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Spectral Theory of Differential and Difference Operators in Hilbert Spaces

By

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To my wife Irene Chemeli, daughter Ivy Auma, father Gabriel Nyamwala and late mum Syprose Awino

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Contents

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That every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therefore the necessary failure of all attempts.

David Hilbert, Paris, Wednesday 8th August, 1900.

Summary

There has been interest in the deficiency indices and the spectral analysis of Sturm-Liouville operators for a long time but fairly little has been done regarding the analysis of higher order differential operators. The main reasons for this limitation of study were lack of finer results on asymptotic integration, a theory that relates the asymptotics of the eigenfunctions to the spectral properties of the underlying Hamiltonians and lack of understanding of the zeros of the characteristic Fourier polynomials associated with differential operators of unbounded coefficients. Eastham and others [31, 32, 33, 32] made attempts to approximate the roots of the Fourier polynomials in their analysis of deficiency indices if the roots fall into groups of different sizes.

This study has developed a spectral theory of higher order differential operators with unbounded coefficients if the eigenvalues of the characteristic Fourier polynomials fall into distinct classes or clusters of different magnitudes. With appropriate smoothness and decay conditions, it has been shown that the spectral properties, deficiency indices and spectra of the underlying differential operators are superpositions of the contributions from the individual clusters. The results have been derived via the method of asymptotic integration. These results are based on quantitative improvement of Levinson's Theorem and the methods are applicable to other classes of linear differential operators.

On the other hand, it has been a well known fact for many years that Sturm-Liouville equations and their discrete counterparts, Jacobi matrices, can be analysed by similar and closely related methods. In fact, the spectral theory of Jacobi matrices and Sturm-Liouville operators has been developed entirely in parallel in recent years. Thus it is not surprising that the theory of Jacobi matrices has been developed very far.

This deep understanding of second order difference operators contrasts with an almost complete absence for the results of higher order difference operators. In that respect, the theory does not even parallel that of differential operators. With appropriate smoothness and decay conditions, it has been shown by asymptotic summation that the minimal difference operator with almost constant coefficients is limit point and its self-adjoint extension operator has no singular continuous spectrum and the absolutely continuous spectrum agrees with that of the constant coefficient limiting operator. In particular, if the characteristic limiting polynomial has $2k$ roots of absolute value one, then the absolutely continuous spectrum has multiplicity k. Finally, the absolutely continuous spectrum of a fourth order difference operator with unbounded coefficients has been calculated. These results extend similar results for even order differential operators to the discrete situation.

Terminology and notation

τ = formal symmetric differential expression
$\mathcal{L}$ = formal symmetric difference expression
T, T^* = minimal / maximal differential operator
L, L^* = minimal / maximal difference operator
ρ_α = spectral measure
$M_\alpha(z)$ = M-matrix
$\mathcal{N}_{T^*+i}$ = Null space of $T^* + i \cdot I$
$\mathcal{N}_{T^*-i}$ = Null space of $T^* - i \cdot I$
$\sigma(H)$ = Spectrum of operator H, H is a self-adjoint operator.
$\sigma_p(H)$ = Point spectrum of H
$\sigma_{ac}(H, k)$ = Absolutely continuous spectrum of H of multiplicity k
$\sigma_{sc}(H)$ = Singular continuous spectrum of H
M_k = The kth-cluster M-factor
$\mathcal{T}$ = Diagonalising matrix
$O(.), o(.)$ = Landau Symbols (the 'big-O' and 'little-o')
USI = Uniformly square integrable

Chapter 1

INTRODUCTION

Since about 1900 a vast amount of research has been pursued for Sturm-Liouville operators. In contrast to this, fairly little has been done regarding the analysis of higher order differential operators. Only in the late seventies and early eighties the deficiency index and partially the essential spectrum became the centre of attention. These results were generally shown with the aid of asymptotic integration. Further analysis of these results, however, was not undertaken. The main reasons for this limitation of the study were for one, a lack of finer results on asymptotic integration and secondly, a theory that relates the asymptotics of the eigenfunctions to the spectral properties of the underlying Hamiltonians.

A new interest in the spectral properties of the differential operators was awakened with the surprising construction of a concrete Hamiltonian with singular continuous spectrum by D. Pearson. About the same time, Weidmann showed by means of oscillation theory and mostly classical estimates that Sturm-Liouville operators

$$\tau y = -y'' + (q_1 + q_2)y \quad \text{on} \quad \mathcal{L}^2([0,\infty))$$

with $q_1', q_2 \in \mathcal{L}^1$ possess an absolutely continuous spectrum. This result was later extended and derived by means of asymptotic integration by Behncke. A further important step in spectral analysis was the concept of subordinancy by Pearson. An attempt to extend this method to higher order differential operators was then undertaken by Remling in his thesis. This attempt, though futile, established asymptotic integration as a valuable tool in spectral analysis in conjunction with the theory of the M-matrix. This latter approach was developed by

Atkinson, Hinton and Shaw and refined later by Hinton and Schneider, thus extending an earlier idea of H. Weyl. With this, Remling could prove some results on the spectral theory of fourth order operators, though unbounded middle terms formed a basic obstacle. Along these lines Behncke, Hinton and Remling finally developed the spectral theory for higher even order operators with bounded coefficients satisfying some regularity conditions. With this and some further results on asymptotic integration by Behncke and Hinton, it was clear that one basic obstacle to analysis of absolutely continuous spectrum of operators with unbounded coefficients is the understanding of the zeros of the polynomials, here the Fourier polynomial, with unbounded coefficients. One such class has been singled out in the analysis of the deficiency index by Eastham and others [31, 32, 33]. For this class, the zeros fall into groups of different size.

The aim of this thesis is to develop a spectral theory of higher order differential operators for which the eigenvalues fall into clusters of different magnitude. The study of such special classes of equations was begun with rather limited results on the deficiency index in the seventies and eighties [31, 32, 33, 61]. The general idea was that the individual clusters will contribute independently to the deficiency index. These results were mostly derived for power coefficients. Here, these results and the classes will be extended by allowing much more general coefficients and by including the analysis of the absolutely continuous spectrum as well, whenever possible. This is made possible by a refinement of Levinson's Theorem . In addition, I will develop the spectral theory of higher order linear difference operators. Apart from a general study of the M-matrix, these results are new. Part of these results have been published jointly with Horst Behncke [21, 22].

1.1 Differential Operators

1.1.1 Background Information

The spectral theory of differential operators has attracted the attention of many researchers during the last decades of the 20th century. This work began with Sturm-Liouville operator $\tau y = -(p(x)y')' + q(x)y$ and also with the Schrödinger operator $\tau y = -y'' + q(x)y$. Here, the concentration will be based on the higher order analogue which has been studied far less. These operators are generated by differential equations of the form

$$\tau y = w^{-1}\{\sum_{k=0}^{n}(-1)^k(p_k y^{(k)})^{(k)}$$
$$- i\sum_{j=1}^{n}(-1)^j((q_j y^{(j)})^{(j-1)} + (q_j y^{(j-1)})^{(j)})\}, \tag{1.1}$$

and their odd order counterparts

$$\tau y = w^{-1}\{(-1)^n i(q_{n+1}(q_{n+1}y^n)')^n + \Sigma_{k=0}^n(-1)^k(p_k y^{(k)})^{(k)}$$
$$- i\Sigma_{j=1}^n(-1)^j((q_j y^{(j)})^{(j-1)} + (q_j y^{(j-1)})^{(j)})\} \tag{1.2}$$

defined on $\mathcal{L}^2((0,\infty), w)$. On $\mathcal{L}^2((0,\infty), w)$, the scalar product is defined with respect to the weight function w. In this thesis, we will assume that the coefficients p_k, q_j of the differential operator are real valued and admit a decomposition of the form $f = f_1 + f_2 + f_3$, $f = p_k, q_j$ [10, 61], with f_1 twice differentiable, f_2 once differentiable and f_3 integrable and measurable. In addition, we will have to demand decay conditions for the derivatives. These decay conditions are needed for asymptotic integration. As such, these conditions are extensions of those used by Behncke, Hinton and Remling [26]. These conditions always guarantee that the initial value problem for these operators has a solution. Even though the above conditions are quite general, we will assume for the proofs that the coefficients are twice differentiable, because the extension to the general case is now routine. For the spectral results, these coefficients will be allowed to be unbounded too and the technique will be to define the operator on $[a,\infty)$ where a is a regular endpoint. Then Remling's results [52] will be used to extend these results to $[0,\infty)$. In addition, one needs $p_n, q_{n+1}, w > 0$ so that the lower order terms can be estimated against the leading term.

Any self-adjoint differential expression with sufficiently smooth real valued coefficients can be written in the form (1.1) or (1.2) [49, Theorem I.15.2]. So (1.1) and (1.2) are the natural starting point. The factors $(-1)^k$ ensure that the kth summand in the differential expressions are nonnegative in the quadratic form sense if $p_k \geq 0$.

In the late 19th and early 20th century, many mathematicians have worked on various aspects of the spectral theory of self-adjoint differential operators. It was Hermann Weyl, however, who developed a unified and a far ranging theory of singular formally self-adjoint second-order differential operator. K. Kodaira in his works [45, 46, 47] unified the results of Weyl and Titchmarsh by methods of functional analysis. His works complete the general theory of second-order differential operators and also presented several points of contacts with the development of the spectral theory of differential operators in the general case. The main divergence being Kodaira's use of a method of contracting hypersurfaces generating Weyl's method of contracting circles. Weyl [62], extended the concept of the Green's kernel (resolvent) to a differential operator of second order. This, he did for second order differential operators defined on bounded intervals and later extended the concept to the semi-axis. He also characterised the essential spectrum of Dirac operators. While the theory of Sturm-Liouville operators is rather well developed and extensive by now, the study of differential operators of higher order has hardly evolved beyond the determination of the deficiency index or the essential spectrum. In fact, the status of spectral theory of such operators until 1990 were concisely summarised in the works of Naimark [50], Glazman [38] and partly Weidmann [61].

This line of studies, however, terminated in the seventies, although it represents only the first step in the understanding of higher order operators. In fact, the study of the deficiency index and essential spectrum is generally considerably simpler than the spectral analysis, because they have much better stability properties [16]. Little is known beyond these books. Their results were mainly concerned with the deficiency indices or essential spectrum. As mentioned earlier, such results are easier to come by, because the deficiency index (essential spectrum) is stable under Kato-bounded (relatively compact) perturbations. In addition, the method of Weyl sequences is an effective tool to analyse the essential spectra. On the other hand, absolutely continuous spectrum is only stable under trace class perturbations. Previously, it has been studied mainly via scattering techniques and physicists generally equated essential spectrum with continuous spectrum.

These results, however, concern only the essential spectrum. So far, the only thorough analysis of some class of fourth order differential operators has been undertaken by C. Remling [52, 53], although he restricted himself to two narrow classes singled out by Eastham [33] in his study of deficiency index. In fact, Remling's results were the first step beyond Sturm-Liouville operators. Meanwhile, Behncke, Hinton and Remling [26] and Behncke [7], have analysed differential operators of order $2k$ with almost constant coefficients by the same methods, generalising the results of Behncke [6], Naimark [50] and Weidmann [59]. It is well known, however, that sparse or oscillatory potentials may lead to a singular continuous spectrum and all sorts of spectral anomalies with Sturm-Liouville operators. Most likely, these results will hold for higher order operators as well. However, techniques like Prüfer transform, transfer-matrices or subordinacy, which are essential for the analysis are not available for these more general classes.

The basic technique that one can use instead of subordinacy analysis is the asymptotic integration of the equations with z, the spectral value, as a parameter. Asymptotic integration theory may be considered as a generalisation of the well known WKB-method of Schrödinger operators. In this case, the asymptotics of the eigenfunctions is typically determined by an exponential factor. This in turn excludes singular continuous spectrum, which is generally connected with non-exponential decay. In addition, singular continuous spectrum is unstable with respect to small finite rank perturbation. This makes asymptotic integration unsuitable for the study of singular continuous spectrum. For the study of spectral theory by means of asymptotic integration, it is necessary to extend the classical Levinson type results to the spectral parameter dependent case. This then leads to estimates of the M-matrix [40, 43]. For this, however, regularity conditions on the coefficients which combine smoothness and decay are always important and simple examples with second order operators, for example, $-y'' + A(\cos x^{\alpha})y = zy$ show that these conditions are almost necessary. Such properties have been used previously by Weidmann [59] and Behncke [6] for Sturm-Liouville operators. Behncke [10] showed that a differential operator of order $2n$ has $\sigma_{ac}(H)$ of spectral multiplicity k if there are $2k$ bounded and $n-k$ exponentially increasing and $n-k$ exponentially decreasing solutions.

Eastham [33, Sect. 3.3], has analysed by asymptotic integration, the asymptotics of solutions of higher order linear differential operators and their deficiency indices. For him, unbounded coefficients posed a serious problem [33, Sect. 3.8]. Meanwhile, improved estimates have remedied this situation. So now, it is essentially the uniform dichotomy condition that poses a serious obstacle to understanding higher order differential operators. This will be extended and generalised to even and odd higher order differential operators with unbounded coefficients. The spectrum of these operators with eigenvalues of distinct sizes will also be investigated. Since now the situation where all eigenvalues of the Fourier polynomial are of the same magnitude is fairly well understood, the challenge lies with the complementary case.

1.1.2 Basic Notations

The study is based on the operators generated by the formal differential expression τ (1.1) or (1.2). Let T be an operator. The symbol $D(T)$ will be used to denote the domain of the operator T. The other notations of this thesis are standard to those of Weidmann [60]. The deficiency index, def$T$, is then defined as the pair

$$\text{def}T = (\text{dim}N_{T^*-i}, \quad \text{dim}N_{T^*+i}).$$

N_{T^*-i} is the null space of $T^* - iI$ and N_{T^*+i} is the null space of $T^* + iI$. Thus N_{T^*-i} is the set of all elements such that $\tau y = iy$. If one uses a nonreal complex spectral parameter z, then for Im$z > 0$, one has dimN_{T^*-i} = dimN_{T^*-z} and dimN_{T^*+i} = dim$N_{T^*-\overline{z}}$. Although the definition of the deficiency indices depend on z, as a consequence of the closed symmetric nature of T, the dimension of the null spaces are independent of z provided that z remains either in the upper or lower half-planes. For Im$z > 0$, N_+ and N_- will denote dim$N_{T^*-\overline{z}}$ and dimN_{T^*-z} respectively. N_+ and N_- may be finite or infinite. Thus def$T = (N_-, N_+)$.

It is worth noting that some authors do switch the definition of the deficiency indices of the operator T. This is evident in the way def$T$ is defined in [61] and [33]. Throughout this study, the deficiency indices results will be given in accordance with the definition given above which is in line with the definition given in [33].
It follows immediately from [60, Theorem 5.21], that T is essentially self-adjoint if and only if $N_- = N_+ = 0$. The deficiency index problem is thus a problem of determining def$T$ or conditions so that $N_+ = N_-$. In the latter case, one would also want to determine the self-adjoint extensions as well as their spectral properties.

The spectrum of a self-adjoint extension will be denoted by $\sigma(H)$ while various components of the spectrum will be denoted as follows: absolutely continuous spectrum $\sigma_{ac}(H)$, discrete spectrum (point spectrum) $\sigma_p(H)$, singular continuous spectrum $\sigma_{sc}(H)$ and essential spectrum

$\sigma_{ess}(H)$. $\sigma_{ac}(H,k)$ will denote absolutely continuous spectrum of H of multiplicity k. In addition, $I \subset \sigma_{ac}(H,k)$ will mean that I belongs to the set $\sigma_{ac}(H,k)$. If $f(x)$, $g(x)$ are complex valued functions with $x \in [a,\infty)$, one writes $f(.) \in \mathcal{L}^p$ if $\int_a^\infty \|f(x)\|^p < \infty$ and

$$\mathcal{L}_0^p = \{f \in \mathcal{L}^p \mid f(x) \to 0, \text{ as } x \to \infty\}.$$

The matrix function $R(.) \in \mathcal{L}^p$, for R_{ij}, will imply that $\int_a^\infty \|R_{ij}(x)\|^p < \infty$, $i,j = 1,2,\ldots,2n$ for even order case or $i,j = 1,\ldots,2n+1$ for odd order case. Assume $f(x)$ to be a coefficient of the differential operator T, then $f(x) = (1+h_f(x))x^\alpha$, $h_f(x) \in \mathcal{F}_l$ such that

$$\mathcal{F}_l = \{h(x) \mid h^{(k)}(x) = o(x^{-k}), \quad 0 \le k \le l\}$$

will be used to denote approximate power type coefficients . The notation $f(x) \approx g(x)$ will mean that for some $a \ge 0$, there exists a constant $C = C(a) > 0$ such that

$$C^{-1} \mid g(x) \mid \le \mid f(x) \mid \le C \mid g(x) \mid \qquad \forall x \in [a,\infty).$$

Similarly, $f(x,z) \approx g(x)$, $z \in A \times [0,\delta)$, $A \subset \mathbb{R}$ and δ sufficiently small, will define the same relation uniformly in z. Equally, denote by $f(x,z) = o(g(x))$ if $g(x) \neq 0$ and $\lim_{x\to\infty} \frac{f(x,z)}{g(x)} = 0$ is true for all z. Finally, the notations $\ll$ and $\gg$ will mean much smaller than and much greater than respectively in the absolute value sense.

