let $P \in \mathbb{R}[t, t^{-1}]$ be a Laurent polynomial with $P(x) > 0$ for all $x \in]0, 1[$. Then P is convenient.

For the proof we may assume that P is a polynomial with $P(x) > 0$ for all $x \in [0, 1]$. By the Fundamental Theorem of Algebra there exists a factorization of P

$$P(t) = \left(\alpha \cdot \prod_{i=1}^{m} (t - x_i)\right) \cdot \left(\prod_{j=1}^{n} (t - y_i) \cdot \prod_{k=1}^{r} \left(t^2 + b_k t + c_k\right)\right)$$

with $x_i > 1$, $y_j < 0$ and irreducible factors $t^2 + b_k t + c_k$, and it is not hard to see that this factorization can be written as a product of convenient polynomials.

In combination with Lemma 2.9 we get the desired characterization of the possible h–vectors.

Corollary 2.14 *(Cor. 4.8 in [B])*
Let $P \neq 0$ be a Laurent polynomial with integer coefficients. The following conditions are equivalent:

a) $P(x) > 0$ *for all* $x \in]0, 1[$.

b) *There exists $d \in \mathbb{N}$, such that there is a finitely generated graded module $M \neq 0$ over a standard graded polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$ with Hilbert series $\frac{P(t)}{(1-t)^d}$.*

Addendum: Further Questions

The $\mathbb{F}$–algebras considered in Paper [B] fulfill three important conditions: They are $\mathbb{Z}$*–graded*, they are *homogeneous* (i. e. generated by elements of degree 1), and they are *polynomial* rings. So there are three obvious directions for an attempt of generalizing the results of [B].

For modules over $R = \mathbb{F}[X_1, \ldots, X_n]$ with the standard–$\mathbb{Z}^n$–grading $\deg(X_i) := e_i$ (the ith unit vector of $\mathbb{Z}^n$) there is also a theory of Hilbert functions and Hilbert series, but the latter are now formal Laurent series in n indeterminates $t_1, \ldots, t_n$. The series H_M may be written in the form

$$\frac{Q_M(t_1, \ldots, t_n)}{(1 - t_1) \cdot \ldots \cdot (1 - t_n)} \quad \text{with} \quad Q_M \in \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]. \tag{10}$$

Using the same filtration argument as in the proof of [4, Prop. 2.13] one can show that for a given Hilbert series there exists a decomposition

$$H_M(t_1, \ldots, t_n) = \sum_{I \subseteq \{1, \ldots, n\}} \frac{Q_I(t_1, \ldots, t_n)}{\prod_{i \in I}(1 - t_i)}, \tag{11}$$

with nonnegative $Q_I \in \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. This result depends strongly on the series being a Hilbert series; it is not clear whether the true analogue of Theorem 2.4, namely the existence of such decomposition for *any* non–negative series of the form (10) can be proven. Anyway, the notion of positivity can probably not be transferred to the $\mathbb{Z}^n$–graded situation in a useful way – one important obstacle is the lack of $\mathbb{Z}^n$–homogeneous regular sequences. Alternatively one could investigate the relation between the Hilbert depth and the maximal number d such that H_M admits a decomposition of the form (11) where $Q_I = 0$ for all subsets with $|I| < d$. This approach is closely related to the notorious *Stanley conjecture*; see [4] for some basic results in this direction.

Similar complications arise in the case where R is positively $\mathbb{Z}$–graded but not homogeneous, i. e. $\deg(X_i) = d_i \geq 1$. Here the Hilbert function of a finitely generated graded R–module is for large n given by a *quasipolynomial*, and the Hilbert series admits a presentation of the form

$$\frac{Q_M(t)}{(1-t^{d_1})\cdot\ldots\cdot(1-t^{d_n})} \quad \text{with} \quad Q_M \in \mathbb{Z}[t, t^{-1}].$$

Again the appearance of different factors in the denominator complicates matters. For a given Hilbert series the same filtration argument as above yields a decomposition parallel to (11), but it is not clear whether a complete analogue of Theorem 2.4 can be found. On the other hand the non–standard $\mathbb{Z}$–graded case has one important advantage over the $\mathbb{Z}^n$–graded case: For any finitely generated graded R–module M of positive depth there exist a homogeneous regular sequence of maximal length (provided that the ground field is infinite). Such a sequence can be composed of elements of degree $d := \operatorname{lcm}(d_1, \ldots, d_n)$, and therefore a modified notion of positivity, namely

$$\mathrm{p}_d(H) := \max\left\{ r \in \mathbb{N} \;\middle|\; (1-t^d)^r H(t) \text{ is non–negative} \right\},$$

seems to be useful in this case. Easy examples show that in general $\mathrm{p}_d(H_M)$ exceeds the Hilbert depth of M, so if a characterization of the Hilbert depth by an arithmetic invariant can be given in this situation, it will be more complicated than Theorem 2.6. The special case of modules over the ring $\mathbb{F}[X, Y]$ with $\deg(X) = \alpha$ and $\deg(Y) = \beta$ coprime is considered in [15], where a second condition for positive Hilbert depth, interestingly related to the *numerical semigroup* $\langle \alpha, \beta \rangle$, is deduced.

Finally one could investigate if similar results on possible Hilbert functions and h–vectors or on the Hilbert depth are also available for certain non–polynomial homogeneous $\mathbb{F}$–algebras. Here Theorem 2.4 is still at disposal, but it is not sufficient for our purpose, because another serious problem appears: The reasoning in the polynomial case depended strongly on the fact that *any* series $t^i/(1-t)^m$ can be realized as the Hilbert series of some module. This is a consequence of the polynomial ring having h–vector $Q = 1$ and cannot be extended to an $\mathbb{F}$–algebra with a more complicated h–vector. Even if the h–vector is as symmetric and neat as for example in the case of a *complete intersection*, see [18, Cor. 3.4], it is not obvious that the decomposition of a Hilbert series provided by Theorem 2.4 is helpful in any way. Probably an extension of the results to homogeneous $\mathbb{F}$–algebras with other h–vectors requires a more elaborated decomposition theorem.

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A NOTE ON THE DIMENSION THEORY OF $\mathbb{Z}^n$–GRADED RINGS

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Abstract

In this note we want to generalize some of the results in [Br73] from polynomial rings in several indeterminates to arbitrary $\mathbb{Z}^n$–graded commutative rings. We will prove an analogue of Jaffard's Special Chain Theorem and a similar result for the height of a prime ideal $\mathfrak{p}$ over its graded core $\mathfrak{p}^*$.

Introduction

In their article [Br73] Brewer *et al.* consider a polynomial ring $R[X_1, \ldots, X_n]$ over an arbitrary commutative ring R. They show that for any prime ideal $\mathfrak{p} \subset R[X_1, \ldots, X_n]$ and its re–extended contraction $\mathfrak{p}^{ce} := (\mathfrak{p} \cap R)R[X_1, \ldots, X_n]$ the relation

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^{ce}) + \mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^{ce}) \tag{1}$$

holds. This statement provides easy proofs of some further results in the dimension theory of polynomial rings.

In this note we want to generalize these results to the case of arbitrary $\mathbb{Z}^n$–graded rings. Our approach is motivated by the following simple observation: Let $R[X_1, \ldots, X_n]$ be equipped with the standard $\mathbb{Z}^n$–grading, and let $\mathfrak{p} \subset R[X_1, \ldots, X_n]$ be a prime ideal which does not contain the indeterminates, then the graded core $\mathfrak{p}^*$ of $\mathfrak{p}$ is nothing but the re–extended contraction $\mathfrak{p}^{ce}$, and therefore

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + \mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^*)\,. \tag{2}$$

We will show that this equation holds for any prime ideal in an arbitrary $\mathbb{Z}^n$–graded ring; moreover, even the proof of (1) given in [Br73] can be carried over to the general situation. In the sequel we discuss some consequences of (2), including a graded–rings–analogue of Jaffard's Special Chain Theorem and a related result on $\mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^*)$.