1.1.3 Hamiltonian Systems

It is advantageous in spectral analysis to write a higher order equation as a Hamiltonian system or a first order system. Hamiltonian systems are first order systems with a particular structure that allow an extension of the Weyl M-function calculus. Let T^* and T be maximal and minimal operators respectively, generated by (1.1) or (1.2). In order to define T^* and T, one needs the quasiderivatives. With the assumption that $n \ge 2$ in (1.1) or (1.2) and some obvious reindexing, $k \leftrightarrow n-k$, the quasiderivatives as defined by Walker [58] are given by

$$(1.3) \quad y^{[k]} = y^{(k)} \quad \text{for} \quad 0 \le k \le n-1, \qquad y^{[n]} = p_n y^{(n)} - iq_n y^{(n-1)},$$

$$y^{[n+1]} = -(y^{(n)})' + i(q_n/p_n)y^{[n]} + (p_{n-1} - (q_n^2/p_n))y^{(n-1)} - iq_{n-1}y^{(n-2)},$$

$$y^{[n+k]} = -y^{[n+k-1]'} + p_{n-k}y^{[n-k]} + i(q_{n-k+1}y^{(n-k+1)} - q_{n-k}y^{(n-k-1)})$$

for $2 \leq k \leq n-1$ for even order differential expression. Thus for $n \geq 2$, one has

$$\tau y = w^{-1}[-(y^{[2n-1]})' + iq_1 y' + p_0 y].$$

The quasiderivatives of odd order differential expressions are defined by

$$y^{[k]} = y^{(k)} \text{ for } 0 \leq k \leq n-1, \qquad y^{[n]} = -\theta q_{n+1} y^{(n)},$$

where $\theta = \frac{(1+i)}{\sqrt{2}}$

(1.4) $$y^{[n+1]} = -(\theta q_{n+1})(y^{[n]\prime}) + i(\theta p_n/q_{n+1})y^{[n]} - iq_n y^{(n-1)},$$

$$y^{[n+2]} = -y^{[n+1]\prime} - (\theta q_n/q_{n+1})(y^{[n]}) + p_{n-1}y^{(n-1)} - iq_{n-1}y^{(n-2)}, \text{if}, n \geq 2,$$

$$y^{[n+k+1]} = -y^{[n+k]'} + p_{n-k}y^{[n-k]} + i(q_{n-k+1}y^{(n-k+1)} - q_{n-k}y^{(n-k-1)})$$

for $2 \leq k \leq n-1$.
In this case, one gets for $n \geq 2$

$$\tau y = w^{-1}[-(y^{[2n]})' + iq_1 y' + p_0 y].$$

Thus $D(T^*)$, the domain of T^*, associated to τ consists of all functions y for which $y^{[k]}$ with $0 \leq k \leq 2n-1$ ($0 \leq k \leq 2n$) are absolutely continuous and $T^* y \in \mathcal{L}^2([0,\infty), w) = \mathcal{L}^2_w$. Precisely, this domain is given by

$D(T^*) = \{y \in \mathcal{L}^2((0,\infty) : w) : \quad y^{[0]}, y^{[1]}, \ldots, y^{[2n-1]}$ are absolutely continuous in $((0,\infty) : w)$, $\tau y \in \mathcal{L}^2((0,\infty) : w)\}$, $\tau y = T^* y$ for $y \in D(T^*)$.

$D(T^*)$ is thus the maximal possible domain in $\mathcal{L}^2((0,\infty) : w)$ for which the quasiderivatives make sense. The domain of odd order differential operator can be defined similarly. It is shown in [61] that T^* is densely defined and closed. An operator defined by restricting the domain of the maximal operator only to those functions y with compact support is known as pre-minimal operator. It will be denoted by T_1 and its domain is defined by

$$D(T_1) = \{y \in D(T^*) : \quad y \quad \text{has compact support in} \quad (0,\infty)\}.$$

$T_1 y = \tau y = T^* y$ for $y \in D(T_1)$. For unbounded domains, T_1 is not closed but it is densely defined. The closure of the pre-minimal operator T_1, $\overline{T}_1$, is the minimal operator generated by (1.1) or (1.2) and will be denoted by T. It is obvious that $T \subset T^*$. One can show, however, that $T = T^{**}$. These relations imply that T is symmetric.

Remark 1.1.1. The formulation of these quasiderivatives does not require smoothness or reality conditions, but requires symmetry as shown by Walker [58]. The main advantage to be gained from the quasiderivatives is the formulation of a spectral theory for τ which does not depend too much on the differentiability of the $p_k(x)$ and $q_j(x)$. However, the domain of these operators depends on the differentiability of these coefficients. Thus, the assumption is always that one can compute quasiderivatives from the functions themselves and the coefficients.

If T is an even order differential operator and $n \leq N_- = N_+ \leq 2n$, then a closed symmetric self-adjoint extension of T exists. Classical von Neumann theory shows that these are characterised uniquely by an isometry $V : N(T^* - z) \to N(T^* + z)$, $\mathrm{Im} z > 0$, (von Neumann first and second formulae) [60, Theorem 8.11 and 8.12] such that the domain of the self-adjoint extension H_V of T, $D(H_V)$, with respect to the isometric mapping V is given by the following von Neumann formulae.

$$D(H_V) = D(T) \dotplus N(T^* - z) \dotplus N(T^* + z) \quad \text{or}$$

$$D(H_V) = D(T) \dotplus \{y + Vy : y \in N(T^* - z)\}.$$

Here, the symbol $\dotplus$ denotes the direct sum in the vector space sense. Therefore, every self-adjoint realisation H_V of τ, which is a differential operator in a natural sense, is a restriction of T^* so that $T \subset H_V = H_V^* \subset T^*$.

Walker [58] had shown that every formally self-adjoint equation of order $2n$ (1.1) and likewise of order $2n + 1$ (1.2) can be written as Hamiltonian systems of the form

$$\mathcal{J} y'(x) = [z\mathcal{A}(x) + \mathcal{B}(x)]y(x), \tag{1.5}$$

where $\mathcal{A}(x)$, $\mathcal{B}(x)$ and $\mathcal{J}$ are matrices of size $2n \times 2n$ ($(2n+1) \times (2n+1)$ for an odd order case) and y is a function with values in $\mathbb{C}^{2n}(\mathbb{C}^{2n+1})$. Furthermore, in (1.5), one has for even order operators

$$\mathcal{J} = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}, \qquad \mathcal{A} = \operatorname{diag}(w, 0, \dots, 0),$$

with an assumption that $\mathcal{A}(x), \mathcal{B}(x)$ are locally integrable in the underlying interval $[a, \infty)$ and that $\mathcal{B}(x) = \mathcal{B}^*(x)$, $\mathcal{A}(x) > 0$ (in the positive definite sense), almost everywhere. For odd order operators, one takes

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & -I_n \\ 0 & i & 0 \\ I_n & 0 & 0 \end{pmatrix}.$$

It is this form of $\mathcal{J}$ that made the inversion of the last n coordinates of the quasiderivatives necessary. There are of course many ways of writing (1.1) and (1.2) as first order system and (1.5) is just but one of them. For this to make sense, the system in (1.5) will have to satisfy the following regularity condition.

Condition 1.1.2 (Regularity Conditions). (i) If y satisfies $\mathcal{J}y' - \mathcal{B}y = z_0\mathcal{A}y$ for some y and some z_0 with $\|y\|_{\mathcal{A}} = 0$, then $y = 0$ and $\mathcal{J}y' - \mathcal{B}y = \mathcal{A}f$ with $\|y\|_{\mathcal{A}} = 0$, then $\|f\|_{\mathcal{A}} = 0$.
In this case, this condition holds automatically for all $z \in \mathbb{C}$ (see, [40]).

(ii) In order to express the higher order quasiderivatives by the lower ones, we demand that our system should satisfy the following regularity condition as well. The equation

$$\begin{bmatrix} 0_s & 0 \\ 0 & I_{2n-s} \end{bmatrix} (\mathcal{J}y' - \mathcal{B}y) = 0 \tag{1.6}$$

can be solved uniquely for $y_{s+1}, \dots, y_{2n}$ in terms of $y_1, \dots, y_s$ and formal derivatives of these first s components. For this condition, see Hinton and Schneider [40].

The $(2n - s)$ equations in (1.6) coincide with the last $(2n - s)$ components of (1.5). Thus, the action of the differential operator to be defined is described by the formal differential operator τ

$$\tau y = \begin{bmatrix} \mathcal{A}_1^{-1}(x) & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{J}y' - \mathcal{B}y), \tag{1.7}$$

where in our case $\mathcal{A}_1 \in \mathbb{C}^{s \times s}$ and $\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{bmatrix}$ with
$\mathcal{A} = \operatorname{diag}(w(x), 0, \dots, 0)$. The equation $\tau y = zy$ is equivalent to (1.5).

For (1.7) to make sense, one needs (1.6) satisfied for the type of operators studied here. Thus, the differential operator generated by τ is defined on the Hilbert space $\mathcal{H}_{\mathcal{A}} = P_s\mathcal{L}^2([0,\infty))$ consisting of the equivalence classes of measurable $\mathbb{C}^s$-valued functions f with

$$\int_0^\infty f^*(x)\mathcal{A}(x)f(x)d(x) < \infty.$$

The regularity condition, Condition 1.1.2, is thus needed to get a well defined operator via (1.7). In the definition of the Hilbert space $\mathcal{H}_{\mathcal{A}}$, P_s is used to denote the projection on the first s components of a vector in $\mathbb{C}^{2n}$. The scalar product is then given by

$$\langle f, g\rangle_w = \int_0^\infty \overline{f}_1(x)g_1(x)w(x)d(x),$$

where f_1 and g_1 are the first components of the vector functions f and g respectively. Moreover, a square integrable solution y of (1.5) can be viewed as a Hilbert space element.

In order to write a higher order equation as a first order system, let $u = (y^{[0]}, \dots, y^{[n-1]}, y^{[2n-1]}, \dots, y^{[n]})^t$. Then the equation $\tau y = zy$ leads to the following system for even order differential operator

$$u' = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} u = \mathcal{C}u. \tag{1.8}$$

The nonzero matrix elements of A, B and C are given by

$$A_{j,j+1} = 1, \quad A_{n,n} = i\frac{q_n}{p_n}, \quad B_{n,n} = p_n^{-1},$$

$$C_{j,j} = p_{j-1}, \quad C_{j,j+1} = iq_j = -C_{j+1,j} \quad j = 1, \dots, n.$$

Here, p_0 and p_{n-1} should be read as $p_0 - zw$ and $p_{n-1} - \frac{q_n^2}{p_n}$, where z is the spectral parameter.
With a similar argument, if $u = (y^{[0]}, \dots, y^{[n]}, y^{[2n]}, \dots, y^{[n+1]})^t$, then one obtains a first order system of (1.2) of the form

$$u' = \mathcal{C}u \text{ with } \mathcal{C} = \begin{pmatrix} A & b & 0 \\ c^t & r & b^t \\ C & c & -A^* \end{pmatrix}, \quad r = i\frac{p_n}{q_{n+1}^2},$$

where the nonzero entries of the $n \times n$ matrices A, C and n-vectors b, c are given by

$$A_{j,j+1} = 1, \quad j = 1, \dots, n-1; \quad b_n = -(\theta q_{n+1})^{-1}, c_n = -\theta q_n / q_{n+1},$$

$$C_{11} = p_0 - zw, \;\; C_{jj} = p_{j-1}, j = 2, \dots, n, \;\; C_{j+1,j} = -iq_j,$$

$$C_{j,j+1} = iq_j, j = 1, \dots, n-1, \;\; \theta = \frac{1}{\sqrt{2}}(1+i).$$

The first order system (1.8) and its odd order counterpart are unitarily equivalent to the Hamiltonian systems and thus their spectral results are identical provided the regularity conditions hold.

Throughout, the analysis will be done in the formal framework of linear Hamiltonian systems. In [43], Hinton and Shaw and in [40], Hinton and Schneider have developed the theory of M-matrix of Hamiltonian systems that one can use to compute the spectra of H. It follows that the basic tool is the M-matrix which turns out to be almost the Borel transform of the spectral measure of the underlying Hamiltonian. Through a standard inversion theorem, the spectral measure can be reconstructed from the M-matrix. Similarly, the M-matrix can be obtained from the eigenfunctions of H. So the M-matrix is the ideal tool that connects spectral properties of H with those of its eigenfunctions.

The underlying interval generally is $[0, \infty)$, though in the context of asymptotic integration, we may use $[a, \infty)$, where a is a very large boundary regular endpoint. It follows from Remling's results [52] that the spectral results from $[a, \infty)$ can be extended to $[0, \infty)$. So in general, we will be rather casual about the interval. With the unbounded coefficients, the limit point conditions are rather rare and quite often one finds

$$\text{def}T = (n+r, n+r), \qquad 0 \le r \le n. \tag{1.9}$$

In this case, r further boundary conditions at ∞ are needed.

Weidmann [61, Theorem 4.1] has given the necessary and sufficient conditions for the existence of self-adjoint extensions whenever τ is defined on a bounded interval (compact situation). If the differential operator has equal deficiency indices, its self-adjoint extensions can be described more explicitly in terms of boundary conditions. This essentially needs the aid of Green's formula which one obtains by integration by parts. Green's formula similarly follows from Lagrange's identity whose version for even order differential operators is stated in the following

Theorem 1.1.3. *Let $f, g : (a,b) \to \mathbb{C}^n$ be such that $f = f^{(0)}, \dots, f^{(2k-1)}$, $g = g^{(0)}, \dots, g^{(2k-1)}$ ($n = 2k$) are absolutely continuous. Then for almost all $x \in (a,b) \subset (0,\infty)$, we have the Lagrange identity*

$$\langle w(x)\tau f(x), g(x)\rangle - \langle w(x) f(x), \tau g(x)\rangle = \frac{d}{dx}[f,g]_x,$$

where

$$[f,g]_x = \Sigma_{j=1}^{k}\{\langle f^{(j-1)}(x), g^{(2k-j)}(x)\rangle - \langle f^{(2k-j)}(x), g^{(j-1)}\rangle\}.$$

The proof can be obtained from [61, Theorem 2.2]. One can easily state the odd order version of Theorem 1.1.3. Given an interval $[a,b]$, the Green's formula for this interval is given by

$$\int_a^b [y_1^*(x)A(x)\tau y_2(x) - \tau y_1^*(x)A(x)y_2(x)]dx = y_1^*(x)\mathcal{J}y_2(x) \mid_a^b$$

with

$$y_1^*(x)\mathcal{J}y_2(x) \mid_a^b = y_1^*(b)\mathcal{J}y_2(b) - y_1^*(a)\mathcal{J}y_2(a) = [y_1, y_2]_a^b.$$

This shows that for all $y_1, y_2 \in D(T^*)$, $\lim_{x\to\infty} y_1^*(x)\mathcal{J}y_2(x)$ exists for a regular point a. This makes the Hamiltonian form so convenient.