1 Terminology, Notations and Basic Facts

In this paper we only consider commutative rings with unit. A $\mathbb{Z}^n$*–graded ring* is a ring R together with a decomposition $R = \bigoplus_{a \in \mathbb{Z}^n} R_a$ of the underlying additive group such that $R_a \cdot R_b \subseteq R_{a+b}$ holds for any $a, b \in \mathbb{Z}^n$. Any $x \in R$ admits a unique presentation $x = \sum_{a \in \mathbb{Z}^n} x_a$; the summands ($\neq 0$) are called the *homogeneous components* of x. A *graded ideal* is an ideal $I \subseteq R$ with the property that for any $x \in I$ all homogeneous components of x belong to I as well; this is equivalent with I being generated by homogeneous elements. For any ideal $I \subseteq R$ the *graded core* I^* is defined as the ideal generated by all homogeneous elements of I.

It is the largest graded ideal contained in I. If S denotes the set of homogeneous elements not contained in the prime ideal $\mathfrak{p}$, then the ring $R_{(\mathfrak{p})} := S^{-1}R$ is called the *homogeneous localization at* $\mathfrak{p}$. It is $\mathbb{Z}^n$–graded in a natural way, and it is **local*, i. e. every proper graded ideal of $R_{(\mathfrak{p})}$ is contained in the unique *maximal ideal $\mathfrak{p}^*R_{(\mathfrak{p})}$.

We note two basic facts about prime ideals in $\mathbb{Z}^n$–graded rings:

Lemma 1.1
Let R be a $\mathbb{Z}^n$–graded ring.

a) A graded ideal $I \subset R$ is a prime ideal if and only if for any homogeneous elements $x, y \in R$, $xy \in I$ implies $x \in I$ or $y \in I$.

b) Let $\mathfrak{p} \subset R$ be a prime ideal, then $\mathfrak{p}^*$ is also a prime ideal.

c) Any minimal prime ideal of R is graded.

PROOF: a) We only have to show that the condition on I is sufficient: If I were not prime, there would be elements $x = \sum_a x_a$ and $y = \sum_b y_b$ not in I with $xy \in I$. Since $x, y \notin I$, there must be degrees $a, b \in \mathbb{Z}^n$ with $x_a, y_b \notin I$. Let $p, q \in \mathbb{Z}^n$ be the lowest of these degrees with respect to the lexicographic ordering $<_L$ of $\mathbb{Z}^n$, i. e. $x_p, y_q \notin I$, but $x_a, y_b \in I$ for all $a <_L p$ and $b <_L q$.

Let us consider the $(p+q)^{\text{th}}$ homogeneous component of xy. Since the lexicographic ordering is compatible with the group structure of $\mathbb{Z}^n$, we have

$$(xy)_{p+q} = \sum_{a+b=p+q} x_a y_b = x_p y_q + \sum_{\substack{a+b=p+q \\ a<_L p}} x_a y_b + \sum_{\substack{a+b=p+q \\ b<_L q}} x_a y_b.$$

By the choice of p and q the two sums on the right hand side are in I, and, the ideal I being graded, also $(xy)_{p+q} \in I$. Therefore $x_p y_q$ must be in I too, but by assumption this leads to the contradiction $x_p \in I$ or $y_q \in I$.

b) This follows from part a), since a homogeneous $x \in R$ belongs to $\mathfrak{p}^*$ iff it belongs to $\mathfrak{p}$.

c) By part b), any non–graded prime ideal contains another prime ideal. □

Lemma 1.2
Let $\mathfrak{p}$ be a prime ideal in a $\mathbb{Z}^n$–graded ring R. Then

a) $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) \leq n$.

b) $\operatorname{ht}(\mathfrak{p}) = \infty \Longleftrightarrow \operatorname{ht}(\mathfrak{p}^*) = \infty$.

PROOF: a) We may replace R with $R/\mathfrak{p}^*$ and assume $\mathfrak{p}^* = 0$. Then R is an integral domain, and $\mathfrak{p}$ remains prime after passing to the homogeneous localization $R_{(0)}$. The ring $R_{(0)}$ is **simple*, i. e. it does not have any non–trivial graded ideal (see [GW78b], §2). By Theorem 1.1.4 of [GW78b] such a ring is isomorphic to a Laurent polynomial ring $k[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]$ over a field k in $m \leq n$ indeterminates; so we get $\operatorname{ht}_R(\mathfrak{p}) = \operatorname{ht}_{R_{(0)}}(\mathfrak{p}R_{(0)}) \leq \dim R_{(0)} = m \leq n$.

b) Any chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}$ induces a chain $\mathfrak{p}_0^* \subseteq \mathfrak{p}_1^* \subseteq \ldots \subseteq \mathfrak{p}^*$. By part a) at most $n+1$ consecutive ideals in the second chain can be identical. Therefore $\mathrm{ht}(\mathfrak{p}) \geq k$ implies $\mathrm{ht}(\mathfrak{p}^*) \geq \lfloor \frac{k}{n+1} \rfloor$. This shows '$\Rightarrow$', and the reverse implication is trivial. □

For a graded prime ideal $\mathfrak{p} \subset R$ we call the supremum of the lengths of descending chains

$$\mathfrak{p} = \mathfrak{p}_r \supset \mathfrak{p}_{r-1} \supset \ldots \supset \mathfrak{p}_1 \supset \mathfrak{p}_0$$

of graded prime ideals the **height* of $\mathfrak{p}$ and denote it by $^*\mathrm{ht}(\mathfrak{p})$. The supremum of the *heights of all graded prime ideals of R is called the **dimension* of R and denoted by $^*\dim R$. With the same argument as in the proof of part b) in Lemma 1.2, we get the following estimate.

Corollary 1.3
Let R be a $\mathbb{Z}^n$–graded ring and $\mathfrak{p} \subset R$ a graded prime ideal. Then

$$^*\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}) \leq (n+1)\,^*\mathrm{ht}(\mathfrak{p}).$$

□

Given a group homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^k$, one may view a $\mathbb{Z}^n$–graded ring R as $\mathbb{Z}^k$–graded by defining $\bigoplus_{\varphi(a)=b} R_a$ to be the homogeneous component of degree $b \in \mathbb{Z}^k$ in this new grading. We call this the φ–induced grading of R. The graded core of an ideal $I \subseteq R$ with respect to the φ–induced grading will be denoted by $I^{*\varphi}$. The most important case is $\varphi = p_i$, the projection onto the i^{th} factor of $\mathbb{Z}^n$.

An easily proved lemma in this context is the following.

Lemma 1.4
Let $U_1 \oplus U_2$ be a direct–sum decomposition of $\mathbb{Z}^n$ with projections $\pi_i : \mathbb{Z}^n \to U_i$. Then for any ideal I of a $\mathbb{Z}^n$–graded ring we have $(I^{*\pi_1})^{*\pi_2} = I^* = (I^{*\pi_2})^{*\pi_1}$. In particular an ideal I is $\mathbb{Z}^n$–graded if and only if it is π_1– and π_2–graded.

Proof: An element $x \in I$ is $\mathbb{Z}^n$–homogeneous if and only if it is π_1– and π_2–homogeneous. □

2 The Main Theorem

Theorem 2.1 *(Height Theorem)*
Let $\mathfrak{p}$ be a prime ideal in a $\mathbb{Z}^n$–graded ring. Then

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{p}^*\right).$$

Proof: By Lemma 1.2 the equation holds trivially if $\mathrm{ht}(\mathfrak{p}) = \infty$, and there is also nothing to show if $\mathfrak{p} = \mathfrak{p}^*$. We may therefore assume that $\mathfrak{p}$ is a non–graded prime ideal of finite height.

1) First we prove the assertion for $n = 1$. In this case we have $\mathrm{ht}\left(\mathfrak{p}/\mathfrak{p}^*\right) = 1$ by Lemma 1.2, so it is enough to show $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + 1$. This is done by induction on $\mathrm{ht}(\mathfrak{p})$:

The equation holds for $\mathrm{ht}(\mathfrak{p}) = 1$, because $\mathfrak{p}^* \subset \mathfrak{p}$ implies $0 \leq \mathrm{ht}(\mathfrak{p}^*) < \mathrm{ht}(\mathfrak{p}) = 1$, and therefore we must have $\mathrm{ht}(\mathfrak{p}^*) = 0$. Assuming that the equation is valid for prime ideals of height at most $d \geq 1$, we have to show its validity for a prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = d + 1$. Thereto let $\mathfrak{q} \subset \mathfrak{p}$ be a prime ideal with $\mathrm{ht}(\mathfrak{q}) = d$. We distinguish two cases:

a) If $\mathfrak{q}$ is graded, then $\mathfrak{q} = \mathfrak{p}^*$, so we are done at once.

b) Otherwise we have $\mathfrak{q}^* \neq \mathfrak{q}$. By the induction hypothesis this means $\mathrm{ht}(\mathfrak{q}^*) = d-1$. Since $\mathfrak{q} \subset \mathfrak{p}$, we have $\mathfrak{q}^* \subseteq \mathfrak{p}^*$. If this inclusion were not proper, we would have $\mathfrak{p}^* = \mathfrak{q}^* \subset \mathfrak{q} \subset \mathfrak{p}$ contradicting $\mathrm{ht}\left(\mathfrak{p}/\mathfrak{p}^*\right) = 1$. Therefore $\mathfrak{q}^* \subset \mathfrak{p}^*$, and this means $\mathrm{ht}(\mathfrak{p}^*) \geq \mathrm{ht}(\mathfrak{q}^*) + 1 = d$. But the reverse inequality is clear by $\mathfrak{p}^* \neq \mathfrak{p}$, hence we have $\mathrm{ht}(\mathfrak{p}^*) = d$, as desired.

2) We may therefore proceed by induction on n. Assuming that the result is already proved for $\mathbb{Z}^k$–graded rings with $k \leq n$, we have to show it for a $\mathbb{Z}^{n+1}$–graded ring:

Let $\mathfrak{p}$ be a non–graded prime ideal of finite height in the $\mathbb{Z}^{n+1}$–graded ring R. We consider the $\mathbb{Z}^n$–grading of R induced by the projection $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n,\ (a_1, \ldots, a_n, a_{n+1}) \mapsto (a_1, \ldots, a_n)$, and set $\mathfrak{q} := \mathfrak{p}^{*\pi}$. The induction hypothesis for R as a $\mathbb{Z}^n$–graded ring yields

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{q}\right).$$

There remains nothing to show if $\mathfrak{p}^* = \mathfrak{q}$. Otherwise we have $\mathfrak{p}^* = \mathfrak{q}^{*p_{n+1}} \subset \mathfrak{q}$. Considering the p_{n+1}–induced $\mathbb{Z}$–grading of R, the induction hypothesis yields

$$\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}\left(\mathfrak{q}^{*p_{n+1}}\right) + 1 = \mathrm{ht}(\mathfrak{p}^*) + 1,$$

and hence

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{q}\right) = \mathrm{ht}(\mathfrak{p}^*) + 1 + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{q}\right).$$

Since $\mathfrak{p}^* \subset \mathfrak{q}$, we have $\mathrm{ht}\left(\mathfrak{p}/\mathfrak{q}\right) < \mathrm{ht}\left(\mathfrak{p}/\mathfrak{p}^*\right)$. This yields

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + \Big(1 + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{q}\right)\Big) \leq \mathrm{ht}(\mathfrak{p}^*) + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{p}^*\right),$$

and the result follows, as the reverse inequality is trivial. □

As an immediate consequence we get the following result on the dimension of modules – for the special case of a Noetherian ring R this assertion is proved in [GW78b] along different lines.

Corollary 2.2 *(comp. Prop. 1.2.2 in [GW78b])*
Let M be a finitely generated $\mathbb{Z}^n$–graded module over a $\mathbb{Z}^n$–graded ring R. If $\mathfrak{p} \in \operatorname{Supp} M$, then

$$\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + \mathrm{ht}\left(\mathfrak{p}/\mathfrak{p}^*\right).$$

PROOF: Since the ideal $\operatorname{Ann} M$ is graded, the result follows from Theorem 2.1 applied to the graded ring $R/\operatorname{Ann} M$ once we have checked $\mathfrak{p}^* \in \operatorname{Supp} M$. But this can be done literally as in the $\mathbb{Z}$–graded case, see [BH93], Lemma 1.5.6. □

Theorem 2.1 easily leads to a result on the relation between the dimension and the *dimension of a $\mathbb{Z}^n$–graded ring:

Corollary 2.3
Let R be a $\mathbb{Z}^n$–graded ring, then

$$^*\dim(R) \le \dim R \le (n+1)\,^*\dim(R) + n.$$

PROOF: The first inequality is trivial. The second one follows since for any prime ideal $\mathfrak{p} \subset R$ we have

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}\,(\mathfrak{p}^*) + \mathrm{ht}\,(\mathfrak{p}/\mathfrak{p}^*) \le \mathrm{ht}\,(\mathfrak{p}^*) + n \le (n+1)\,^*\mathrm{ht}(\mathfrak{p}^*) + n \le (n+1)\,^*\dim(R) + n$$

by the Height Theorem, Lemma 1.2 and Corollary 1.3. □

In special cases the lower bound for the *dimension of R can be improved. Rather than formulate an exact analogue of Corollary 2 in [Br73], we give a simplified statement.

Corollary 2.4
Let R be a $\mathbb{Z}^n$–graded ring such that $^*\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ holds for any graded prime ideal $\mathfrak{p} \subset R$. Then

$$^*\dim(R) \le \dim R \le \,^*\dim(R) + n.$$

PROOF: The proof virtually remains the same as in the general case, except that the condition on R enables to avoid the estimate of Corollary 1.3. □

The strong prerequisite of the last corollary is fulfilled for important classes of rings:

Lemma 2.5
Let R be a $\mathbb{Z}^n$–graded ring. If R is Noetherian or Prüfer, then $^*\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ holds for any graded prime ideal $\mathfrak{p} \subset R$.

PROOF: In the Noetherian case we show that any chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$ of length $\mathrm{ht}(\mathfrak{p})$ can be replaced by a chain $\mathfrak{p}'_0 \subset \mathfrak{p}'_1 \ldots \subset \mathfrak{p}'_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$ consisting of $\mathbb{Z}^n$–graded prime ideals. This is done by induction on n, starting with the vacuos case $n = 0$. For $n > 0$ let $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ again denote the projection which discards the last entry. By induction we may assume that the ideals $\mathfrak{p}_i$, $i = 1, \ldots, d-1$ are graded in the π–induced grading of R, and by Lemma 1.1, $\mathfrak{p}_0$ is even $\mathbb{Z}^n$–graded. We may therefore employ a second induction and assume that for some $k < d$ the ideals $\mathfrak{p}_0, \ldots, \mathfrak{p}_k$ are $\mathbb{Z}^n$–graded. If $\mathfrak{p}_{k+1}$ is not already $\mathbb{Z}^n$–graded, it must be replaced by a matching $\mathbb{Z}^n$–graded prime ideal:

a) If $\mathfrak{p}_{k+2}$ is $\mathbb{Z}^n$–graded, we find a homogeneous element $x \in \mathfrak{p}_{k+2} \setminus \mathfrak{p}_k$. Let $\mathfrak{q}$ be a minimal prime of the graded ideal $\mathfrak{p}_k + (x)$ contained in $\mathfrak{p}_{k+2}$. Again by Lemma 1.1, $\mathfrak{q}$ is $\mathbb{Z}^n$–graded, and is properly contained in $\mathfrak{p}_{k+2}$, because $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}_k) = 1$ by Krull's Principal Ideal Theorem. Hence we may replace $\mathfrak{p}_{k+1}$ by $\mathfrak{q}$.

b) In the remaining case $\mathfrak{p}_{k+2}$ is not p_n–graded. By Lemma 1.2 we have $\mathrm{ht}\left(\mathfrak{p}_{k+2}/\mathfrak{p}_{k+2}^{*p_n}\right) = 1$, therefore $\mathfrak{p}_{k+2}^{*p_n}$ lies properly between $\mathfrak{p}_k$ and $\mathfrak{p}_{k+2}$. Since $\mathfrak{p}_{k+2}^{*p_n}$ is $\mathbb{Z}^n$–graded by Lemma 1.4, we have found a suitable replacement for $\mathfrak{p}_{k+1}$.