Lemma 1.1.4.

$$D(T) = \{y \in D(T^*) \mid y(a) = 0 \quad \textit{and}$$

$$\left. \lim_{x\to\infty} y^*(x)\mathcal{J}y_1(x) = 0, \quad \forall y_1 \in D(T^*)\right\}.$$

In the remainder, we will consider only separated boundary conditions [61], because the boundary conditions affect at most the point spectrum in our case. The boundary condition at the left endpoint 0 can be described as

$$(\alpha_1, \alpha_2)y(0) = 0,$$

where α_1, α_2 are n by n complex-valued matrices described with $\text{rank}(\alpha_1, \alpha_2) = n$ and

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n, \qquad \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0_n. \tag{1.10}$$

With a a regular boundary point, the boundary conditions at a are given by [40]

$$(\alpha_1, \alpha_2)y(a) = 0. \tag{1.11}$$

In the limit point case, a boundary condition at infinity is not necessary and hence by [61, Theorem 4.8], one has

$$\lim_{x\to\infty} y^*(x)\mathcal{J}y_1(x) = 0 \qquad \forall y, y_1 \in D(T^*).$$

This can be shown by using Green's formula. One thus defines a self-adjoint extension H_α of T by

$$D(H_\alpha) = \{y \in D(T^*) \mid \quad (\alpha_1, \alpha_2)y(0) = 0\}.$$

As noted above, we will only consider separated boundary conditions here because all self-adjoint extensions have unitarily equivalent absolutely continuous parts. If $r \geq 1$ in (1.9), then T is non-limit-point and additional boundary conditions at infinity are needed. These are given as

$$\lim_{x\to\infty} w_k^* \mathcal{J}y(x) = 0.$$

The functions $w_1, \ldots, w_r$ are linearly independent modulo $D(T)$ at infinity and may be chosen as eigenfunctions of $T^*w_j = zw_j$, $z \in \mathbb{C}\backslash\mathbb{R}$. They also satisfy $\lim_{x\to\infty} w_k^*(x)\mathcal{J}w_j(x) = 0$ for $j, k = 1, \ldots, r$ [40]. Define as in [40]

$$F(z) = \left\{f \in \mathbb{C}^{2n} : Y_z f \in \mathcal{L}_A^2[a, \infty)\right\}, \quad \mathrm{Im} z \neq 0 \quad \text{and}$$

$$R_\infty = \{y \in D(T^*) : (y_i^* \mathcal{J}y)(\infty) = 0, \quad \forall y_i \in D(T^*)\}.$$

Here, $Y_z = Y_\alpha(., z)$ and $(y_i^*\mathcal{J}y)(\infty)$ denote the limits $\lim_{x\to\infty} y_i^*(x)\mathcal{J}y(x)$ for $y, y_i \in D(T^*)$. From the results of Schneider [56, Theorem 6.23], it follows that $\dim(D(T^*)/R_\infty) = 2r$. One thus obtains the following result.

Lemma 1.1.5. *Let*

$$D(\widetilde{T^*}) = \left\{y \in D(T^*) : (w_j^*\mathcal{J}y)(\infty) = 0, \quad 1 \leq j \leq r\right\}, \quad \widetilde{T}^*y = T^*y$$

and

$$C(z) = \left\{f \in F(z) : (w_j^*\mathcal{J}Y_z f)(\infty) = 0, \quad 1 \leq j \leq r\right\}, \quad Im z \neq 0$$

then it follows that

(i) $D(\widetilde{T}^*) = R_\infty \dot{+} L(w_1, \ldots, w_r)$

(ii) $\widetilde{T}$ *has deficiency index* (n, n)*, that is,* $dim C(z) = n$ *and*

(iii) $[(Y_{z_1} g)^* \mathcal{J}(Y_{z_2} f)](\infty) = 0$ *for* $f \in C(z_2), g \in C(z_1)$.

For proof, see [1, 40, 56]. The Lemma shows how to construct a limit-point differential operator from the domain of a non-limit-point differential operator. The key idea to introduce the "limit point" operator $\widetilde{T}$, is to extend Remling's results [52] to the non-limit-point condition.

Theorem 1.1.6. *Let* T *be a formally symmetric differential operator of order* $2n$ *on the interval* $[a, \infty)$ *for which* a *is a regular boundary endpoint. Assume the deficiency index of* T *is* $(n+r, n+r)$*. Then the self-adjoint extensions* H *of* T *with separated boundary conditions are defined by the domain*

$$D(H) = \{y \in D(T^*) \mid (\alpha_1, \alpha_2) y(a) = 0,$$

$$\left. \lim_{x \to \infty} w_k^*(x) \mathcal{J} y(x) = 0, \quad k = 1, \ldots, r \right\}.$$

Here, α_1, α_2 *satisfy the conditions in (1.10) and (1.11). The functions* $w_1, \ldots, w_r$ *are linearly independent modulo* $D(T)$ *at infinity and may be chosen as eigenfunctions of* $T^* w_j = z w_j$*,* $z \in \mathbb{C} \setminus \mathbb{R}$*. They also satisfy* $\lim_{x\to\infty} w_k^*(x) \mathcal{J} w_j(x) = 0$ *for* $j, k = 1, \ldots, r$

The proof follows at once from [61, Theorems 4.6, 5.4b and 5.5] and [52]. From the definition of the operator $\widetilde{T}^*$ in Lemma 1.1.5, it is true that for any self-adjoint extension H of T, $T \subset \widetilde{T} \subset H \subset \widetilde{T}^* \subset T^*$ and from Theorem 1.1.6, it follows that $H = H_\alpha$ with

$$D(H_\alpha) = \{y \in D(\widetilde{T}^*) : \quad (\alpha_1, \alpha_2) y(a) = 0\}, \tag{1.12}$$

where α_1, α_2 satisfy the conditions in (1.10). Thus every self-adjoint extension of $\widetilde{T}$ is an extension of T with separated boundary conditions.

1.1.4 M-Matrix

The M-matrix generalises the m-function of Weyl Titchmarsh and thus relates the asymptotics of the eigenfunctions of higher order differential operators to the spectrum of their self-adjoint realisations. We will assume that the regularity conditions, Condition 1.1.2 hold.

Let $Y_\alpha(.,z) = (U_\alpha(.,z), V_\alpha(.,z))$ be the fundamental matrix of (1.5) with initial values

$$Y_\alpha(a,z) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix}, \quad \alpha_1, \alpha_2 \text{ satisfy (1.10) and (1.11).} \tag{1.13}$$

U_α, V_α are $2n$ by n complex-valued matrices whose every column solves $\tau u = zu$. Note that $V_\alpha(.,z)$ satisfy the boundary conditions at a. Therefore, the columns of $Y_\alpha(.,z)$ span the $2n$-dimensional vector space of solutions of (1.5).

In the limit point case, self-adjoint extensions are realised by fixing boundary conditions at a. Now fix the boundary conditions to the right through $\alpha = (\alpha_1, \alpha_2)$ and using the techniques of Hinton and Shaw [43], for Im$z \neq 0$, the M-matrix, $M_\alpha(z) \in \mathbb{C}^{n\times n}$ is defined by

$$\chi_\alpha(x,z) = Y_\alpha(x,z) \begin{bmatrix} I_n \\ M_\alpha(z) \end{bmatrix} \in \mathcal{L}^2_{\mathcal{A}}[a,\infty).$$

$M_\alpha(z)$ is analytic for Im$z \neq 0$ and Im$M_\alpha(z)$ is positive definite in the upper half plane. The columns of $\chi_\alpha(x,z)$ form a basis for the square integrable solutions of (1.5). $M_\alpha(z)$ is thus a Herglotz function and all the properties of the classical m-function for a second order equation are satisfied for this more general M-matrix [5, 40, 43]. In the non-limit-point situation, the construction of M-matrix is based on $\widetilde{T}$.

Theorem 1.1.7. *For all $z \in \mathbb{C}$ with $Imz \neq 0$, there exists a unique regular $n \times n$ matrix $M = M_\alpha(z)$ with*

(i) *The columns of $\chi_\alpha(.,z) = U_\alpha(.,z) + V_\alpha(.,z)M_\alpha(z)$ are square integrable solutions of (1.5) in the domain of $\widetilde{T}^*$.*

(ii) $\lim_{x\to\infty} w_j^*(x)\mathcal{J}\chi_\alpha(x,z) = 0$, $1 \leq j \leq r$, *the functions w_j are those in Theorem 1.1.6.*

(iii) $\lim_{x\to\infty}(\chi_\alpha(x,z_1)^*\mathcal{J}\chi_\alpha(x,z_2)) = 0$, $\forall Imz_1, Imz_2 \neq 0$ *where $(U_\alpha(.,z), V_\alpha(.,z))$ is the fundamental matrix of (1.5) and with initial values of (1.13).*

(iv) *$M(z)$ is a Herglotz function.*

The proof of this theorem follows from [40, Theorem 4.7] and [1, Theorem 2.4]. The theorem also implies that the M-matrix, $M_\alpha(z)$ is uniquely determined. One therefore obtains the following result as a Corollary to Theorem 1.1.7.

Corollary 1.1.8. *For $Imz_i \neq 0$ $(i = 1, 2)$, it follows that*

$$M_\alpha(z_1) - M_\alpha^*(z_2) = (z_1 - z_2)\int_a^\infty \chi_\alpha^*(x, \overline{z}_2)\mathcal{A}(x)\chi_\alpha(x, z_1)dx.$$

It is also true for $Imz \neq 0$ that $M_\alpha^*(\overline{z}) = M_\alpha(z)$ and

$$\begin{aligned} \langle \chi_\alpha(x,z), \chi_\alpha(x,z)\rangle &= \int_a^\infty \chi_\alpha^*(x,z)\mathcal{A}(x)\chi_\alpha(x,z)dx \\ &= \frac{ImM_\alpha(z)}{Imz} \end{aligned} \tag{1.14}$$

where $ImM_\alpha(z) = \frac{(M_\alpha(z) - M_\alpha^*(z))}{2i}$.

The proof of this corollary can be obtained from [1, Corollary 2.5].

One can show with the aid of the associated resolvent that $M_\alpha(z)$ is holomorpic in both the upper and lower half planes, see Lemma 1.1.11. Now let

$$\mathcal{J}y' = [z\mathcal{A}(x) + \mathcal{B}(x)]y + \mathcal{A}(x)f(x) \tag{1.15}$$

be an inhomogeneous differential problem with $f(.) \in \mathcal{L}^1_{loc}[a, \infty)$. Then its solution, that is, the resolvent of self-adjoint realisation H can be derived via the following lemma.

Lemma 1.1.9. *The inhomogeneous equation (1.15) with initial values $y(a) = y_a \in \mathbb{C}^{2n}$ has a unique solution given by*

$$y(x,z) = Y(x,z)\left[\begin{bmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{bmatrix} y_a - \mathcal{J}\int_a^x Y^*(t,\overline{z})\mathcal{A}(t)f(t)dt\right] \tag{1.16}$$

where $Y(., z), Y(., \overline{z})$ denote the fundamental matrices of the homogeneous equation (1.5) with initial values (1.13).

For the proof, see Weidmann [61, Theorem 2.1] and Hinton and Schneider [40, Theorem 5.1].

Now let Im$z \neq 0$ and $f(.) \in \mathcal{L}^2_{\mathcal{A}}[a, \infty)$, one then obtains the following representation of the resolvent.

Lemma 1.1.10. *The resolvent of an arbitrary self-adjoint extension H of T is given by*

$$((H - z)^{-1}P_s f)(x) = P_s \int_a^\infty G(x,t,z)\mathcal{A}(t)f(t)dt \tag{1.17}$$

with

$$G(x,t,z) = \begin{cases} \chi_\alpha(x,z)V_\alpha^*(t,\overline{z}) : \ t < x \\ V_\alpha(x,z)\chi_\alpha^*(t,\overline{z}) : \ t > x \end{cases}$$

For the proof, see [40]. It follows from this lemma that if Im$z \neq 0$, $f(.) \in \mathcal{L}^2_{\mathcal{A}}[a,\infty)$ and

$$y(x,z,f) = \int_a^\infty G(x,t,z)\mathcal{A}(t)f(t)dt$$

$$= \chi_\alpha(x,z)\int_a^x V_\alpha^*(t,\overline{z})\mathcal{A}(t)f(t)dt + V_\alpha(x,z)\int_x^\infty \chi_\alpha^*(t,\overline{z})\mathcal{A}(t)f(t)dt,$$

then the columns of χ_α are square integrable and $y(.,z,f)$ is well defined for every $x \in [a,\infty)$. Since $M_\alpha(z) = M_\alpha^*(\overline{z})$, it follows that

$$G(x,t,z) = \begin{cases} Y(x,z)\begin{bmatrix} 0_n & I_n \\ 0_n & M_\alpha(z) \end{bmatrix} Y^*(t,\overline{z}) : \ t < x \\ Y(x,z)\begin{bmatrix} 0_n & 0_n \\ I_n & M_\alpha(z) \end{bmatrix} Y^*(t,\overline{z}) : \ t > x \end{cases}.$$

$G(.,.,z)$ is thus continuous in (x,t) for $x \neq t$. For $t = x$, there may be a jump of $\mathcal{J}$. This is seen as follows. Recall that $Y^*(x,\overline{z})\mathcal{J}Y(x,z) = \mathcal{J}$ and hence for a fixed (x,z), it follows that $\lim_{\delta\to 0^+}(G(x,x+\delta,z) - G(x,x-\delta,z)) = \mathcal{J}$. This jump definition around $\mathcal{J}$ satisfies the self-adjoint boundary conditions at $x = a$ as a function of x. Thus y is also absolutely continuous and satisfies the self-adjoint boundary conditions at a and thus solves the inhomogeneous equation (1.15). This shows that M_α is essentially given by the matrix elements of the resolvent. In fact, one has the following

Lemma 1.1.11. *[52, Lemma 3.1] For* $Imz \neq 0$, *it follows that*

$$M_\alpha(z) = ReM_\alpha(i) + z\|\chi_\alpha(.,i)\|^2 + (z^2+1)\langle\chi_\alpha(.,i),(H_\alpha - z)^{-1}\chi_\alpha(.,i)\rangle. \tag{1.18}$$

Here, $\|\chi_\alpha(.,i)\|^2$ *is an* $n \times n$ *matrix with matrix elements* $(\|\chi_\alpha\|^2)_{kl} = \langle P_s(\chi_\alpha)_k, P_s(\chi_\alpha)_l\rangle_{\mathcal{A}}$ *where* $(\chi_\alpha)_k$ *denotes the kth column of* χ_α. *The final term in the right hand side of (1.18) is interpreted analogously.*

Remark 1.1.12. Lemma 1.1.11 shows that the M-function is essentially the Borel transform of the spectral measure ρ_α of H_α. $\chi_\alpha = Y\begin{bmatrix} I_n \\ M_\alpha(z) \end{bmatrix}$ relates it to square integrable eigenfunctions and (1.19) below allows the reconstruction of ρ_α. It is thus not surprising that the M-function is essentially given by an expectation value of the resolvent. From Lemma 1.1.11, one relates $R = \langle \chi_\alpha, (H_\alpha - z)^{-1}\chi_\alpha \rangle$ with $R_{kl} = \langle P_s(\chi_\alpha)_k, (H_\alpha - z)^{-1} P_s(\chi_\alpha)_l \rangle_{\mathcal{A}}$. The lemma shows also holomorphy of $M_\alpha(z)$ for $\text{Im}z \neq 0$ as well as Herglotz property. The absolutely continuous spectrum of the self-adjoint extensions is independent of the boundary conditions and thus the index α will be dropped.

Since $M(z)$ is holomorphic for $\text{Im}z \neq 0$ and satisfies $\text{Im}M(z) \geq 0$, it implies that $M(z)$ has a Herglotz representation with a uniquely determined matrix-valued non-decreasing function $\rho_\alpha(t)$ on $\mathbb{R}$ whose components are right continuous. The density of the absolutely continuous part of ρ_α is given by

$$\rho_\alpha(a,b) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_a^b \text{Im}M_\alpha(t + i\epsilon)dt$$

for an interval (a,b) provided that $\rho_\alpha(\{a\}) = \rho_\alpha(\{b\}) = 0$, is the spectral Borel measure of H, that is, there is a unitary transformation $U : \mathcal{H}_A \to \mathcal{L}^2(\mathbb{R}, d\rho_\alpha)$ with $UHU^* = T_{id}$. H is thus unitarily equivalent to the operator multiplication by the independent variable in the space $\mathcal{L}^2(\mathbb{R}, d\rho_\alpha)$ (see, [3]), $(UHU^*g)(x) = xg(x)$.
The measure ρ_α can be recovered from $M_\alpha(z)$ using the following formula

$$d\rho_\alpha(t) = \frac{1}{\pi} w^* \lim_{\epsilon \to 0^+} \text{Im}M_\alpha(t + i\epsilon)dt. \tag{1.19}$$

It is this representation of $M_\alpha(z)$ in terms of spectral measure matrix ρ_α which connects it with the spectral properties of the operator H_α.

A transformation at the boundary point a of α to $\beta = (\beta_1, \beta_2)$ satisfying the assumptions in (1.10) and (1.11) and with the fundamental matrix $Y(a,z) = Y(a,z,\beta)$ leads to the following relations [44, 52]

$$M_\beta(z) = (-\gamma_2 + \gamma_1 M_\alpha(z))(\gamma_1 + \gamma_2 M_\alpha(z))^{-1}, \tag{1.20}$$

where $\gamma_1 = \beta_1\alpha_1^* + \beta_2\alpha_2^*$, $\gamma_2 = \beta_2\alpha_1^* - \beta_1\alpha_2^*$.