In the Prüfer case we show that any prime ideal $\mathfrak{q}$ contained in a graded prime ideal $\mathfrak{p}$ must itself be graded. Let $x \in \mathfrak{q}$ be non–zero, and consider the graded ideal I generated by the components of x. By Proposition 2.5 of [AA82] the inverse I^{-1} is a graded R–submodule of the homogeneous localization $R_{(0)}$. Since $II^{-1} = R$, there must be a homogeneous $h \in I^{-1}$ and a component x_s of x such that $hx_s \notin \mathfrak{p}$, and this also means $hx \notin \mathfrak{p}$. But for an arbitrary component x_a of x we have $(hx)x_a = (hx_a)x \in \mathfrak{q}$, and therefore $x_a \in \mathfrak{q}$, as desired. □

We close this section with the announced analogue of Jaffard's Special Chain Theorem.

In his monograph [Ja60] P. Jaffard introduced the notion of a *special chain* of prime ideals in a polynomial ring: A chain $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in $R[X_1, \ldots, X_n]$ is called a special chain, if for every $i = 0, \ldots, r$ the ideal $(\mathfrak{p}_i \cap R)R[X_1, \ldots, X_n]$ also appears in this chain. Again using the graded core as a substitute for the re–extended contraction, we define an analogue for general $\mathbb{Z}^n$–graded rings as follows.

Definition 2.1
A chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in a $\mathbb{Z}^n$–graded ring is called a special chain*, if for any $i = 0, \ldots, r$ there is a $j \leq i$ with $\mathfrak{p}_i^* = \mathfrak{p}_j$; i. e. the graded cores of all members also appear in this chain.*

With this definition Jaffard's Special Chain Theorem (see [Ja60], Chapitre II, Théorème 3, and [Br73], Corollary 3) can be adopted verbatim:

Theorem 2.6 *(Jaffard's Special Chain Theorem – graded analogue)*
Let $\mathfrak{p}$ be a prime ideal of height d in a $\mathbb{Z}^n$–graded ring R. Then there is a special chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$$

of length d in R. In particular, if R is finite–dimensional, then $\dim R$ can be realized as the length of a special chain of prime ideals of R.

We will deduce this statement from the Height Theorem, but, as one checks in an instant, the reverse implication also holds. Hence these results are equivalent.

Proof: We use induction on $d = \mathrm{ht}(\mathfrak{p})$, the case $d = 0$ being trivial by Lemma 1.1. For $d > 0$ we distinguish two cases.

i) If $\mathfrak{p}$ is graded, we choose a prime ideal $\mathfrak{q} \subset \mathfrak{p}$ with $\mathrm{ht}(\mathfrak{q}) = d - 1$. By induction there is a special chain of length $d - 1$ ending with $\mathfrak{q}$, and of course this chain remains special if we add the graded ideal $\mathfrak{p}$.

ii) Otherwise we have $k := \mathrm{ht}(\mathfrak{p}^*) < \mathrm{ht}(\mathfrak{p})$, and therefore the induction hypothesis yields a special chain
$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{k-1} \subset \mathfrak{p}_k = \mathfrak{p}^*.$$
By Theorem 2.1 we can extend this chain to a chain
$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_k = \mathfrak{p}^* \subset \mathfrak{p}_{k+1} \subset \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$$
of length d, and this extension is still a special chain, since for $i = k + 1, \ldots, d$ we have $\mathfrak{p}_i^* = \mathfrak{p}^*$.

□

3 A Result on $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*)$

In this final section we show that the Height Theorem also provides a relation between a prime ideal and its graded core, which may be viewed as a relative version of Jaffard's Theorem. First we introduce a denomination for another specific sort of prime ideal chains:

Definition 3.1
A chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in a $\mathbb{Z}^n$–graded ring is called a ladder*, if for any $i = 1, \ldots, r$ there is a $j_i \in \{1, \ldots, n\}$ with $\mathfrak{p}_i^{*p_{j_i}} = \mathfrak{p}_{i-1}$.*

We note an immediate consequence of Theorem 2.1.

Lemma 3.1
Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ be a ladder in a $\mathbb{Z}^n$–graded ring R. Then $\operatorname{ht}(\mathfrak{p}_i/\mathfrak{p}_0) = i$ for $i = 0, \ldots, r$.

Proof: The assertion is trivial for $i = 0$. For $i > 0$ the ideal $\mathfrak{p}_0$ is p_{j_i}–graded, therefore we may consider the p_{j_i}–induced $\mathbb{Z}$–grading of $R/\mathfrak{p}_0$. Since $(\mathfrak{p}_i/\mathfrak{p}_0)^{*p_{j_i}} = \mathfrak{p}_i^{*p_{j_i}}/\mathfrak{p}_0$, the Height Theorem yields

$$\operatorname{ht}(\mathfrak{p}_i/\mathfrak{p}_0) = 1 + \operatorname{ht}\left(\mathfrak{p}_i^{*p_{j_i}}/\mathfrak{p}_0\right) = 1 + \operatorname{ht}(\mathfrak{p}_{i-1}/\mathfrak{p}_0),$$

and the result follows by induction. □

Theorem 3.2
Let R be a $\mathbb{Z}^n$–graded ring and $\mathfrak{p} \subset R$ a prime ideal. Then there exists a ladder

$$\mathfrak{p}^* = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$$

of length d if and only if $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) = d$.

Proof: The 'only if'–part is clear by the previous lemma. The converse is proved by induction on $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*)$, again starting with the vacuous case $\mathfrak{p} = \mathfrak{p}^*$. For $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) = d > 0$ the ideal $\mathfrak{p}$ is non–graded, so there is an index $j \in \{1, \ldots, n\}$ with $\mathfrak{p}^{*p_j} \subset \mathfrak{p}$. The same argument as in the previous proof yields $\operatorname{ht}(\mathfrak{p}^{*p_j}/\mathfrak{p}^*) = d - 1$, and therefore the induction hypothesis can be applied to $\mathfrak{p}^{*p_j}$. □

Theorem 3.2 can be useful for extending assertions related to a prime ideal and its graded core from the $\mathbb{Z}$–graded to the $\mathbb{Z}^n$–graded case by means of induction. Under certain circumstances this method also applies to results outside dimension theory. As an illustrative example we consider a theorem in the range of homological invariants of graded modules.

Let R be an arbitrary Noetherian ring, M an R–module, and $\mathfrak{p} \subset R$ a prime ideal. For any integer $i \geq 0$ the i^{th} *Bass number of M at $\mathfrak{p}$* is defined as

$$\mu_i(\mathfrak{p}, M) := \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}),$$

here $k(\mathfrak{p})$ denotes the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. In the special case of a Noetherian local ring $(R, \mathfrak{m}, k)$ and a finitely generated R–module $M \neq 0$ of depth t, we call $\mu_t(\mathfrak{m}, M) = \dim_k \operatorname{Ext}^t_R(k, M)$ the *type* of M and denote it by $r(M)$.

Theorem 3.3 *(Theorem 1.2.3 in [GW78b])*
Let R be a Noetherian $\mathbb{Z}^n$–graded ring, M a $\mathbb{Z}^n$–graded R–module, and $\mathfrak{p}$ a non–graded prime ideal of R with $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) = d$. Then $\mu_{i+d}(\mathfrak{p}, M) = \mu_i(\mathfrak{p}^*, M)$ for every integer $i \geq 0$, and $\mu_i(\mathfrak{p}, M) = 0$ for $0 \leq i < d$.

Proof: The result is well–known for $\mathbb{Z}$–graded rings, see [GW78a], Theorem 1.1.2. In the $\mathbb{Z}^n$–graded situation we choose a ladder $\mathfrak{p}^* = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{d-1} \subset \mathfrak{p}_d = \mathfrak{p}$ by Theorem 3.2. Considering the p_{j_d}–induced grading of R, we may apply the $\mathbb{Z}$–version of the statement to $\mathfrak{p}_d$ and $\mathfrak{p}_{d-1} = \mathfrak{p}_d^{*p_{j_d}}$ and get

$$\mu_{i+d}(\mathfrak{p}_d, M) = \mu_{i+d-1}\left(\mathfrak{p}_d^{*p_{j_d}}, M\right) = \mu_{i+d-1}(\mathfrak{p}_{d-1}, M) \quad \text{and} \quad \mu_0(\mathfrak{p}_d, M) = 0,$$

so the assertion follows by induction on d. □

Corollary 3.4 *($\mathbb{Z}^n$–graded version of Theorem 1.5.9 in [BH93])*
Let R be a Noetherian $\mathbb{Z}^n$–graded ring, M a finitely generated $\mathbb{Z}^n$–graded R–module, and $\mathfrak{p}$ a non–graded prime ideal in $\operatorname{Supp} M$ with $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*) = d$. Then

$$\operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}^*} + d \quad \textit{and} \quad r(M_{\mathfrak{p}}) = r(M_{\mathfrak{p}^*}).$$

□

Corollary 3.5
Let R be a Noetherian $\mathbb{Z}^n$–graded ring, M a finitely generated $\mathbb{Z}^n$–graded R–module, and $\mathfrak{p} \subset R$ a prime ideal. Then $M_{\mathfrak{p}}$ is Cohen–Macaulay if and only if $M_{\mathfrak{p}^*}$ is.

Proof: We may assume $\mathfrak{p} \in \operatorname{Supp} M$, since otherwise $\mathfrak{p}^* \notin \operatorname{Supp} M$. Let $d = \operatorname{ht}(\mathfrak{p}/\mathfrak{p}^*)$, then $\dim M_{\mathfrak{p}} = \dim M_{\mathfrak{p}^*} + d$ by Corollary 2.2 and $\operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}^*} + d$ by the preceding result, and consequently $\dim M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}}$ if and only if $\dim M_{\mathfrak{p}^*} = \operatorname{depth} M_{\mathfrak{p}^*}$. □

Corollary 3.6
Let R be a Noetherian $\mathbb{Z}^n$–graded ring and $\mathfrak{p} \subset R$ a prime ideal. Then $R_{\mathfrak{p}}$ is Gorenstein if and only if $R_{\mathfrak{p}^*}$ is.

Proof: A Noetherian local ring is Gorenstein if and only if it is a Cohen–Macaulay ring of type 1, see [BH93], Theorem 3.2.10; therefore the assertion is a direct consequence of the two preceding corollaries. □

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REMARKS ON HILBERT SERIES OF GRADED MODULES OVER POLYNOMIAL RINGS

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Abstract

In this article we discuss a result on formal Laurent series and some of its implications for Hilbert series of finitely generated graded modules over standard–graded polynomial rings: For any integer Laurent function of polynomial type with non–negative values the associated formal Laurent series can be written as a sum of rational functions of the form $\frac{Q_j(t)}{(1-t)^j}$, where the numerators are Laurent polynomials with non–negative integer coefficients. Hence any such series is the Hilbert series of some finitely generated graded module over a suitable polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$. We give two further applications, namely an investigation of the maximal depth of a module with a given Hilbert series and a characterization of Laurent polynomials which may occur as numerator in the presentation of a Hilbert series as a rational function with a power of $(1-t)$ as denominator.

MSC Codes: 13D40, 16W50.

1 Introduction

Let $\mathbb{F}$ be an arbitrary field. We consider the ring $R = \mathbb{F}[X_1, \ldots, X_n]$, equipped with the standard $\mathbb{Z}$–grading, i. e., $\deg(X_i) = 1$ for $i = 1, \ldots, n$. Furthermore let $M \neq 0$ be a finitely generated $\mathbb{Z}$–graded R–module. Every homogeneous component of M is a finite–dimensional $\mathbb{F}$–vectorspace, and since R is positively graded and M is finitely generated, $M_k = 0$ for $k \ll 0$. Hence the *Hilbert function* of M

$$H(M, -) : \mathbb{Z} \to \mathbb{Z}, \ k \mapsto \dim_{\mathbb{F}} (M_k),$$

is a well–defined *integer Laurent function* (see [2], Definitions 5.1.1 and 5.1.12). The formal Laurent series *associated to* $H(M, -)$

$$H_M(t) = \sum_{k \in \mathbb{Z}} H(M, k) t^k = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{F}} M_k) \, t^k \in \mathbb{Z}[[t]][t^{-1}] \tag{1}$$

is called the *Hilbert series* of M.

By a classical theorem of Hilbert (see [1, Thm. 4.1.3] or [2, Thm. 5.1.21]), $H(M, -)$ is a function of *polynomial type*: There is a polynomial $P \in \mathbb{Q}[t]$ of degree $\dim(M) - 1$, such that $\dim_{\mathbb{F}} (M_k) = P(k)$ for $k \gg 0$. As a consequence of this theorem and a well–known result in the theory of generating functions, see for example Lemma 4.1.7 of [1] or Theorem 5.2.10 of [2], H_M may be written as a fraction of the form

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d} \tag{2}$$

where $Q_M \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial with $Q(1) \neq 0$, and d equals the Krull dimension of M. (Of course (2) and any similar equation in this article is to be understood as an equation in the ring of formal Laurent series – i. e., we do not claim any kind of convergence.) Two important assertions about the Laurent polynomial Q_M are easily checked: The value $Q_M(1)$, called the *multiplicity* of M, is positive, see [1, Prop. 4.1.9] or [2, Prop. 5.4.2]. The coefficient of the lowest non–vanishing term in Q_M equals the vector space dimension of the lowest non–vanishing degree of M and is therefore also positive, see [2, Thm. 5.2.20]. For more results on Hilbert functions and Hilbert series we refer the reader to the literature, especially to our main sources [1, Chapter 4], and [2, Chapter 5].

The central result of this article is a kind of refinement of equation (2): For any integer Laurent function of polynomial type with non–negative values the associated formal Laurent series can be written as a sum of rational functions of the form $\frac{Q_j(t)}{(1-t)^j}$, where the numerators $Q_j \in \mathbb{Z}[t,t^{-1}]$ have non–negative coefficients. As an immediate consequence any such series is the Hilbert series of some finitely generated graded module over a suitable polynomial ring. The result also provides an essential ingredient in the our subsequent investigation of the following questions:

- What is the maximal depth of a finitely generated graded R–module with a specific Hilbert series?
- Which Laurent polynomials may appear in (2) as numerators?