It follows that all the spectral information is contained in $M(z)$. Thus Lemma 5.1 of [52] says that if there are eigenfunctions with poles and they satisfy the boundary conditions or if there is singular continuous

spectrum around, then $| M(z_+) |= \infty$ which implies that $\mathrm{Im}M(z_+) = 0$ according to the multiplicity. In particular, if $\sigma_{sc}(H) = \emptyset$, then each bounded eigenfunction decreases the rank of $\mathrm{Im}M(z_+)$. This would likewise hold for eigenfunctions associated to $\sigma_{sc}(H)$.
This result can be extended to non-limit-point situation since from Lemma 1.1.5, Theorem 1.1.6 and Theorem 1.1.7, one can construct $\chi_\alpha(.,z)$ from the basis of $N(\widetilde{T}^* - z)$ for $\mathrm{Im}z \neq 0$. One therefore has

Lemma 1.1.13. *Let $\{u_1, \ldots, u_s\}$ such that $u_j \in N(\widetilde{T}^* - \lambda)$ ($1 \leq j \leq s$) be linearly independent and λ real with $L(\{u_j\}) \cap D(H_\alpha) = \{0\}$ for a self-adjoint boundary condition α. Then there are s linearly independent vectors $w_j \in \mathbb{C}^n$ such that*

$$w_j^* ImM_\alpha(\lambda + i\epsilon)w_j = O(\epsilon) \quad when \quad \epsilon \to 0^+.$$

Thus $rankM_\alpha \leq n - s$.

Lemma 1.1.14. *For all $\lambda \in \mathcal{B}$, $\mathcal{B} \subset \mathbb{R}$ a real Borel set. Let $L(w_1(\lambda), \ldots, w_s(\lambda))$ be an s-dimensional subspace satisfying $\lim_{\epsilon\to0^+} w_j^*(\lambda)ImM_\alpha(\lambda+i\epsilon)w_j(\lambda) = 0$, $\forall w_j \in L(w_1(\lambda), \ldots, w_s(\lambda))$ and $\lambda \in \mathcal{B}$. Then the spectral multiplicity of the operator of multiplication by a variable T_{id} in the space $\mathcal{L}^2(\mathcal{B}, d\rho_\alpha)$ is at most $n - s$.*

Proof. This follows immediately from [52, Lemma 5.2] together with spectral representation, and Lemma 1.1.13. □

Theorem 1.1.15. *[52, Theorems 6.1 and 6.3] Let $dimN(\widetilde{T}^* - \lambda) = s$, $\forall\lambda \in \mathcal{B}$, $\mathcal{B}$ a real Borel set. For the boundary condition α, there is no eigenvalue $\lambda \in \mathcal{B}$ of H_α. Let $w_j \in \mathbb{C}^n$ be the corresponding s-linearly independent vectors from Lemma 1.1.13. Assume that $\limsup_{\epsilon\to0^+} \|M_\alpha(\lambda + i\epsilon)\| < \infty$ and $\liminf_{\epsilon\to0^+} v^* ImM_\alpha(\lambda + i\epsilon)v > 0$ for $v \notin L(\{w_j(\lambda)\})$, $1 \leq j \leq s$. Then it follows that $\rho_{sc}^{a,\beta}(\mathcal{B}) = 0$ for an arbitrary left hand endpoint a and an arbitrary boundary condition β, that is, the support of ρ_{sc} lies outside $\mathcal{B}$.*

The asymptotic method that is applied for this analysis makes the essential spectrum coincide with absolutely continuous spectrum while singular continuous spectrum will be absent as implied by Theorem 1.1.15.

1.1.5 Asymptotic Integration

The aim of asymptotic integration is to determine the asymptotics of the eigenfunctions of differential operators. For second order operators, this problem has been studied extensively. The basic problem posed by a differential system (1.8) is that, the solution vectors $u(x)$ cannot normally be written explicitly as expressions involving the entries of the given square matrix $\mathcal{C}(x)$. It is of course this difficulty which creates the challenge and interest in developing a wide range of techniques for investigating the properties of solutions.

One of the most important results in asymptotic integration theory which helps in solving this problem is the Levinson's Theorem. It states that the solutions of a system

$$u'(x) = \{\Lambda(x) + R(x)\}u(x), \qquad \Lambda(x) = \mathrm{diag}(\lambda_i(x)) \tag{1.21}$$

look like the solutions of the unperturbed system $u' = \Lambda u$, if $R(x)$ is sufficiently small and $\Lambda(x) = \mathrm{diag}(\lambda_i(x))$ satisfies a dichotomy condition. In Levinson's original result, small means absolutely integrable. The dichotomy condition amounts to: $\mathrm{Re}\lambda_i(x)$ and $\mathrm{Re}(\lambda_k(x) - \lambda_j(x))$, $k \neq j$, have approximately constant sign modulo $\mathcal{L}^1$ for large x. Actually, the condition is slightly weaker, but for us this suffices. Meanwhile, Levinson's Theorem has been extended in several directions by modifying the dichotomy condition as well as the allowed perturbations. These permissible perturbations will be called Levinson's terms in order to simplify the use of Levinson's Theorem.

For more details, see the book of Eastham [33] or [25]. In spectral theory, the matrix elements and $\lambda_i(x)$ will generally depend also on the spectral parameter z. Thus, one writes $\lambda_i = \lambda_i(x, z)$ for this. In this case, it will be important to prove Levinson's Theorem uniformly in z in order to control the z-dependence of the solutions. The following z-uniform version will suffice [24, 25, 53].

Theorem 1.1.16. *Let $\Lambda(x, z) = diag(\lambda_1(x, z), \ldots, \lambda_{2n}(x, z))$ and $R(x, z)$ be $2n \times 2n$ matrices which for all x, are analytic functions of $z \in \Omega \subset \mathbb{C}$. For any unequal pair of indices i and j, $i, j \in [1, \ldots, 2n]$, assume that $\Lambda = diag(\lambda_1(x, z), \ldots, \lambda_{2n}(x, z))$ satisfies the dichotomy condition uniformly in z, that is, for every unequal pair i, j, $a \leq t \leq x < \infty$, we have*
$Re\{\lambda_i(x, z) - \lambda_j(x, z)\}$ has constant sign modulo $\mathcal{L}^1([a, \infty))$ for all

$z \in \Omega$.
Moreover, assume that $\|R(x,z)\| \le \rho(x)$ with $\rho \in \mathcal{L}^1([a,\infty))$. Then

$$Y'(x,z) = [\Lambda(x,z) + R(x,z)]Y(x,z)$$

has solutions $y_k(x,z)$, $1 \le k \le 2n$, with the asymptotic form

$$Y_k(x,z) = (e_k + r_k(x,z)).\exp(\int_a^x \lambda_k(t,z)dt)$$

where e_k denotes the kth unit vector and $r_k(x,z)$ depends analytically on $z \in \Omega$ and tends to 0 z-uniformly as $x \to \infty$.

The proof of this theorem follow closely that of Theorem 1.3.1 in [33]. One can also obtain similar proofs in [24, 25, 53]. Thus, the theorem gives the form of solutions of the perturbed system whenever the eigenvalues satisfy the uniform dichotomy condition and the perturbing terms are integrable.

In order to transform the first order system into Levinson's form, in our case, we will need standard Kummer Liouville (K.L)-transformation, two diagonalisations and possibly some further $(I+Q)$-transformations after the two diagonalisations. In particular, these transformations are carried out in that order. The standard K.L-transform will be used to simplify the coefficients of the operator and consequently the remaining discussions on asymptotic integration. Diagonalisations help in transforming $\mathcal{C}$ into diagonal and integrable terms. After diagonalisations, any off-diagonal terms that are not integrable could be reduced into integrable terms by applying some more $(I+Q)$-transformations. These are fairly standard and have been investigated in [6, 26, 33]. Finally, before applying Levinson's Theorem, one needs to prove a uniform dichotomy condition for the spectral parameter z.

In order to transform (1.8) into Levinson's form, (1.8) has to be diagonalised. This requires the eigenvalues of $\mathcal{C}$. Expansion of $\det(\mathcal{C}-\lambda\cdot I) = \mathcal{P}$, leads to the characteristic polynomial

$$\mathcal{P}(x,z,\lambda) = \sum_{k=0}^{n}(-1)^k p_k(x)\lambda^{2k} + 2i\sum_{j=1}^{n}(-1)^{j-1}q_j(x)\lambda^{2j-1} - zw \tag{1.22}$$

for even order differential operators while odd order differential operators give

$$\mathcal{P}(x,z,\lambda) = iq_{n+1}\lambda^{2n+1} + \sum_{k=0}^{n}(-1)^k p_k(x)\lambda^{2k} + 2i\sum_{j=1}^{n}(-1)^{j-1}q_j(x)\lambda^{2j-1} - zw. \tag{1.23}$$

Since these polynomials do not have real coefficients, it is advantageous to replace the eigenvalue parameters λ by $-i\nu$. Then the corresponding Fourier polynomials are

$$\mathcal{P}_F(x,z,\nu) = \sum_{k=0}^{n} p_k\nu^{2k} + 2\sum_{j=1}^{n} q_j\nu^{2j-1} - zw \quad \text{and} \tag{1.24}$$

$$\mathcal{P}_F(x,z,\nu) = q_{n+1}\nu^{2n+1} + \sum_{k=0}^{n} p_k\nu^{2k} + 2\sum_{j=1}^{n} q_j\nu^{2j-1} - zw. \tag{1.25}$$

These are polynomials with real coefficients if z is real, reflecting the symmetry of T.

1.1.6 Transformation of the System into Levinson's Form

Kummer-Liouville Transformation:
The main function of Kummer-Liouville (K.L)-transformation is to make the coefficients as simple as possible. It goes back to the transformation $y(x) = \mu(x)z(t)$ which has been used mainly for second order operators. Just as for second order operators, the starting point is

$$y(x) = \mu(x)z(t), \quad t = f(x) \quad \text{with} \quad f'(x) = \gamma > 0. \tag{1.26}$$

At the system level, this amounts to:

$$u(x) = F(x)v(t), \quad t = f(x) \quad \text{with} \quad f' = \gamma > 0, \tag{1.27}$$

where F is a $2n \times 2n$ or $2n+1 \times 2n+1$ matrix for T even or odd order differential operator respectively. In principle, the matrix elements of F can be computed explicitly, see [2, 10]. For more details on the general Kummer-Liouville transformation, see [2, 10, 17].

In our case, we will use a K.L-transformation which transforms p_n and w into 1 for even order operators, and q_{n+1} and w into 1 for odd order operators. This K.L-transformation will be called standard K.L-transformation. In this case, one simply has

$$\gamma = (wp_n^{-1})^{\frac{1}{2n}}, \quad w\mu^2 = \gamma \quad \text{in the even order case and} \tag{1.28}$$

$$\gamma = (wq_{n+1}^{-2})^{\frac{1}{2n+1}}, \quad w\mu^2 = \gamma \quad \text{in the odd order case.}$$

The second relation in each case guarantees that the K.L-transformation even defines a unitary map $U : \mathcal{L}^2_w \to \mathcal{L}^2$ with

$$Uy(x) = y(x)\mu^{-1}(x) = y(g(t)\mu)^{-1}(g(t)) \quad \text{and} \quad g = f^{-1}$$

which makes it ideal for spectral analysis.

Diagonalising Transformations:
The main role of diagonalisations is to transform the matrix $\mathcal{C}$ in (1.8) into a diagonal form.

By considering the resultant or the discriminant of $\mathcal{P}$ and $\partial_\lambda \mathcal{P}$, one can show that there are only finitely many spectral values z for which $\mathcal{P}(\lambda, z)$ has multiple roots. Let $\omega_1 < \omega_2 < \cdots < \omega_k$ denote all of the real spectral values z leading to multiple roots. Following [10], the analysis will be restricted to small complex neighbourhoods of $z_0 \in (\omega_i, \omega_{i+1})$, $i = 0, \ldots, k$, where $\omega_0 = -\infty$ and $\omega_{k+1} = \infty$. For a given $z_0 \in (\omega_i, \omega_{i+1})$, one can now choose $\epsilon > 0$ and $a > 0$ so that $\mathcal{P}(x, \lambda, z) = 0$ has no multiple roots for any

$$z \in \mathcal{K}_\epsilon(z_0) = \{z \mid \ | z - z_0 | \leq \epsilon, \quad \mathrm{Im} z \geq 0\} = \mathcal{K}$$

and $x \geq a$. This is possible because the roots of $\mathcal{P}$ depend analytically on the coefficients. Throughout the proof of some results, it may be necessary to adjust a and ϵ repeatedly. Thus (1.24) and (1.25) have $2n$ and $2n+1$ distinct roots respectively. It is well known that the matrix $\mathcal{T}_0$ having the eigenvectors $\varrho(\lambda, x)$ of $\mathcal{C}$ as its columns, $\mathcal{C}(x)\varrho(x, \lambda) = \lambda(x)\varrho(x, \lambda)$, diagonalises $\mathcal{C}$. The eigenvectors ϱ of $\mathcal{C}$ for the eigenvalue λ are given by, for $1 \leq k \leq n$,

$$\varrho_k = \lambda^{k-1}$$

$$\varrho_{n+k} = -iq_k\lambda^{k-1} + \sum_{r=k}^{n}(-1)^{k+r}p_r\lambda^{2r-k} + \sum_{r=k}^{n-1}(-1)^{k+r}2iq_{r+1}\lambda^{2r-k+1}.$$

It is, however, advantageous to base the transformation $\mathcal{T}$ (diagonalising matrix) on the eigenvectors $w_k = w(x, \lambda_k)$ [33, Sect. 3.1 and 3.3], where

$$w_k(x) = M_k^{-\frac{1}{2}} \varrho(\lambda_k, x) = M_k^{-\frac{1}{2}} (1, \lambda_k, \lambda_k^2, \ldots), \tag{1.29}$$

$$M_k = \partial_\lambda \mathcal{P}(x, z, \lambda_k).$$

Since the roots are distinct, M_k is invertible for all $k = 1, \ldots, 2n$ for even order case or $k = 1, \ldots, 2n+1$ for odd order case. Thus, the transformation $\mathcal{T} = (w_1, w_2, \ldots, w_{2n})$ and

$$u = \mathcal{T} v \tag{1.30}$$

lead to

$$v' = (\Lambda - \mathcal{T}^{-1}\mathcal{T}')v \qquad \text{with} \qquad \Lambda = \operatorname{diag}(\lambda_i(x, z)). \tag{1.31}$$

The matrix elements of $\mathcal{T}^{-1}\mathcal{T}'$ are given by $(\mathcal{T}^{-1}\mathcal{T}')_{ii} = 0$. This is a consequence of the normalisation of the eigenvectors with the factors $M_k^{-\frac{1}{2}}$. Then

$$\begin{aligned}(\mathcal{T}^{-1}\mathcal{T}')_{jk} = (\lambda_k - \lambda_j)^{-1} M_j^{-\frac{1}{2}} M_k^{-\frac{1}{2}} (&\sum_{l=0}^{n} (-1)^l p_l' \lambda_k^l \lambda_j^l \\ &- i \sum_{l=1}^{n} (-1)^l q_l' (\lambda_k^l \lambda_j^{l-1} + \lambda_k^{l-1} \lambda_j^l))\end{aligned} \tag{1.32}$$

[20, 33]. If the coefficients are sufficiently smooth so that $\mathcal{T}^{-1}\mathcal{T}'$ is small, the system (1.31) is almost diagonal. The matrix $\mathcal{T}^{-1}\mathcal{T}'$ can be split up into a Levinson part R and a smooth part S. S is a differentiable remainder while R is absolutely integrable by assumption. Under appropriate conditions on the coefficients, the differentiable part S satisfies $S_{ij}(x, z) \to 0$ for $x \to \infty$ and $S_{ij}(x, z) \in \mathcal{L}^2$ uniformly for $z \in \mathcal{K}_\epsilon(z_0)$. For this we require decay conditions like

$$f' = o(1), \quad f' \in \mathcal{L}^2, \quad (f')^2, f'' \in \mathcal{L}^1, \quad f = p_k, q_j.$$

These decay conditions are of course adopted to asymptotic integration and will be made more precise in Chapter two. Thus the system can be written as

$$v' = (\Lambda + S + R)v \quad \text{with} \quad \Lambda = \operatorname{diag}(\lambda_k(x, z)) \tag{1.33}$$

which can be diagonalised further.