2 A Decomposition Theorem

In this section we formulate and prove the announced result on decomposition of certain Laurent series into a sum of rational functions of a special type. First we introduce the following simplifying notions: A formal Laurent series $H \in \mathbb{R}[[t]][t^{-1}]$ is called *non–negative*, if all of its coefficients are non–negative, and an integer Laurent function is *of polynomial type* d, if it agrees with a polynomial function of degree d for large arguments. Further we recall the technique of *shifting* a module: For any graded R–module M and any integer i the shifted module $M(i)$ is defined as the graded R–module with component $\big(M(i)\big)_k = M_{k+i}$ for all $k \in \mathbb{Z}$. Obviously we have

$$H_{M(i)}(t) = t^{-i} \cdot H_M(t),$$

and $M(i)$ has the same dimension and depth as M (see [2, Prop. 5.2.15, Prop. 5.4.5]).

Theorem 2.1
Let f be an integer Laurent function of polynomial type $d-1$ which takes non–negative values. Then the associated Laurent series H_f admits a presentation

$$H_f(t) = \sum_{j=0}^{d} \frac{Q_j(t)}{(1-t)^j} \quad \text{with non–negative } Q_j \in \mathbb{Z}[t,t^{-1}]. \tag{3}$$

Proof: We use induction on d. In the exceptional case $d = 0$ we have $f(k) = 0$ for $k \gg 0$, hence $H_f(t) =: Q_0(t)$ is itself a Laurent polynomial. For $d = 1$ there is $K \in \mathbb{N}$ such that $f(k) = f(K)$ for all $k \geq K$, and we get

$$H_f(t) = \sum_{k=b}^{K-1} f(k)t^k + \sum_{k=K}^{\infty} f(K)t^k = \sum_{k=b}^{K-1} f(k)t^k + \frac{f(K)t^K}{1-t},$$

thus the assertion is valid with $Q_0(t) = \sum_{k=b}^{K-1} f(k)t^k$ and $Q_1(t) = f(K)t^K$.

Next let f be of polynomial type $d \geq 1$, and assume that the result is already proven for functions of polynomial type $d-1$ with non–negative values. We consider the series

$$(1-t)H_f(t) = f(b)t^b + \sum_{k=b+1}^{\infty} \big((f(k) - f(k-1)\big)t^k = \sum_{k=b}^{\infty} \Delta f(k)t^k.$$

The difference function Δf is of polynomial type $d-1$ (see [2, Cor. 5.1.11]), and since for large k the value of f is given by the value of a polynomial function of positive degree, there must be a $K \in \mathbb{Z}$ with $\Delta f(k) \geq 0$ for all $k \geq K$. Therefore we may apply the induction hypothesis to the series $\sum_{k=K}^{\infty} \Delta f(k)t^k$. This yields

$$\sum_{k=K}^{\infty} \Delta f(k)t^k = \sum_{j=0}^{d} \frac{\tilde{Q}_j(t)}{(1-t)^j}$$

with non–negative $\tilde{Q}_j \in \mathbb{Z}[t, t^{-1}]$. Furthermore we have

$$(1-t)\sum_{k=b}^{K-1} f(k)t^k = \sum_{k=b}^{K-1} \Delta f(k)t^k - f(K-1)t^K,$$

and so

$$\begin{aligned}(1-t)H_f(t) &= \sum_{k=b}^{K-1} \Delta f(k)t^k + \sum_{k=K}^{\infty} \Delta f(k)t^k \\ &= (1-t)\sum_{k=b}^{K-1} f(k)t^k + f(K-1)t^K + \sum_{j=0}^{d} \frac{\tilde{Q}_j(t)}{(1-t)^j}.\end{aligned}$$

After dividing by $(1-t)$ we get

$$\begin{aligned}H_f(t) &= \sum_{k=b}^{K-1} f(k)t^k + \frac{f(K-1)t^K}{1-t} + \sum_{j=0}^{d} \frac{\tilde{Q}_j(t)}{(1-t)^{j+1}} \\ &= \sum_{k=b}^{K-1} f(k)t^k + \frac{f(K-1)t^K + \tilde{Q}_0(t)}{1-t} + \sum_{j=2}^{d+1} \frac{\tilde{Q}_{j-1}(t)}{(1-t)^j},\end{aligned}$$

so the assertion is valid with Laurent polynomials $Q_0(t) := \sum_{k=b}^{K-1} f(k)t^k$, $Q_1(t) :=$ $f(K-1)t^K + \tilde{Q}_0(t)$, and $Q_j := \tilde{Q}_{j-1}$ for $j = 2, \dots, d+1$. □

The importance of this result is confirmed by the following simple observation:

Lemma 2.2
Let $H \neq 0$ be a formal Laurent series such that there is a decomposition

$$H(t) = \sum_{j=r}^{d} \frac{Q_j(t)}{(1-t)^j}$$

with non–negative numerators $Q_j \in \mathbb{Z}[t, t^{-1}]$ and $Q_r \neq 0$. Then there exists a finitely generated graded R–module with Hilbert series H and depth r.

Proof: Let

$$H(t) = \sum_{j=r}^{d} \frac{Q_j(t)}{(1-t)^j} = \sum_{j=r}^{d} \frac{\sum_{k=p_j}^{q_j} h_{jk} t^k}{(1-t)^j}.$$

be a decomposition of H with non–negative coefficients h_{jk}, then the module

$$N := \bigoplus_{j=r}^{d} \left(\bigoplus_{k=p_j}^{q_j} \Big(R/(X_{j+1}, \ldots, X_n)(-k) \Big)^{h_{jk}} \right)$$

has the required properties. □

Hence Theorem 2.1 yields a characterization of Hilbert functions of finitely generated graded R–modules. (Compare to Macaulay's theorem on Hilbert functions of *affine* $\mathbb{F}$*–algebras*, [1, Thm. 4.2.10] or [2, Thm. 5.5.32].)

Corollary 2.3
Let f be an integer Laurent function with associated Laurent series H_f. The following conditions are equivalent:

a) The function f is of polynomial type and takes non–negative values.

b) The series H_f admits a presentation as in Theorem 2.1.

c) The series H_f is the Hilbert series of a finitely generated graded module over some standard–graded polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$.

d) The function f is the Hilbert function of a finitely generated graded module over some standard–graded polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$.

□

For brevity we introduce a name for the formal Laurent series considered in this section:

Definition 2.1
A Laurent series is called viable*, if it is associated to an integer Laurent function of polynomial type with non–negative values.*

3 Hilbert Depth and Positivity

Let H be a viable Laurent series, then, by Hilbert's theorem, all finitely generated graded R–modules with Hilbert series H are of the same dimension; we will denote this number by $\dim(H)$. On the contrary, there is no similar statement on depth, that is, there may be modules with the same series, but different depths: A very simple example is given by the $\mathbb{F}[X]$–modules

$$\mathbb{F}[X] \text{ and } \Big(\mathbb{F}[X]/(X)\Big) \oplus \Big(\mathbb{F}[X](-1)\Big).$$

The depth of a finitely generated graded R–module with a given Hilbert series H is bounded above by its dimension $\dim(H)$; therefore the following definition makes sense.

Definition 3.1
i) Let H be a viable Laurent series. The number

$$\operatorname{Hdep}(H) := \max\left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a f. g. gr. } R\text{–module } N \\ \text{with } H_N = H \text{ and } \operatorname{depth}(N) = r. \end{array} \right\}$$

is called the Hilbert depth *of H.*

ii) Let M be a finitely generated graded R–module, then we write $\operatorname{Hdep}(M)$ *instead of* $\operatorname{Hdep}(H_M)$.

We want to show that the Hilbert depth coincides with the arithmetic invariant of H defined by

$$\operatorname{p}(H) := \max\{r \in \mathbb{N} \mid (1-t)^r H(t) \text{ is non–negative}\},$$

which will be called the *positivity* of H.

Theorem 3.1
Let H be a viable Laurent series, then $\operatorname{Hdep}(H) = \operatorname{p}(H)$.

Proof: The inequality $\operatorname{Hdep}(H) \le \operatorname{p}(H)$ follows from standard facts about regular sequences: Let N be a finitely generated graded R–module with $H_N = H$. Furthermore let $\mathbb{K}$ be an infinite extension field of $\mathbb{F}$. Then $N' := N \otimes_{\mathbb{F}} \mathbb{K}$ is a finitely generated graded module over $R' := R \otimes_{\mathbb{F}} \mathbb{K}$ with the same depth and Hilbert series as N (see [1, Prop. 1.2.16] resp. [2, Cor. 5.2.19]). Since R' is a standard graded algebra over the infinite field $\mathbb{K}$, Proposition 1.5.12 of [1] yields the existence of a maximal N'–regular sequence $\underline{a} = a_1, \dots, a_r$ that consists of elements of degree 1. The Hilbert series of $N'/\underline{a}N'$ is then given by

$$H_{N'/\underline{a}N'}(t) = (1-t)^r H_{N'}(t) = (1-t)^r H_N(t) = (1-t)^r H(t)$$

(see [2], Cor. 5.2.17 or [3], Cor. 3.2), and this series is non–negative. Hence we have

$$\operatorname{depth}(N) \le \operatorname{p}(H)$$

for all finitely generated graded R–modules with Hilbert series H.

The converse can be deduced from the results of the previous section: Let H be a viable series with $\dim(H) =: d$. Then the series $(1-t)^{\operatorname{p}(H)} H(t)$ is non–negative by the definition of $\operatorname{p}(H)$, and its coefficients are given by an integer Laurent function of polynomial type $d-1-\mathrm{p}(M)$. Hence by Theorem 2.1 there are non–negative $\tilde{Q}_j \in \mathbb{Z}[t,t^{-1}]$, such that

$$(1-t)^{\mathrm{p}(H)}H(t) = \sum_{j=0}^{d-\mathrm{p}(H)} \frac{\tilde{Q}_j(t)}{(1-t)^j}.$$

Division by $(1-t)^{\mathrm{p}(H)}$ and setting $Q_j := \tilde{Q}_{j-\mathrm{p}(H)}$ yields a decomposition

$$H_M(t) = \sum_{j=\mathrm{p}(H)}^{d} \frac{Q_j(t)}{(1-t)^j},$$

and so $\mathrm{Hdep}(H) \geq \mathrm{p}(H)$ by Lemma 2.2. □

Theorem 3.1 has a consequence for the behaviour of the Hilbert depth with respect to short exact sequences:

Corollary 3.2
Let $0 \to U \to M \to N \to 0$ *be an exact sequence of finitely generated graded modules over* $\mathbb{F}[X_1,\ldots,X_n]$, *then* $\mathrm{Hdep}(M) \geq \min\{\mathrm{Hdep}(U), \mathrm{Hdep}(N)\}$.

Proof: Let $r = \min\{\mathrm{Hdep}(U), \mathrm{Hdep}(N)\} = \min\{\mathrm{p}(H_U), \mathrm{p}(H_N))\}$, then

$$(1-t)^r H_M(t) = (1-t)^r \left(H_U(t) + H_N(t)\right) = (1-t)^r H_U(t) + (1-t)^r H_N(t)$$

is non–negative, hence $\mathrm{Hdep}(M) = \mathrm{p}(H_M) \geq r$. □

Contrary to the corresponding result on depth, see [1, Prop. 1.2.9], there are no similar estimates for $\mathrm{Hdep}(U)$ and $\mathrm{Hdep}(N)$, as the next and closing example of this section shows; it also illustrates that $\mathrm{Hdep}(M)$ may even exceed $\max\{\mathrm{Hdep}(U), \mathrm{Hdep}(N)\}$.

Example 3.1
Let $\mathfrak{m} := (X_1,\ldots,X_n) \subset \mathbb{F}[X_1,\ldots,X_n]$, *then* $\mathrm{Hdep}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$.

Proof: For all $r \in \mathbb{N}$ we have

$$\begin{aligned}(1-t)^r H_{\mathfrak{m}}(t) = nt \quad &+ \sum_{k=2}^{r} \left[\binom{n+k-1-r}{k} + (-1)^{k-1}\binom{r}{k}\right] t^k \\ &+ \sum_{k=r+1}^{\infty} \binom{n+k-1-r}{k} t^k,\end{aligned}$$

as one shows easily by induction on r. Since

$$r \leq \left\lceil \frac{n}{2} \right\rceil \implies r \leq \frac{n+1}{2} \implies r \leq n+1-r \leq n+k-1-r \quad \text{for } k = 2,\ldots,r,$$

this series is non–negative for $r \leq \lceil \frac{n}{2} \rceil$, and hence $\mathrm{Hdep}(\mathfrak{m}) \geq \lceil \frac{n}{2} \rceil$. Inspecting the second coefficient we see

$$\binom{n+1-r}{2} - \binom{r}{2} = \frac{n(n+1)}{2} - rn \geq 0 \implies r \leq \frac{n+1}{2} \implies r \leq \left\lceil \frac{n}{2} \right\rceil,$$

and this shows $\mathrm{Hdep}(\mathfrak{m}) \leq \lceil \frac{n}{2} \rceil$. □

4 The Numerator of a Hilbert Series

In this section we investigate which Laurent polynomials can be realized as numerator of the Hilbert series of a graded module over some standard graded polynomial ring. A first answer to this question is given by the results of the second section:

Lemma 4.1
Let P be a Laurent polynomial with integer coefficients. The following conditions are equivalent:

a) There exists $d \in \mathbb{N}$ such that the Laurent series $\frac{P(t)}{(1-t)^d}$ is non–negative.

b) There exists $d \in \mathbb{N}$ such that there is a finitely generated graded module over a polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$ with Hilbert series $\frac{P(t)}{(1-t)^d}$.

PROOF: The coefficients of the series $\frac{P(t)}{(1-t)^d}$ are given by an integer Laurent function of polynomial type (again [1, Prop. 4.1.7] or [2, Thm. 5.2.10]). Hence Corollary 2.3 yields a) $\Rightarrow$ b), and the converse is trivial. □

We will to show that the fairly abstract condition a) is equivalent to a simple property of P regarded as a real valued function of a real variable. Another application of Theorem 2.1 leads us to the following statement.

Lemma 4.2
Let $P \neq 0$ be a Laurent polynomial with integer coefficients. If P satisfies the equivalent conditions of Lemma 4.1, then $P(x) > 0$ for all $x \in]0, 1[$.

PROOF: Without loss of generality we may assume $P(1) \neq 0$. By assumption there is $d \in \mathbb{N}$ such that $\frac{P(t)}{(1-t)^d}$ is a non–negative Laurent series. Its coefficients are given by an integer function of polynomial type $d-1$, hence Theorem 2.1 yields a decomposition of the form (3), and after multiplication by $(1-t)^d$ we get

$$P(t) = \sum_{j=0}^{d} Q_j(t) \cdot (1-t)^{d-j}$$

with non–negative Laurent polynomials $Q_0, \ldots, Q_d$. The Laurent polynomial on the right hand side takes positive values for $0 < x < 1$. □

In the sequel we want to prove the converse of this assertion. First we introduce a name for Laurent polynomials which satisfy condition a) of Lemma 4.1 – for technical reasons we also allow non–integer coefficients.

Definition 4.1
A Laurent polynomial $P \in \mathbb{R}[t, t^{-1}]$, $P \neq 0$, is called convenient*, if there exists $d \in \mathbb{N}$, such that*

$$\frac{P(t)}{(1-t)^d}$$

is a non–negative Laurent series.

The following elementary property of convenient Laurent polynomials will be important.

Lemma 4.3
Let $P, Q \in \mathbb{R}[t, t^{-1}]$ be convenient. Then $P \cdot Q$ is also convenient.