If the system (1.31) is not in Levinson form, a further diagonalising transformation will be needed. This will make sense in general, only if the higher derivatives decay faster. To discuss these further transformations, write (1.31) in the form (1.33) with $\Lambda = \mathrm{diag}(\lambda_i(x, z))$ and assume that the coefficients of the perturbing matrix S are differentiable while R is integrable. While the general method to diagonalise $\Lambda + S$ has been described in [14], a simplified transformation will be used here, that is,

$$v = (I + B)v_1 \tag{1.34}$$

with $B_{ij} = (\lambda_j - \lambda_i)^{-1} S_{ij}$. For this, one needs

$$(\lambda_j - \lambda_i)^{-1} S_{ij} = o(1) \tag{1.35}$$

so that one can form $(I + B)^{-1}$. The transformed system is then

$$v_1' = (\Lambda + S_1 + (1 + B)^{-1} R(1 + B))v_1 \tag{1.36}$$

with $S_1 = -(I+B)^{-1}(B' - SB)$. In general, one should now adjoin the diagonal part of S_1 to Λ, because purely off-diagonal perturbations are easier to handle (cf. [11]). (1.36) shows that further diagonalisations require additional differentiability of the coefficients. But it is only a small part of the terms in S_1, which cannot be differentiated further. These are the terms in B' in which p_k'' or q_j'' appear. Leaving those terms aside, adjoin them to R, the diagonalisation procedure can be repeated. These diagonalisations should be repeated until the off-diagonal terms are suitably small. This of course requires that the terms involving p_k'' and q_j'' are suitably small.

$(I + Q)$*-transformation*:
Basically, there are only two transformations, that is, the Kummer-Liouville transformation which involves transforming the independent variable x and the diagonalisations that transform the system into diagonal form and remainder term. $(I + Q)$-transformations have two functions namely; turning conditionally integrable terms into integrable terms and making the remainder terms after the diagonalisations smaller.

A $(I + Q)$-transformation as described in Eastham's book [33] is of the K.L type but with $F(x) = I + Q$ and $t = x$. Since the coefficients are assumed not to possess conditionally integrable terms, only $(I + Q)$-transformations that make remainder terms after two diagonalisations smaller will be discussed.

Assume that after the diagonalisations there are some off-diagonal terms that are not integrable. Then in this case, one can apply more $(I + Q)$-transformations to make them smaller. The starting point for further $(I+Q)$-transformations is equation (1.31). Assume $-\mathcal{T}^{-1}\mathcal{T}' = R$ is the remainder matrix. Adjoin its diagonal elements to the diagonal elements of Λ such that $R_{ii} = 0$, $i = 1, \ldots, 2n$. Here the explanation will be given for even order operators and the same will hold for odd order operators but with the indices i, j running from 1 up to $2n + 1$. Then with the substitution $v = (I + Q)v_1$, equation (1.31) becomes

$$v_1' = (\Lambda + (1 + Q)^{-1}(\Lambda Q - Q\Lambda + R - Q' + RQ))v_1. \tag{1.37}$$

Necessary for this $(I+Q)$-transformation is $Q = o(1)$. Any further $(I+Q)$-transformation with this assumption will make the remainder terms smaller. These can be repeated severally until all remainder terms are integrable. It is, therefore, necessary to show that the contributions to the diagonals as a result of these $(I + Q)$-transformations do not affect the asymptotics of the solutions. Behncke [13] showed that any decay in excess of integrability for R is reflected in the decay of the remainder terms.

Since the diagonal part of R is trivial by assumption, the equation $\Lambda Q - Q\Lambda + R = Q'$ reduces to $(2n - 1) \cdot 2n$ scalar equations, for which the solutions can be estimated well in terms of the coefficients R_{ij}. These equations are

$$Q_{ij}' = (\lambda_i - \lambda_j)Q_{ij} + R_{ij}, \qquad i \neq j, \qquad i, j = 1, \ldots, 2n. \tag{1.38}$$

For (1.38), solutions are needed that decay at infinity. One now writes (1.38) as

$$q' = \lambda q + r. \tag{1.39}$$

If the dichotomy condition holds for Λ, the solutions of (1.38) respectively (1.39), which are small at infinity are given by

$$q(x, z) = -\int_x^\infty \exp(\mu(x) - \mu(t))r(t)dt \qquad \text{if} \qquad \mathrm{Re}\lambda \geq 0 \tag{1.40}$$

$$q(x,z) = \int_a^x \exp(\mu(x) - \mu(t)) r(t) dt \qquad \text{if} \qquad \text{Re}\lambda < 0 \tag{1.41}$$

where $\mu(x) = \int_a^x \lambda(t) dt$. It has been shown in [13] that if h increases monotonically to ∞ on $[a, \infty)$ and $rh \in \mathcal{L}^1$, then $q = O(h^{-1})$. This implies that $Q(x) \to 0$ as $x \to \infty$ and any decay for R in excess of integrability is passed on to the remainder terms.

In spectral theory, the eigenvalues as well as the remainder terms R_{ij} depend uniformly on the spectral parameter z or even more often $\eta = \text{Im}z$. It is thus important to write (1.39) in terms of z. This can be written as

$$q' = (i\lambda_1 + \eta\lambda_2) q + r \tag{1.42}$$

$$\text{with} \quad \lambda_1, \lambda_2 \quad \text{real valued}, \quad 0 \le \eta \le \epsilon > 0,$$

where η takes the role of Imz. It is likewise shown in [13] that if R satisfies stronger decay conditions, so will r_{ij} the elements of R.

In general, the new remainder term RQ will be smaller. This holds already if $Q \to 0$ and the results of [13] allow to control it. This gives an improvement of (1.36) which needs small off-diagonal terms. This type of transformation may be repeated as well. With all these transformations, one is finally led to a system $w' = (\widetilde{\Lambda} + \widetilde{R})w$ to which Levinson's Theorem can be applied. Here, $\widetilde{\Lambda}$ is Λ plus diagonal terms arising from the various transformations. Thus, the kth solution of this system is

$$w = (e_k + r_k(x,z)) \exp \int_0^x \widetilde{\lambda_k}(t,z) dt. \tag{1.43}$$

The remainder term r_k results from the terminal Levinson system, while $\widetilde{\lambda}_k$ is λ_k plus further diagonal terms arising from the various transformations. Since the coefficients are twice differentiable, after all the procedures of the asymptotic integration, one has the following general solutions of (1.1), (1.2) or (1.8)

$$\begin{aligned} u_k(x,z) = \mathcal{T}(I+B)(I+Q_1)\cdots(I+Q_r) \\ \cdot(e_k + r_k(x,z)) \exp \int_a^x \widetilde{\lambda}_k(t,z) dt. \end{aligned} \tag{1.44}$$

Here, $\mathcal{T}$ and $(I+B)$ are diagonalising matrices, while the Q_i arise from the $(I+Q)$-transformations. In order to analyse (1.44), it is useful to demand *"higher order derivatives of the coefficients decay faster"* .

This can be expressed, for example, for the coefficients $f = p_k, q_j$ as

$$\frac{f^{(k)}}{f} \in \mathcal{L}^{\frac{2}{k}} \qquad k = 1, 2, \tag{1.45}$$

that is, each derivative introduces a factor $\mathcal{L}^k$. If the coefficients are differentiable more often, this condition can be weakened in an obvious fashion. The consequences of such a smoothness and decay condition in addition to (1.35) are

$$B = o(1), \qquad Q_{j+1} = o(Q_j) = o(B), \qquad r_k = o(B), \tag{1.46}$$

so that the solution (1.44) is essentially given by

$$u_k(x, z) = \mathcal{T}(I + B)(e_k + o(1)) \exp(\int_a^x \widetilde{\lambda}_k(t, z)dt). \tag{1.47}$$

Since one is mainly interested in y_k, the first component of u_k and its square integrability, one has with $\mathcal{T}_{1k} = M_k^{-\frac{1}{2}}$

$$\begin{aligned} y_k(x, z) = {} & (M_k^{-\frac{1}{2}} + \sum_j M_j^{-\frac{1}{2}} B_{jk})(e_k + r(x, z)) \\ & \cdot \exp(\int_a^x \widetilde{\lambda}_k(t, z)dt). \end{aligned} \tag{1.48}$$

When the coefficients are bounded, the term $M_k^{-\frac{1}{2}}$ dominates $\sum_j M_j^{-\frac{1}{2}} B_{jk}$. Therefore, the form factor, $M_k^{-\frac{1}{2}} + \sum_j M_j^{-\frac{1}{2}} B_{jk}$, can be estimated by $M_k^{-\frac{1}{2}}$. With unbounded coefficients and eigenvalues of different magnitude, however, the term $\sum_j M_j^{-\frac{1}{2}} B_{jk}$ may be dominant. It also follows that the correction terms to the eigenvalues are integrable under normal circumstances but in case of slower decay and higher order smoothness this may not be true. It is an open problem, whether such higher order corrections can substantially alter the square integrability and thus for example change the spectrum from absolutely continuous to discrete. We conjecture that this will not occur.

1.2 Difference Operators

1.2.1 Background Information

The theory of difference equations has grown at an accelerated pace in the past decade. It now occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole. It is undisputable fact that difference equations appeared much earlier than differential equations and were instrumental in paving way for the development of the latter. It is only recently that difference equations have started receiving as much attention as they deserve.

A very important aspect of the qualitative study of the solutions of difference systems is periodicity. Even though higher order equations are expressible as difference systems, they merit some special attention. It is easy to incoporate the method of variation of constants, the concept of exact and adjoint equations and Lagrange's and Green's identities into this analysis. This is followed by method of generating functions, which is a very elegant technique for obtaining the closed form of solutions of higher order difference equations.

It has been known for many years that Sturm-Liouville equations and their discrete counterparts, Jacobi matrices, can be analysed by similar and closely related methods. In fact, the spectral theory of Jacobi matrices and Sturm-Liouville operators has been developed entirely in parallel in recent years. As an example, note the recent far reaching results of Remling concerning the so called Oracle Theorem [54, 55]. Many Schrödinger type results have their discrete counterparts and often a result in the discrete or continuous sector leads to a result in the other area. It is not surprising that the theory of Jacobi matrices has been developed very far. This deep understanding of second order operators contrasts with an almost complete absence for results of higher order difference operators. In that respect, the theory does not even parallel that of differential operators. For example, there are no results on the deficiency index for higher order difference operators.

While the Titchmarsh-Weyl M-function has long been known to be a valuable tool for spectral analysis of differential operators, presently, there are only three papers that have developed the general M-function theory for difference systems. These are the papers of Clark and

Gesztesy [29], Y. Shi [63] and Fischer and Remling [36]. The theory of M-function for difference operators as developed in these papers are equivalent but the approach in [63] will be applicable in this study since its results are closer to the traditional approach of Hinton and Shaw [43]. Thus, the analysis will be developed parallel to that of Shi [63].

The spectral analysis will be done for higher even order difference operators generated by the discrete analogue of (1.1). Contrary to differential equations, there does not seem to be a theory of odd order difference operators, not even for simple constant coefficient third order operators. Thus the difference equation, which is the discrete analogue of (1.1), is given by

$$\begin{aligned}\mathcal{L}y(t) = w^{-1}(t)\Bigg\{&\sum_{i=0}^{n}(-1)^i\Delta^i[p_i(t)\Delta^i y(t-i)]\\ &- i\sum_{j=1}^{n}(-1)^j[\Delta^{j-1}(q_j(t)\Delta^j y(t-j))\\ &+ (\Delta^j(q_j(t)\Delta^{j-1}y(t-j+1))]\Bigg\}, \end{aligned} \tag{1.49}$$

defined on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$ with $w(t), p_n(t) > 0$ and $t \in \mathbb{N}$. The study will be done in two stages: general even higher order difference operators with both odd and even coefficients, $p_k(t)$ and $q_j(t)$, and finally, fourth order difference operators with unbounded coefficients.

In the definition of the difference equation (1.49), the notation Δ will denote the forward difference operator, that is, $\Delta f(t) = f(t+1) - f(t)$. The underlying function space will be $\ell^2_w(\mathbb{N})$ or $\ell^2_w(\mathbb{Z})$. Thus, the terms $\Delta^k(p_k(t)\Delta^k y(t-k))$ actually represent $\Delta^k(p_k(t)\Delta^{*k}y(t))$ and this also shows the symmetry of $\mathcal{L}$. The notations that will be used in this study are largely standard and follow closely those of [63]. Thus like in the differential case, the coefficients will be assumed to be almost constant, real valued and admit a decomposition of the form

$$f = f_0 + f_1 + f_2 + f_3 \quad \text{with} \quad f_0 \quad \text{constant and} \tag{1.50}$$

$$\Delta^2 f_1, (\Delta f_1)^2, \Delta f_2, f_3 \in \ell^1, \qquad f_1, \Delta f_1, f_2 = o(1), \qquad f = p_k, q_j.$$

For the highest order coefficient p_n and the weight function w, assume

$$p_n, w > 0 \quad \text{and normalise} \quad p_{n,0} = w_0 = 1$$

$$p_k(t) \to c_k, \qquad q_j(t) \to d_j \quad \text{as} \quad t \to \infty. \tag{1.51}$$

1.2.2 Hamiltonian System

Discrete Hamiltonian systems originated from the discretisation of continuous Hamiltonian system and from the discrete processes acting in accordance with the Hamiltonian principle such as discrete physical problems, and discrete control problems. Thus, in order to define discrete Hamiltonian system, one introduces quasi-differences, see [21, 63]

$$x_i(t) = \Delta^{i-1} y(t-i), \qquad i = 1, \dots, n,$$

$$u_n(t) = y^{[n]} = p_n(t)\Delta^n y(t-n) - iq_n(t)\Delta^{n-1} y(t-(n-1))$$

$$\begin{aligned} u_k(t) = y^{[2n-k]} &= \sum_{l=k}^{n} (-1)^{l-k} \Delta^{l-k}(p_l(t)\Delta^l y(t-l)) \\ & - i \sum_{l=k+1}^{n} (-1)^{l-k} \Big\{ \Delta^{l-k}(q_l(t)\Delta^{l-1} y(t-l+1)) \\ & + \Delta^{l-k-1}(q_l(t)\Delta^l y(t-l)) \Big\} - iq_k(t)\Delta^{k-1} y(t-k+1) \\ & k = 1, \dots, n-1. \end{aligned} \tag{1.52}$$

These formulae correspond very closely to the expressions in (1.3).

Define the vector valued functions $x(t)$, $u(t)$ and $y(t)$ by

$$x(t) = (x_1(t), \dots, x_n(t)), \quad u(t) = (u_1(t), \dots, u_n(t)),$$

$$y(t) = (x(t), u(t))^{tr}$$

and also the partial shift operator $R(y)(t)$ by $(Ry)(t) = (x(t+1), u(t))^{tr}$, where the superscript tr denotes transpose. (1.49) can be written in its discrete linear Hamiltonian form, see [63]

$$\mathcal{J}\Delta y(t) = [zW(t) + P(t)]R(y)(t), \tag{1.53}$$

where $t \in \mathbb{Z}^+ := [0, \infty) = \{t\}_{t=0}^{\infty}$, $W(t)$ and $P(t)$ are $2n \times 2n$ complex Hermitian matrices, $W(t) = \text{diag}(w(t), 0, \dots, 0)$, $w(t) > 0$, is the weighted function, $x(t), u(t) \in \mathbb{C}^n$, $\mathcal{J}$ is a canonical symplectic matrix, that is,

$$\mathcal{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \qquad \text{and} \qquad P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}.$$

The nonzero elements of $n \times n$ matrices A, B and C are given by

$$A_{j,j+1} = 1, \quad A_{n,n} = i\frac{q_n}{p_n}, \quad B_{n,n} = p_n^{-1},$$

$$C_{j,j} = p_{j-1}, \quad C_{j,j+1} = iq_j = -C_{j+1,j}, \quad j = 1, \ldots, n.$$

Here, p_0 and p_{n-1} should be read as $p_0 - zw$ and $p_{n-1} - \frac{q_n^2}{p_n}$, where z is the spectral parameter.

Let $\ell_w^2([0,\infty))$ be a Hilbert space with weight function w and define this Hilbert space using the vector valued functions $x(t)$, $u(t)$ and $y(t)$ by

$$l_w^2([0,\infty)) = \Big\{y : y = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C} \quad \text{and} \\ \sum_{t=0}^{\infty} (Ry^*)(t)W(t)(Ry)(t) < \infty \Big\}.$$

Then the scalar product for the vector valued functions of the system is, see [63]

$$\sum_{t=0}^{\infty} \overline{y_1}(t+1)w(t)y(t+1) = \langle y_1, y\rangle_w, \quad y, y_1 \in \ell_w^2([0,\infty)).$$

Thus, one defines the maximal difference operator L^* on $\mathbb{N}$, generated by (1.49), by its domain

$$D(L^*) = \big\{y \in \ell_w^2[0,\infty) : \text{there exists} \quad f \in \ell_w^2[0,\infty) \\ \text{such that} \quad \mathcal{J}\Delta y(t) - P(t)(Ry)(t) = W(t)f(t), \quad t \in [0,\infty)\big\},$$

$$L^*y = f, \quad \text{if} \quad \mathcal{J}\Delta y(t) - P(t)(Ry)(t) = W(t)f(t).$$

In this case, one defines a pre-minimal operator generated by (1.49) by

$$D(L') = \{y \in D(L^*) : \text{there exists} \quad n \in \mathbb{N} \\ \text{such that} \quad y(0) = y(t) = 0, \quad \forall t \geq n+1\}, \quad L^*y = L'y.$$

The closure of the pre-minimal operator L', $\overline{L'}$, is defined as the minimal difference operator generated by (1.49), that is, the minimal operator is a restriction of maximal operator L^* by further Dirichlet boundary conditions at 0. This minimal operator will be denoted by

L. It follows that L and L^* are symmetric, $L \subset L^*$ and $L = L^{**}$ as expected.

However, it is advantageous to write (1.53) in the form

$$\begin{aligned} \mathcal{J}\Delta \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} &= \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \\ &= \begin{pmatrix} -C(t) + zW & A^*(t) \\ A(t) & B(t) \end{pmatrix} \begin{pmatrix} x(t+1) \\ u(t) \end{pmatrix}. \end{aligned} \tag{1.54}$$

One can then write (1.54) as a first order system. In order to ensure the existence, uniqueness and continuation of the solution of the initial value problem of (1.54), it will always be assumed that $I_n - A$ is invertible in $\mathbb{N}$ ($\mathbb{Z}$). In this case, (1.54) becomes

$$\begin{aligned} \begin{pmatrix} x(t+1) \\ u(t+1) \end{pmatrix} &= S(t,z) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \\ &= \begin{pmatrix} E & EB \\ CE & I - A^* + CEB \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \end{aligned} \tag{1.55}$$

Here, $E = (I_n - A)^{-1}$.

Just as in Section 1.1.2, define the deficiency indices of the operator L by

$$\mathrm{def}L = (\mathcal{N}_{L^*-i}, \mathcal{N}_{L^*+i}) = (N_-, N_+).$$

Thus $n \leq N_\pm \leq 2n$ and if $N_- = N_+$, then a closed symmetric self-adjoint extension H of L exists. It is worth noting that a regularity condition similar to Condition 1.1.2 will be needed for the spectral analysis of higher order difference operators, that is, there exists an n_0 such that for all nontrivial solutions $y(.,z)$ of (1.54) and all $z \in \mathbb{C}$ [63, (1.7)]

$$\sum_{t=0}^{n} (Ry(t,z))^* W(t) (Ry(t,z)) > 0, \quad n \geq n_0.$$

For the difference operators with equal deficiency indices, the self-adjoint extensions can be described more explicitly in terms of boundary conditions. These can be defined via Green's identity. Let $[a, \infty)$ be the underlying interval of integral numbers. Then Green's identity is given by

$$
\begin{aligned}
\langle y_1, Ly_2\rangle - \langle Ly_1, y_2\rangle &= \sum_{t=a}^{\infty} y_1^*(t+1)\mathcal{A}(t+1)y_2(t+1) \\
&\quad - \sum_{t=a}^{\infty} y_1^*(t)\mathcal{A}(t)y_2(t) \\
&= \lim_{t\to\infty} y_1^*(t+1)\mathcal{J}y_2(t+1) - y_1^*(a)\mathcal{J}y_2(a).
\end{aligned}
$$

One requires that the boundary terms $\lim_{t\to\infty} y_1^*(t+1)\mathcal{J}y_2(t+1)$ and $y_1^*(a)\mathcal{J}y_2(a)$ vanish, see [63]. As for difference equations, one then defines boundary conditions for L in the systems form. For this, one fixes two n by n matrices α_1 and α_2, that is, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^{n\times n}$, with

$$
\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n, \qquad \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0_n. \tag{1.56}
$$

As before, the boundary condition for the left regular endpoint a is given as

$$
(\alpha_1, \alpha_2)y(a) = 0. \tag{1.57}
$$

We will be interested in the limit point case and thus the boundary conditions will be needed for the regular endpoint a only. If L is limit point, then L has self-adjoint extensions. With $a = 0$, the self-adjoint extensions H of L are precisely defined by

$$
D(H) = \{y \in D(L^*) : (\alpha_1, \alpha_2)y(0) = 0\}, \quad L^*y = Hy
$$

and $L \subset H = H^* \subset L^*$.

Here, we will show that with the conditions (1.50), H is limit point.

1.2.3 M-Matrix

Many results of the M-matrix described in Section 1.1.4 carry over to the difference operator case. Thus, let $Y_\alpha(.,z) = (U_\alpha(.,z), V_\alpha(.,z))$ be the fundamental matrix of (1.54) with initial values

$$
Y_\alpha(a,z) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix}, \quad \alpha_1, \alpha_2 \ \text{ satisfy (1.56) and (1.57).} \tag{1.58}
$$

U_α, V_α are $2n$ by n complex-valued matrices whose every column solves $Ly = zy$ and that $V_\alpha(.,z)$ satisfy the self-adjoint boundary conditions at a. Therefore, the columns of $Y_\alpha(.,z)$ span the $2n$-dimensional vector space of solutions of (1.54). Then in the limit point case with Im$z > 0$, one has a matrix $M \in \mathbb{C}^{n\times n}$ such that

$$\chi_\alpha(t,z) = Y_\alpha(t,z)\begin{bmatrix} I_n \\ M(z)\end{bmatrix} = U_\alpha(t,z) + V_\alpha(t,z)M(z),$$

where $\chi_\alpha(t,z)$ satisfies the boundary conditions (1.57). As before, $M(z)$ is determined from the solutions that stay absolutely square summable as Im$z \searrow 0$, it is unique, analytic both in the upper and lower half planes and satisfies $M^*(\overline{z}) = M(z)$, see [63]. Most of the results on the M-matrix in the continuous version that were obtained by Remling [52] carry over to the discrete case and have similarly been derived by Shi [63, Sect. 6]. It has been shown in [63] that if L is limit point at $t = \infty$, then one can construct the M-matrix $M(z)$ for the Hamiltonian restricted to $[a,\infty)$ with Dirichlet boundary conditions. To do this, let $\begin{bmatrix} W_1(a,z) \\ W_2(a,z)\end{bmatrix}$ be a system of n square summable solutions for Im$z > 0$. Then from the theory of Hinton and Shaw [43], it follows that these solutions also arise from $Y(t,z)\begin{bmatrix} I_n \\ M(z)\end{bmatrix}$, where $Y_a(t,z)$ is the fundamental solution of the system satisfying the appropriate boundary conditions at a. In this case, these are $Y_a(a,z) = I_{2n}$. If one compares both sets of solutions, it shows that there is an invertible n by n matrix C such that

$$\chi(a,z) = \begin{bmatrix} W_1(a,z) \\ W_2(a,z)\end{bmatrix} C = Y_a(t,z)\begin{bmatrix} I_n \\ M(z)\end{bmatrix}.$$

This in turn implies $M(z) = W_2(a,z)W_1^{-1}(a,z)$. Now let $F_\alpha(.,z)$ be an n by $2n$ system of square summable solutions of (1.54) satisfying boundary conditions at a and infinity and $z', z \notin \mathbb{R}$, then $F_\alpha(.,z) - F_\alpha(.,z') \in D(H)$ and by results of Remling [52, Lemma 3.1] and Shi [63, Theorem 6.3], it follows that $\langle F_\alpha(.,z), F_\alpha(.,z)\rangle = (\mathrm{Im}z)^{-1}\mathrm{Im}M(z)$ if $z' = z$. Therefore, if $z = \mu + i\epsilon$ for some $\epsilon > 0$, then one has for $\mu_+ = \lim_{\epsilon\to 0+} \mu + i\epsilon$

$$\text{(1.59)} \qquad \begin{aligned} \mathrm{Im}M(\mu_+) &= \lim_{\epsilon\to 0+} \mathrm{Im}M(\mu_+ + i\epsilon) \\ &= \lim_{\epsilon\to 0+} \epsilon\langle F_\alpha(.,\mu_+ + i\epsilon), F_\alpha(.,\mu_+ + i\epsilon)\rangle. \end{aligned}$$

All other properties outlined in Section 1.1.4 are satisfied by this M-matrix. Besides most of the properties have been shown in [63, Sect. 6].

Now let $y(t) \in \ell^2_W([a,\infty))$ and $R(y)(t)$ be vector valued function and partial shift operator respectively as defined in Section 1.2.2 and assume that $\phi(.,z)$ is $2n \times n$ matrix-valued solution of the Hamiltonian system satisfying $\phi(a,z) = \mathcal{J}\alpha^*$, $\alpha = (\alpha_1, \alpha_2)$. Define now the inhomogeneous Hamiltonian system by

$$Ly(t) = zW(t)(Ry)(t) + W(t)(Rf)(t), \qquad t \in [a,\infty), \tag{1.60}$$

with the homogeneous boundary conditions satisfied at $t = a$ and f a function in $\ell^2_w([a,\infty))$.

Theorem 1.2.1. *[63, Theorem 6.11] For any $z \in \mathbb{C}$ with $Imz \neq 0$ and for any $f \in \ell^2_w([a,\infty))$, (1.60) has a unique solution $y(.,z)$ satisfying the inhomogeneous boundary conditions $(\alpha_1,\alpha_2)y(a) = v_0$, $y \in \ell^2_w([a,\infty))$. Furthermore, $y(t,z)$ can be expressed as*

$$y(t,z) = \chi(t,z)v_0 + \sum_{s=a}^{\infty} G(t,s,z)W(s)(Rf)(s), \qquad t \in [a,\infty)$$

where the Green's function is defined as

$$G(t,s,z) = \begin{cases} \chi(t,z)(R\phi)^*(s,\overline{z}) : & a \leq s \leq t-1 \\ \phi(t,z)(R\chi)^*(s,\overline{z}) : & t \leq s < \infty \end{cases}.$$

1.2.4 Asymptotic Summation

Levinson's Theorem is one of the most useful results in the asymptotic integration of linear differential equations as mentioned earlier. For linear systems, one can expect that a good deal of the results extend from the differential calculus setting also to difference calculus setting. This is indeed the case. For example, the Benzaid-Lutz results [27] are the difference equations counterparts of Levinson's Theorem. Also Hartman-Wintner Theorem has its discrete counterpart. These results are concerned with systems of linear difference equations of the form $y(t+1) = A(t)y(t)$ for $t \geq t_0$ and $A(t)$ is an invertible square matrix. Then, the Levinson's-Benzaid-Lutz (LBL)-Theorem states that the solutions of a system

$$y(t+1) = [\Lambda(t) + R(t)]y(t), \tag{1.61}$$

where $\Lambda(t)$ is diagonal and invertible, look like the solutions of the unperturbed system $y(t+1) = \Lambda(t)y(t)$, if $R(t)$ is sufficiently small and

$\Lambda(t) = \text{diag}(\lambda_i(t))$ satisfies a dichotomy condition. In the Benzaid-Lutz results [27], small means absolutely summable, that is, for all $i = 1, \ldots, 2n$, $\lambda_i^{-1} R(t) \in \ell^1$. The dichotomy condition in this case will amount to: for any pair of indices i and j, such that $i \neq j$, assume there exists δ with $0 < \delta < 1$ such that $\mid \lambda_i(t) \mid \geq \delta$ for all $t \geq t_0$, then either $\mid \frac{\lambda_i(t)}{\lambda_j(t)} \mid \geq 1$ or $\mid \frac{\lambda_i(t)}{\lambda_j(t)} \mid \leq 1$ for a large t. The LBL-result has been extended in many directions by either strengthening the dichotomy condition and weakening the decay conditions of $R(t)$ and vice versa. For more details, see [27] or [15]. As in the continuous case, in the spectral theory of the difference operators, the matrix elements and $\lambda_i(t)$ will generally depend also on the spectral parameter z. Thus one writes $\lambda_i = \lambda_i(t, z)$ for this. In this case, it will be important to prove LBL-Theorem uniformly in z in order to control the z-dependence of the solutions. The following theorem, which will be called Levinson's-Benzaid-Lutz Theorem throughout this study, will suffice for the application of asymptotic summation to the system (1.54).

Theorem 1.2.2. *Let $\Lambda(t, z) = diag\{\lambda_1(t, z), \ldots, \lambda_{2n}(t, z)\}$ for $t \geq t_0$. Assume*

(i) $\lambda_i(t, z) \neq 0$ *for all* $1 \leq i \leq 2n$ *and* $t \geq t_0$

(ii) $R(t, z)$ *is a* $2n \times 2n$ *matrix defined for all* $t \geq t_0$*, satisfying* $\sum_{t=0}^{\infty} \mid \frac{1}{\lambda_i(t,z)} \mid \| R(t, z) \| < \infty$*, for all* $i = 1, 2, \ldots, 2n$

(iii) $\Lambda(t, z)$ *satisfies the following uniform dichotomy condition. For any pair of indices* i *and* j*, such that* $i \neq j$*, assume there exists* δ *with* $0 < \delta < 1$ *such that* $\mid \lambda_i(t) \mid \geq \delta$ *for all* $t \geq t_0$*. Then either* $\mid \frac{\lambda_i(t)}{\lambda_j(t)} \mid \geq 1$ *or* $\mid \frac{\lambda_i(t)}{\lambda_j(t)} \mid \leq 1$ *for a large* t*.*

Then the linear system $X(t + 1, z) = [\Lambda(t, z) + R(t, z)]X(t, z)$ *has a fundamental matrix satisfying, as* $t \to \infty$

$$X(t, z) = [I + o(1)] \prod_{l=t_0}^{t-1} \Lambda(l, z). \tag{1.62}$$

The proof follows closely from Lemma 2.1 and Theorem 3.3 in [27].

In order to solve (1.55) by asymptotic summation, (1.55) has to be diagonalised. This requires the eigenvalues of $S(t, z)$. Expansion of $\det(S - \lambda \cdot I) = \mathcal{P}(t, \lambda, z)$ leads to the following result

$$\begin{aligned}\mathcal{F}(\lambda, z, t) &= \frac{p_n - iq_n}{\lambda^n}\mathcal{P}(\lambda, t, z)\\ &= \sum_{k=0}^{n} p_k(1-\lambda)^k(1-\lambda^{-1})^k\\ &+ \sum_{j=1}^{n} q_j(1-\lambda)^{j-1}(1-\lambda^{-1})^{j-1}(i\lambda + (i\lambda)^{-1}).\end{aligned} \tag{1.63}$$

In this case, p_0 should be read as $p_0 - zw$. This shows that with λ also λ^{-1} is a root. If one assumes that $q_j(t) = 0$ for all $j = 1, \ldots, n$, then a special form of (1.63) for even order difference operators with even coefficients only may be obtained. In this case, $\mathcal{F}$ is a function of $\zeta = \lambda + \lambda^{-1}$ only.

1.2.5 Transformation of the System

In order to transform (1.55) into Levinson-Benzaid-Lutz form or LBL-form

$$y(t+1, z) = (\Lambda(t, z) + R(t, z))y(t, z), \tag{1.64}$$

$$\Lambda(t) = \mathrm{diag}(\lambda_i(t, z), \ldots, \lambda_{2n}(t, z) \quad \text{and} \quad \lambda_i^{-1}(t, z)R_{ij}(t, z) \in \ell^1,$$

we will need two diagonalisations and some $(I + Q)$-transformations.

Diagonalisations:
The eigenvalues are determined from (1.63) by analysing $\mathcal{P}(t, \lambda, z)$ and $\mathcal{F}(t, \lambda, z)$ with the aid of limiting expressions $\mathcal{P}_0(t, \lambda, z)$ and $\mathcal{F}_0(t, \lambda, z)$. These are obtained by replacing the coefficients by their limiting values $p_{k,0}$ and $q_{j,0}$. By considering the resultant or the discriminant of $\mathcal{P}_0$ and $\partial_\lambda \mathcal{P}_0$, one can show that there are only finitely many spectral values z for which $\mathcal{P}_0(\lambda, z)$ has multiple roots. Let $\omega_1 < \omega_2 < \cdots < \omega_k$ denote all of the real spectral values z leading to multiple roots. Following [10], the analysis will be restricted to small complex neighbourhoods of $z_0 \in (\omega_i, \omega_{i+1})$, $i = 0, \ldots, k$, where $\omega_0 = -\infty$ and $\omega_{k+1} = \infty$. For a given $z_0 \in (\omega_i, \omega_{i+1})$, one can now choose $\epsilon > 0$ and $a > 0$ so that $\mathcal{P}(t, \lambda, z) = 0$ has no multiple roots for any

$$z \in \mathcal{K}_\epsilon(z_0) = \{z \mid \ \mid z - z_0 \mid \leq \epsilon, \quad \mathrm{Im} z \geq 0\} = \mathcal{K}$$

and $t \geq a$. This is possible because the roots of $\mathcal{P}_0$ depend analytically on the coefficients. Throughout the study, it may be necessary to adjust a and ϵ repeatedly.