PROOF: Let $d, e \in \mathbb{N}$, such that $\frac{P(t)}{(1-t)^d}$ and $\frac{Q(t)}{(1-t)^e}$ are non–negative Laurent series, then the same holds for the product $\frac{P(t) \cdot Q(t)}{(1-t)^{d+e}}$. □

We will show that any $P \in \mathbb{R}[t, t^{-1}]$ which takes positive values in $]0, 1[$ is convenient. This is easily seen for linear polynomials:

Lemma 4.4
Let $P = \alpha t + \beta \in \mathbb{R}[t]$ with $P(x) > 0$ for all $x \in]0, 1[$. Then P is convenient.

PROOF: By assumption we have $\beta \geq 0$, so there is nothing to show for $\alpha > 0$. For $\alpha < 0$ the series

$$\frac{\alpha t + \beta}{1 - t} = \frac{(-\alpha) \cdot (1 - t) + \alpha + \beta}{1 - t} = (-\alpha) + \frac{P(1)}{1 - t}$$

is already non–negative. □

Next we consider quadratic polynomials. This will be the critical step towards the general case.

Lemma 4.5
Let $P(t) = \alpha t^2 + \beta t + \gamma \in \mathbb{R}[t]$ with $P(x) > 0$ for all $x \in]0, 1[$. Then P is convenient.

PROOF: We may assume $P(0) > 0$ and $P(1) > 0$ – otherwise we are actually in the linear case. Also the case $\alpha < 0$ can immediately be reduced to the linear case because of the identity

$$\alpha t^2 + \beta t + \gamma = (-\alpha t)(1 - t) + (\alpha + \beta)t + \gamma.$$

Let therefore $\alpha > 0$. We may moreover assume $\alpha = 1$, since division of a Laurent series by a positive number does not affect the signs of the coefficients. We distinguish three cases by the position of the zeros of P.

i) P has two zeros $x_1 \leq x_2 < 0$: This means $\beta = -(x_1 + x_2) > 0$, so P itself is non–negative.

ii) P has two zeros $1 < x_1 \leq x_2$: Then we have

$$\beta + 2\gamma = -(x_1 + x_2) + 2x_1 x_2 = x_1(x_2 - 1) + x_2(x_1 - 1) > 0,$$

and therefore the series

$$\frac{t^2 + \beta t + \gamma}{(1 - t)^2} = \gamma + \sum_{k=1}^{\infty} \Big((\beta + 2\gamma) + (k - 1) \cdot P(1) \Big) t^k$$

has no negative coefficients.

iii) P has no real zeros: For all $d \in \mathbb{N}$, $d \geq 3$, we have

$$\begin{aligned}
\frac{P(t)}{(1-t)^d} &= \left(t^2+\beta t+\gamma\right) \cdot \sum_{k=0}^{\infty} \binom{k+d-1}{d-1} t^k \qquad (4)\\
&= \gamma + d\gamma t + \sum_{k=2}^{\infty} \gamma \cdot \binom{k+d-1}{d-1} t^k \\
&+ \beta t + \sum_{k=1}^{\infty} \beta \cdot \binom{k+d-1}{d-1} t^{k+1} + \sum_{k=0}^{\infty} \binom{k+d-1}{d-1} t^{k+2} \\
&= \gamma + (d\gamma+\beta)t \\
&\quad + \sum_{k=2}^{\infty} \left[\gamma \cdot \binom{k+d-1}{d-1} + \beta \cdot \binom{k+d-2}{d-1} + \binom{k+d-3}{d-1}\right] t^k \\
&= \gamma + (d\gamma+\beta)t + \sum_{k=2}^{\infty} T_d(k) \cdot \frac{\prod_{j=1}^{d-3}(k+j)}{(d-1)!} \cdot t^k
\end{aligned}$$

with

$$\begin{aligned}
T_d(k) &:= \gamma(k+d-1)(k+d-2) + \beta(k+d-2)k + k(k-1) \\
&= \gamma\Big(k^2+(2d-3)k+(d-1)(d-2)\Big) + \beta\Big(k^2+(d-2)k\Big) + k^2 - k \\
&= (\gamma+\beta+1)k^2 + \Big((2d-3)\gamma + (d-2)\beta - 1\Big)k + \gamma(d-1)(d-2) \\
&= (\gamma+\beta+1)k^2 + \Big((2\gamma+\beta)d - (3\gamma+2\beta+1)\Big)k + \gamma(d-1)(d-2).
\end{aligned}$$

We consider this term as a polynomial in k and compute its discriminant in terms of d:

$$\begin{aligned}
\Delta(d) &= \Big((2\gamma+\beta)d - (3\gamma+2\beta+1)\Big)^2 - 4\cdot(\gamma+\beta+1)\cdot\gamma(d-1)(d-2) \\
&= (2\gamma+\beta)^2 d^2 - 4\gamma(\gamma+\beta+1)d^2 + \text{terms of lower degree in } d \\
&= (\beta^2-4\gamma)d^2 + \ldots
\end{aligned}$$

Since P has no real zeros, its discriminant $\beta^2 - 4\gamma$ is negative. Therefore $\Delta(d)$ is also negative for $d \gg 0$, and hence for large d the quadratic polynomial $T_d \in \mathbb{R}[k]$ has no real zeros. Since its leading coefficient $\gamma+\beta+1 = P(1)$ is positive, T_d takes only positive values for all $x \in \mathbb{R}$. Therefore the series (4) is non–negative for $d \gg 0$, and P is convenient. □

The generalization to arbitrary Laurent polynomials can be obtained by standard arguments:

Theorem 4.6
Let $P \in \mathbb{R}[t,t^{-1}]$ be a Laurent polynomial with $P(x) > 0$ for all $x \in\ ]0,1[$. Then P is convenient.

PROOF: Since multiplication by a power of t or of $(1-t)$ does not affect either of the conditions, we may assume that P is a polynomial with $P(x) > 0$ for all $x \in [0,1]$. By the preceding three lemmata it is enough to show that P admits a presentation as a product of polynomials of degree ≤ 2 which take positive values in $[0,1]$. The Fundamental Theorem of Algebra yields a factorization of P into linear and irreducible quadratic real polynomials. Since P has no zeros in $[0,1]$, this factorization can be written in the form

$$P(t) = \left(\alpha \cdot \prod_{i=1}^{m} (t - x_i)\right) \cdot \left(\prod_{j=1}^{n} (t - y_i) \cdot \prod_{k=1}^{r} \left(t^2 + b_k t + c_k\right)\right)$$

with $x_i > 1$, $y_j < 0$ and irreducible factors $t^2 + b_k t + c_k$. Let us denote the terms in the big brackets by P_1 resp. P_2. The polynomial P_2 already has the desired form, so we only have to consider P_1. Since P and P_2 take positive values in $[0,1]$, the same holds for P_1, while each factor $t - x_i$ takes negative values in $[0,1]$. So m has to be even if and only if $\alpha > 0$. For even m we may write

$$P_1(t) = \Big((t - x_1)(t - x_2)\Big) \cdot \ldots \Big((t - x_{m-1})(t - x_m)\Big) \cdot \alpha,$$

and for odd m we have

$$P_1(t) = \Big((t - x_1)(t - x_2)\Big) \cdot \ldots \Big((t - x_{m-2})(t - x_{m-1})\Big) \cdot \Big(\alpha(t - x_m)\Big).$$

In both cases we arrive at a factorization of the desired form also for P_1, and hence P is shown to be convenient. □

In combination with the Lemmata 4.1 and 4.2 this assertion yields the following characterization of the possible numerators of H_M.

Corollary 4.7
Let $P \neq 0$ be a Laurent polynomial with integer coefficients. The following conditions are equivalent:

a) $P(x) > 0$ for all $x \in\,]0,1[$.

b) There exists $d \in \mathbb{N}$, such that there is a finitely generated graded module $M \neq 0$ over a standard graded polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$ with Hilbert series $H_M(t) = \frac{P(t)}{(1-t)^d}$.

□

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