Since $\mathcal{P}(t,\lambda,z)$ has $2n$ distinct roots, one can determine the corresponding eigenvectors. Because the coefficients are assumed to be almost constant, in the ideal situation of constant coefficients, a time shift corresponds to a multiplication by λ. Somewhat, more generally, one has

$$\Delta \to (\lambda-1), \qquad y(t+k) \to \lambda^k, \quad k \in \mathbb{Z}.$$

Here $\to$ means "replace by" or "corresponds to". With this and (1.52), one obtains

$$\begin{aligned} x_i &= (\lambda-1)^{i-1}\lambda^{-1}, \qquad i=1,\dots,n, \\ u_n &= p_n(\lambda-1)^n\lambda^{-n} - iq_n(\lambda-1)^{n-1}\lambda^{-(n-1)} \end{aligned}$$

$$\begin{aligned} u_k = &\sum_{l=k}^{n}(-1)^{l-k}(\lambda-1)^{2l-k}(p_l\lambda^{-l} \\ &- i\sum_{l=k+1}^{n}(-1)^{l-k}(\lambda-1)^{2l-k-1}\lambda^{-l}\left(q_l\lambda+q_l\right) - iq_k(\lambda-1)^{k-1}\lambda^{-k+1} \end{aligned}$$

$$k=1,\dots,n-1.$$

These eigenvectors are indeed bounded. Since, by assumption, the coefficients converge to their limiting values, the eigenvectors will converge to limiting vectors likewise

$$\begin{pmatrix} x(t,\lambda) \\ u(t,\lambda) \end{pmatrix} \to \begin{pmatrix} x \\ u \end{pmatrix}, \qquad \lambda(t) \to \lambda(\infty).$$

It is well known that the matrix $T(t)$ formed with the eigenvectors as columns will diagonalise $S(t,z)$. One thus transforms the system by

$$\begin{pmatrix} x(t,z) \\ u(t,z) \end{pmatrix} = T(t,z)y(t,z).$$

This results into a system of the form

$$\begin{aligned} y(t+1,z) &= T^{-1}(t+1,z)S(t,z)T(t,z)y(t,z) \\ &= (\Lambda+R)(t,z)y(t,z), \end{aligned} \tag{1.65}$$

where $R(t,z) = -T^{-1}(t+1,z)\Delta T(t,z)\Lambda(t,z)$. The expression $R(t,z)$ consists of ℓ^2 and ℓ^1 terms. One can thus split $R(t,z)$ into $V(t,z)$ and $\widetilde{R}(t,z)$, where $V(t,z) \in \ell^2$ and $\widetilde{R}(t,z) \in \ell^1$. (1.65) therefore simplifies to

$$y(t+1,z) = (\Lambda(t,z) + V(t,z) + \widetilde{R}(t,z))y(t,z), \tag{1.66}$$

$$V(t,z) \to 0, \quad t \geq t_0.$$

Using results of Behncke [15], one can diagonalise (1.66) further. A matrix $(I + B(t))$ formed with the eigenvectors of $(\Lambda + V)(t)$ can be used to diagonalise (1.66). By adjoining the diagonals of $V(t)$ to $\Lambda(t)$ and dropping the spectral parameter z temporarily, $B(t)$ is constructed as follows [15]:

$$B_{ii} = 0, \quad B_{ij} = (\lambda_j - \lambda_i)^{-1} V_{ij}, \quad i \neq j, \tag{1.67}$$

$$i, j = 1, \dots, 2n, \quad t \geq t_0.$$

The corrections to the eigenvalues-diagonal are given by $(\Lambda_2)_{ii} = (VB)_{ii}$, $i = 1, \dots, 2n$. Applying the transformation $y(t+1) = (I + B(t))v(t)$ on (1.66), one obtains

$$\begin{aligned} v(t+1) = \big\{&[(I + B(t+1))^{-1}(I + B(t))][\Lambda(t) + \Lambda_2(t)] \\ &+ (I + B(t+1))^{-1}R(t)(I + B(t))\big\}\, v(t). \end{aligned} \tag{1.68}$$

$(I + Q)$-Tranformation:
Although $(I+Q)$-transformations can be used to eliminate conditionally summable terms, in our case it will be assumed that these terms are absent. It follows that $(I+Q)$-transformations will only be needed to make remainder terms after two diagonalisations smaller. Assume the system to be in the form (1.66) which is comparable to [27, Case III] and moreover, let $\widetilde{R}_{ii} = 0, i = 1, \dots, 2n$. Applying the transformation $y(t) = (I + Q(t))v(t)$ to (1.66), one obtains

$$\begin{aligned} v(t+1) = \big\{&\Lambda(t) + (I + Q(t+1))^{-1}[-Q(t+1)\Lambda(t) + \Lambda(t)Q(t) + V(t) \\ &+ V(t)Q(t) + \widetilde{R}(t)(I + Q(t))]\big\}\, v(t). \end{aligned}$$

This equation simplifies greatly if one assumes that

$$V(t) = -\Lambda(t)Q(t) + Q(t+1)\Lambda(t).$$

For this kind of assumption to make sense, one requires the solutions $Q(t)$ of this equation to satisfy the condition $Q(t) \to 0$ as $t \to \infty$. Concretely as in [27, (3.18)], one has

$$Q_{ij}(t+1) = Q_{ij}(t)\frac{\lambda_i(t)}{\lambda_j(t)} + \frac{V_{ij}(t)}{\lambda_j(t)}, \qquad i \neq j. \tag{1.69}$$

It has been shown in [15, Lemma 1 and Theorem 2] that whenever (1.69) satisfies uniform dichotomy condition, then the solution $Q(t) \to 0$ as $t \to \infty$ and any decay in R in excess of summability is passed onto the remainder.

With all these transformations, one is finally led to a system of $w(t+1) = (\widetilde{\Lambda} + \widetilde{R})w(t)$ to which LBL-Theorem can be applied. The kth solution of this system is

$$w_k(t,z) = (e_k + r_k(t,z)) \prod_{s=t_0}^{t-1} \widetilde{\lambda}_k(s). \tag{1.70}$$

One can perform backward transformations to obtain the general solutions of (1.49) or (1.64). In (1.70), $\widetilde{\lambda}_k$ is λ_k plus further diagonal terms arising from various transformations. One also formally requires that

"higher order of the differences to decay faster."

Because the coefficients are assumed to have second order difference, the above statement somewhat translates to

$$\Delta^l f \in \ell^{\frac{2}{l}}, \qquad l = 1, 2, \qquad f = p_k, q_j. \tag{1.71}$$

The consequences of such a smoothness and decay conditions are

$$Q_1 = o(B), \quad Q_{j+1} = o(Q_j), \quad r_k = o(B),$$

so that the solutions (1.70) in the general form are essentially given by

$$\begin{aligned} y_k(t,z) &= (T(t,z)(I + B(t,z)))(e_k + o(1)) \prod_{s=t_0}^{t-1} \widetilde{\lambda}_k(s,z) \\ &= (\varrho_k(t,z) + r_k(t,z)) \prod_{s=t_0}^{t-1} \lambda_k(s,z), \end{aligned} \tag{1.72}$$

where $\varrho_k(t,z)$ is the normalised eigenvector.

Chapter 2

SPECTRAL THEORY OF HIGHER ORDER DIFFERENTIAL OPERATORS

With the smoothness and decay conditions (1.45), the major difficulties in determining the asymptotics of the eigenfunctions and the spectral analysis lie with the roots of the characteristic polynomial. This and the behaviour of these polynomials near these roots, which is a necessary ingredient for the dichotomy condition, are now the major obstacles to an understanding of the asymptotics of the eigenfunctions of these differential operators. Thus, any class of polynomials, which readily leads to a solution of these properties, results into a class of differential operators with an accessible spectral theory. Such a particular class of polynomials was investigated in connection with the study of the deficiency index by Eastham and others [31, 32, 33, 34, 61]. These authors, however, never investigated the more ambitious spectral theory.

The studies on the deficiency indices were only limited to few cases and the results were geared towards deficiency indices with the assumption that one diagonalisation will transform the system form into Levinson's form, Eastham [33, Sect. 3.2, 3.3 and 3.11]. The basic idea in the approach of [33, Sect. 3.3] is to consider polynomials which lead to eigenvalues with different orders of magnitude. This is achieved by considering polynomials for which the coefficients can be ordered in their respective size. Thus, the characteristic polynomial can be broken up into segments

$$p_n = p_{k_l}, \dots, p_{k_{l-1}}, \dots, p_{k_{l-2}}, \dots, p_{k_1}, \dots, p_0$$

with two consecutive p_{k_j} $(j = 0, 1, \dots, l)$ forming clusters and each cluster leading to eigenvalues λ of the characteristic polynomial, which can be computed approximately from that cluster alone. Moreover, eigenvalues λ and μ associated to different clusters should satisfy $\lambda \gg \mu$ or $\lambda \ll \mu$ [20, 23]. Actually, in these earlier studies, only clusters of size two with no odd order coefficients were considered.

Eastham's results had a further limitation since the asymptotics of eigenfunctions were obtained only in terms of various derivatives. This kind of asymptotic formulae cannot produce spectral results and differing with his techniques, an extension of these results will be carried out in this study where the system will be diagonalised repeatedly in order to achieve Levinson's form with the only assumption that higher order derivatives decay faster. Like in [33], the study will begin in Section 2.1 with eigenvalues of cluster size two for even order differential operator. In Sections 2.2-2.5, this technique will be generalised to eigenvalues in bands of distinct sizes using the technique of partial cluster polynomial determined by pivot coefficients associated with various coefficients of differential equation (1.1) and (1.2). The length of the intervals will be assumed to be both odd and even. Thus, the following assumptions will be made to guarantee the existence of the solutions of the system.

(i) In Section 2.1, it will be assumed that p_k, q_j ,$0 \leq k \leq n$, $1 \leq j \leq n$, are twice differentiable, while p_k are nowhere zero in the interval $[a, \infty)$ where $a > 0$.

(ii) In Sections 2.2-2.5, it will be assumed that p_k and q_j, $0 \leq k \leq n$, $1 \leq j \leq n$ or $1 \leq j \leq n+1$ for even or odd order differential operators respectively are twice differentiable while r_{k_j}, the pivot coefficients, $(1 \leq j \leq l)$ associated with these coefficients are twice differentiable and nowhere zero in $[a, \infty)$ and in Section 2.5, in the case of decomposition method, it will be assumed that these coefficients are nowhere zero in $\mathbb{R}$.

(iii) It will also be assumed throughout this chapter that the coefficients are at least twice differentiable and

"higher order derivatives if they exist decay faster".

The latter condition will have to be made more precise, however.

(iv) It will further be assumed that the pivot coefficients $r_{k_j}(x)$ satisfy $| r_{k_j}(x) |> 0$ for $x \geq a > 0$. This condition restricts the analysis initially to the interval $[a, \infty)$. However, by the results of Remling [52], these results can be extended to $[0, \infty)$.

(v) In this chapter, it will either be taken that $p_0 \approx w$ or $w = o(p_0)$ and quite often zw will be absorbed into p_0 unless stated otherwise.

Section 2.6 is about the extension of the results in Sections 2.1-2.5 to approximate power type coefficients and also an extension to differential operators with coefficients having a certain form of decomposition. Even on the level of deficiency index alone, the results of Section 2.1 are far reaching extensions of those in [33, Sect. 3.8 and 3.11]. The results on the absolutely continuous spectrum are new and can be found in a joint paper with Behncke [23]. These results were envisioned and sketched, however, in the paper of Behncke and Hinton [20].

2.1 Clusters of Size Two

Consider the differential equation (1.1) with the coefficients p_k and q_j twice differentiable and p_k nowhere zero in $[0, \infty)$ satisfying

$$\frac{p_{k-1}}{p_k} = o(\frac{p_k}{p_{k+1}}) \quad \text{and} \quad q_k = O((p_{k-1}p_k)^{\frac{1}{2}}), \tag{2.1}$$

$k = 1, \ldots, n-1$. Define $\{p_k, q_k, p_{k-1}\}$ as the kth cluster and assume that $| \frac{p_{k-1}}{p_k} |^{\frac{1}{2}} = m_k$ is not integrable. In the remainder, the relation $f \gg g$ or $g \ll f$ if $g = o(f)$ will be interpreted as very much larger than in the absolute value sense.

Lemma 2.1.1. *a) For $k = 1, \ldots, n$, the roots of characteristic Fourier polynomial $\mathcal{P}_F(x, \lambda, z)$ (1.24), if ν is replaced by λ and (2.1) is satisfied, are approximately given by the roots of the cluster polynomials $p_k\lambda^2 + 2q_k\lambda + p_{k-1} = 0$*

$$\lambda^0_{k\pm} \approx -\frac{q_k}{p_k} \pm (-\frac{p_{k-1}}{p_k} + (\frac{q_k}{p_k})^2)^{\frac{1}{2}}. \tag{2.2}$$

b) With $m_k = | \frac{p_{k-1}}{p_k} |^{\frac{1}{2}}$, one has $m_{k-1} \ll m_k$ and $\lambda_k = O(m_k)$.

c) For $j < k-1$, $\mid p_j \mid m_k^{2j} \ll \mid p_{k-1} \mid m_k^{2(k-1)} \approx \mid p_k \mid m_k^{2k}$. This also holds for $j > k$.

The proof of this is straight forward but can also be found in [33, 34]. The proof of a considerably more general result will be given in Section 2.2. As regards (2.2), assume in addition the roots to be non-degenerate, that is,

$$m_k \approx \mid -\frac{p_{k-1}}{p_k} + (\frac{q_k}{p_k})^2 \mid^{\frac{1}{2}} \qquad k = 1, \dots, n. \tag{2.3}$$

Thus m_k is a measure for the size of the eigenvalues for the kth cluster. The Lemma also shows that the roots of the characteristic polynomial may be determined from (2.2) by iteration. For this, write for the Fourier polynomial

$$\mathcal{P} = p_k\lambda^{2k} + 2q_k\lambda^{2k-1} + p_{k-1}\lambda^{2k-2} + R_k(\lambda)$$

and solve

$$\lambda^2 + 2\frac{q_k}{p_k}\lambda + \frac{p_{k-1}}{p_k} + R_k(\lambda_k^0)\lambda_k^{0(-2(k-1))}p_k^{-1} = 0, \tag{2.4}$$

where λ_k^0 is the first order approximation for $\lambda_{k\pm}$ (Lemma 2.1.1). This approximation is important if one wants to assess the influence of zw on λ_k and thus the contribution of the kth cluster to the deficiency index and spectrum. In (2.4), consider $\widetilde{R_k} = R_k(\lambda_k^0)\lambda_k^{0(-2(k-1))}$ as a perturbation of p_{k-1}. Its contribution is then in first order

$$\Delta_\pm = -\frac{1}{2}\widetilde{R}_k p_k^{-1}(\lambda_{k\pm} + \frac{q_k}{p_k})^{-\frac{1}{2}}. \tag{2.5}$$

In particular, the perturbation term of the spectral parameter can be estimated by

$$\Delta_{z\pm} = \frac{zw}{2p_k\lambda_k^{0(2(k-1))}(\lambda_{k\pm}^0 + \frac{q_k}{p_k})^{\frac{1}{2}}} \approx \frac{zw}{\mid p_k \mid m_k^{2k-1}}. \tag{2.6}$$

Of course, this can also be estimated as

$$\lambda(z_0 + i\eta) = \lambda(z_0) + (\partial_\lambda \mathcal{P}_\lambda(\lambda(z_0)))^{-1} iw\eta = \lambda(z_0) + \frac{iw\eta}{M_k}.$$

Thus a key role in all perturbing terms is played by

$$M_k = \partial_\lambda \mathcal{P}(\lambda_k).$$

M_k will thus be referred to as the M-factors. This is of course an adoption of Eastham's notation and this M has nothing to do with the M-matrix defined earlier. They can be estimated as

Lemma 2.1.2. $| M_k | \approx | p_k \lambda_k^{2k-1} | \approx | p_{k-1} \lambda_k^{2k-3} |$ *and*
$| M_{k+1} | \gg | M_k |$, $k = 1, \ldots, n-1$.

Proof. Clearly

$$| p_k m_k^{2k-1} | = | p_k \frac{p_{k-1}}{p_k} | m_k^{2k-3} = | p_{k-1} m_k^{2k-3} | \,.$$

Now $| p_k m_{k+1}^{2k-1} | \gg | p_k m_k^{2k-1} |$ shows by induction
$| p_n m_n^{2n-1} | \gg | p_{n-1} m_{n-1}^{2n-3} |$ and $| M_n | \gg | M_{n-1} | \gg \ldots \gg | M_1 |$ follows because the leading terms for the kth cluster in $\mathcal{P}$ arise in that cluster itself.

□

Lemma 2.1.2 essentially uses conditions based on the p_k, because, in some sense, they determine the clusters, while the q_j only satisfy (2.1). Lemma 2.1.2 also shows that for the kth cluster, the contribution of the spectral parameter can be estimated by

$$\Delta_{z\pm} \approx \frac{zw}{| M_k |}.$$

This result is of course of critical importance for the analysis of the deficiency indices and the dichotomy condition.

An important ingredient for asymptotic integration as mentioned in Chapter one is the dichotomy condition. In this case, one needs that for any unequal pair of indices k and j, sign $\mathrm{Im}(\lambda_k(x,z) - \lambda_j(x,z))$ is constant modulo $\mathcal{L}^1$. This is because we are determining the eigenvalues from the Fourier polynomial. Because of spectral analysis, this will be needed uniformly in the spectral parameter z, $z \in \mathcal{K}_\epsilon(z_0)$.

Lemma 2.1.3. *The roots of the Fourier polynomial satisfy the uniform dichotomy condition.*

Proof. Since the dichotomy condition within each cluster is trivial because of (2.3), it suffices to analyse $\mathrm{Im}(\lambda_k(x,z) - \lambda_j(x,z))$ for $k > j$. It is worth mentioning here that the roots are those of the Fourier polynomial. If $\mathrm{Im}\lambda_k \approx m_k$, there is nothing to show. Thus, one may assume $\mathrm{Im}\lambda_k(x,z) \approx \frac{\eta w}{M_k}$, $\eta = \mathrm{Im} z$. If then $\mathrm{Im}\lambda_j \approx \frac{\eta w}{M_j}$, the dichotomy condition follows from Lemma 2.1.2. Now $m_j \geq | \frac{\eta w}{M_j} | \gg | \frac{\eta w}{M_k} |$ finally proves the dichotomy condition.

□

Once the dichotomy condition is settled, it remains to estimate the matrix elements of $\mathcal{T}^{-1}\mathcal{T}'$. These are made up of expressions of the form

$$(\lambda_k - \lambda_j)^{-1} M_j^{-\frac{1}{2}} M_k^{-\frac{1}{2}} \{ \frac{p_l'}{p_l} p_l \lambda_j^l \lambda_k^l, \qquad \frac{q_l'}{q_l} q_l (\lambda_k^l \lambda_j^{l-1} + \lambda_k^{l-1} \lambda_j^l) \}$$

and by antisymmetry, one may assume $k \geq j$. The elements of the matrix $\mathcal{T}^{-1}\mathcal{T}'$ within the eigenvalue blocks can be estimated by

$$\frac{f'}{f} \qquad \text{with} \qquad f = p_k, q_j. \tag{2.7}$$

Next assume cluster l to be distinct from cluster k and j. Then estimate the above terms as

$$m_k^{-\frac{1}{2}} M_k^{-\frac{1}{2}} m_j^{-\frac{1}{2}} M_j^{-\frac{1}{2}} \frac{m_j^{\frac{1}{2}}}{m_k^{\frac{1}{2}}} \{ \frac{p_l'}{p_l} \mid p_l \lambda_j^{2l} \mid^{\frac{1}{2}} \mid p_l \lambda_k^{2l} \mid^{\frac{1}{2}}, \quad \text{or}$$

$$\frac{q_l'}{q_l} \mid \frac{\lambda_k}{\lambda_j} \mid^{\frac{1}{2}} \mid q_l (\lambda_k^{2l-1} \mid^{\frac{1}{2}} \mid q_l \lambda_j^{2l-1} \mid \}.$$

Since $m_k \mid M_k \mid \approx \mid p_k \lambda_k^{2k} \mid$ and that the maximal values of $\mathcal{P}(\lambda)$ for λ_k are attained within the cluster k, the above expression can be estimated by

$$\frac{f'}{f} \cdot \frac{m_j^{\frac{1}{2}}}{m_k^{\frac{1}{2}}} \cdot o(1), \qquad \frac{g'}{g} \cdot o(1), \tag{2.8}$$

for $f = p_k$, $g = q_j$, $k \neq j$. In the same manner, one can show that the above expression gives an estimate also for $l = k$ or $l = j$. If the coefficients are of approximate power type, the $o(1)$ factor can be written as $x^{-\epsilon}$ for some $\epsilon > 0$ even.

Remark 2.1.4. The estimates (2.7) and (2.8) are clearly not optimal. $(\mathcal{T}^{-1}\mathcal{T}')_{jk}$ will mainly involve p_k, p_j, q_k and q_j, while coefficients from clusters, which are further apart, appear with smaller and smaller $o(1)$ factors.

While (2.7) and (2.8) suggest a condition like $\frac{f^{(l)}}{f} \in \mathcal{L}^{\frac{2}{l}}$ for $f = p_k, q_j$, $l = 1, 2$, the condition

$$\frac{f'}{f} . m_1^{-1} = o(1), \qquad f = p_k, q_j, \tag{2.9}$$

is critical for the application of second diagonalising transformation. For power coefficients, this implies that $| p_1 | \leq o(p_0 x^2)$. For general coefficients, (2.9) is difficult to assess if one does not have any more detailed information on the smallest eigenvalues. It may be possible to avoid (2.9) if one also introduces separate diagonalising transformations for the small eigenvalue block. The perturbation term after the second diagonalisation is

$$-(I+B)^{-1}(B' + (\mathcal{T}^{-1}\mathcal{T}')B) = R_1. \tag{2.10}$$

Note that we assumed the coefficients p_k and q_j to be twice differentiable and also in chapter one, it was stated that only two diagonalisations will be required. This then implies that one needs only one matrix B for the second diagonalisation. Thus the perturbing term $R_1 = o(1)$ (1.35). However, if one has to apply more than two diagonalisations or $(I+Q)$-transformations, then one requires

$$\frac{f''}{f}.m_1^{-1} \in \mathcal{L}^1, \quad \frac{f'}{f}m_1^{-\frac{1}{2}} \in \mathcal{L}^2, \quad f = p_k, q_j. \tag{2.11}$$

A further $(I+Q)$-transformation makes the remainder terms after the two diagonalisations smaller so that a chain of such transformations will make the remainder term very small, that is, Levinson's term. It thus remains to estimate the corrections to the eigenvalues, that is, the diagonal terms of the transformation matrices. These are given by

$$\sum_j (\mathcal{T}^{-1}\mathcal{T}')_{ij} B_{ji} = (\mathcal{T}^{-1}\mathcal{T}')_{ij}(\lambda_j - \lambda_i)^{-1}(\mathcal{T}^{-1}\mathcal{T}')_{ji} = \Lambda_{2ii}$$

because $B_{ii} = 0$. By (2.11), these terms are integrable and therefore irrelevant for the square integrability of the eigenfunctions. Thus, the eigenfunctions are approximately given by

$$\begin{aligned} u_k &\approx \mathcal{T}(I+B)\cdot(e_k + o(1))\cdot \exp\int_a^x \lambda_k(t)dt \\ &= G_k(x)\cdot \exp\int_a^x \lambda_k(t)dt, \end{aligned} \tag{2.12}$$

where $G_k(x) = \mathcal{T}(I+B)\cdot(e_k + o(1))$.

In general, the growth of the eigenfunctions is mainly determined by the exponential factor. In order to interpret (2.12) more easily, we shall just assume that. For this, we have to consider two cases for all k.

a) If $\frac{w}{M_k} \in \mathcal{L}^1$, then we need $\frac{w}{M_k} \cdot \exp \epsilon m_k x$ nonintegrable for any $\epsilon > 0$.

b) If $\frac{w}{M_k} \notin \mathcal{L}^1$, then we need $\frac{w}{M_k} \cdot \exp -\epsilon m_k x \in \mathcal{L}^1$ for any $\epsilon > 0$.

This condition will be referred to as the regularity condition on the form factors G_k. For power bounded coefficients, this amounts to

$$p_k, q_j = O(x^N), \quad \mid \frac{w}{M_k}(x) \mid \leq Cx^{-1-\epsilon} \quad \text{or} \quad \mid \frac{w}{M_k}(x) \mid \geq Cx^{1+\epsilon} \tag{2.13}$$

in order to exclude the occurrence of simple logarithmic terms in the exponents. This is, of course, only a technical constraint to simplify the representation. The methods expounded here are applicable in much more general circumstances. This assumption avoids the interference of the form factors G_k with the exponents. It can be avoided at the cost of further technical assumptions. (2.12) shows that all clusters contribute individually to the deficiency index and spectrum. Thus, since $\sigma_{sc}(H) = \emptyset$, one has

$$\text{def}T = \sum_{k=1} (\text{def}T)_k$$

and

$$\sigma_{ac}(H) = \cup_{k=1}^{n} \sigma_{ac}(H)_k. \tag{2.14}$$

Here, the sum respectively the union goes by all clusters $k = 1, \dots, n$ and the union in (2.14) is to be understood as additive with respect to the multiplicities.

Lemma 2.1.5. *If $\frac{w}{M_k}$ is integrable, then the form factor $(\mathcal{T}B)_k$ is square integrable.*

Proof. Assume that $\frac{w}{M_k}$ is integrable, then we need to show that $(\mathcal{T}B)_k$ is square integrable [32, (2.8)] and [33, Theorem 3.9.1]. Thus, one has to analyse for $k > j$

$$\begin{aligned}(\mathcal{T}B)_{kj} = \sum_j \sum_l M_j^{-1} M_k^{-\frac{1}{2}} (\lambda_j - \lambda_k)^{-2} ((-1)^l p_l' \lambda_k^l \lambda_j^l \\ - i q_l' (\lambda_j^l \lambda_k^{l-1} + \lambda_k^l \lambda_j^{l-1})) = \sum A_{k,j,l} + B_{k,j,l}.\end{aligned} \tag{2.15}$$

More specifically, one has to show that each summand, $A_{k,j,l}$ and $B_{k,j,l}$ is square integrable if $\frac{w}{M_k}$ is integrable. This result will be shown for

$A_{k,j,l}$ only. Since for $l < j$, $| p_{l+1}\lambda_k^{2(l+1)} |>| p_l\lambda_k^{2l} |$ and for $l > k$, $| p_{l+1}\lambda_k^{2(l+1)} |<| p_l\lambda_k^{2l} |$, one may even assume $j \leq l \leq k$. The proof will now be by induction on n. If also $\frac{w}{M_{k-1}}$ is integrable, the problem reduces to $k-1$, because $| \frac{A_{k,j,l}}{A_{k-1,j,l}} |= o(1)$ for $k-1 \geq l \geq j$. Thus, assume that $k = n$ is the only term for which $\frac{w}{M_n}$ is integrable. By using a K.L-transform if necessary, one may also assume $| p_n |=| p_0 |= 1$. Then $w | A_{k,j,l} |^2$ can be estimated by

$$w \mid M_k^{-1} \mid m_k^{-4} \mid M_j^{-2} \mid\mid \frac{p_l'}{p_l} \mid^2\mid p_l\lambda_k^{2l} \mid\mid p_l\lambda_j^{2l} \mid\leq$$

$$\leq\mid M_j^{-1} \mid m_k^{-2} \mid \frac{p_l'}{p_l} \mid^2 w(m_k^{-1} \mid M_j^{-1} \mid\mid p_l\lambda_j^{2l} \mid)(m_k^{-1} \mid M_k^{-1} \mid\mid p_l\lambda_k^{2l} \mid).$$

Since the terms in brackets are bounded and since $| \frac{p_l'}{p_l} |$ is w-square integrable, it remains to show that $m_k^{-2} \mid M_j^{-1} \mid\leq m_k^{-2} \mid M_1^{-1} \mid$ is bounded. Since $M_1 = O(1)$ and since $w \mid M_k^{-1} \mid= w(\mid p_k \mid m_k^{2k-1})^{-1} = wm_k^{-(2k-1)}$ is integrable, while w is not, this follows easily.

□

The proof provided in Lemma 2.1.5 can easily be extended to clusters of arbitrary lengths and the extension of this result will be pursued in Section 2.4.

Theorem 2.1.6. *Consider the differential operator T generated by (1.1), where the coefficients p_k, q_j, $k = 0, \ldots, n$ and $j = 1, \ldots, n$ are twice differentiable, p_k are nowhere zero in $[a, \infty)$ and satisfy (2.1) and (2.11). For the general coefficients, assume the regularity conditions on the form factors are satisfied while for the power type coefficients, assume (2.13) to be satisfied. Moreover, if $p_0 \approx w$, assume the coefficients of the lowest cluster to be almost constant. Then the clusters $k = 1, \ldots, n$ contribute individually to the deficiency index and spectrum, that is, (2.14) holds. Specifically, one has for the k-th cluster, $k > 1$:*

a) $Im\lambda_k \approx m_k$, then one of the two eigenfunctions is z-uniformly square integrable and depends analytically on z so that

$$(defT)_k = (1, 1), \qquad \sigma_{ac}(H)_k = \emptyset.$$

b) *$Im\lambda_k \approx \frac{\eta w}{M_k}$, $\eta = Imz$, $\frac{w}{M_k}$ not integrable. Then $(\mathcal{T}B)_k$ is not square integrable and $(defT)_k = (1,1)$ and this cluster contributes to the absolutely continuous spectrum, $\sigma_{ac}(H,1)_k = \mathbb{R}$. There is an index $r \leq n$ so that $\frac{w}{M_k}$ is integrable for $k \geq r$ and nonintegrable for $1 \leq k < r$.*

c) *$Im\lambda_k \approx \frac{\eta w}{M_k}$, $\frac{w}{M_k}$ integrable then the form factor $(\mathcal{T}B)_k$ is square integrable and both eigenfunctions of cluster k are z-uniformly square integrable. In particular, $(defT)_k = (2,2)$ and $\sigma_{ac}(H)_k = \emptyset$.*

Cluster 1, the smallest cluster, plays a particular role.

d) *If $w = o(p_0)$, the above statements remain valid for cluster 1.*

e) *If $w \approx p_0$ and $p_1 > 0$, let $\bar{c} = sup(\frac{p_0}{w} - \frac{q_1^2}{p_1 w})$ and $\underline{c} = inf(\frac{p_0}{w} - \frac{q_1^2}{p_1 w})$, then if $m_1 = \mid \frac{w}{p_1} \mid^{\frac{1}{2}}$ is nonintegrable, $(defT)_1 = (1,1)$ and $\sigma_{ac}(H,1)_1 \supset [\bar{c}, \infty)$, while $\sigma_{ess}(H)_1 \cap (-\infty, \underline{c}) = \emptyset$.*

f) *With $w \approx p_0$ and $p_1 < 0$, and $\underline{c}$, $\bar{c}$ as above $(defT)_1 = (1,1)$ and $(-\infty, \underline{c}] \subset \sigma_{ac}(H,1)_1$, $(\bar{c}, \infty) \cap \sigma_{ess}(H_1) = \emptyset$.*

Proof. The results about the deficiency index follow directly from (2.12). The square integrability of the form factor $(\mathcal{T}B)_k$ follows immediately from Lemma 2.1.5.

□

The theorem above can be extended in various ways. Here, a Levinson term should signify an expression which is ultimately transformed into $\mathcal{L}^1$. In the proof of Theorem 2.1.6, it has been assumed that Levinson terms are irrelevant for the asymptotics and hence condition (2.9) can be weakened somehow. This can be illustrated easily for coefficients of approximate power type.

The example below shows that condition (2.9), the regularity condition, can be weakened for low lying clusters. That is, the regularity conditions are not necessary for low lying clusters but one has to pay a price for this. If the coefficients are unbounded, then only discrete spectrum can be obtained. The regularity condition can be weakened by performing eigenvalue block transformation for the low lying clusters after the first diagonalisation until the off-diagonal elements are sufficiently small, and a second diagonalisation and further $(I+Q)$-transformations can be carried out as usual. More precisely, for approximate power type coefficients and the lowest cluster, one has

Example 2.1.7. Consider (2.1) with $p_k \sim x^{\alpha_k}$ and assume that (2.9) is violated by the first cluster so that one has instead the following inequality

$$\alpha_n - \alpha_{n-1} < \alpha_{n-1} - \alpha_{n-2} < \ldots < \alpha_2 - \alpha_1 < 2 < \alpha_1 - \alpha_0.$$

By applying a K.L transformation, one can even achieve $\alpha_n = \alpha_0 = 0$. Now the smallest eigenvalue block after the first diagonalisation has the form

$$A = \begin{pmatrix} \lambda_+ & a \\ -a & \lambda_- \end{pmatrix}$$

with $\lambda_\pm$ given by (2.2), $k = 1$ and a determined by $(\mathcal{T}^{-1}\mathcal{T}')_{1,2}$. This may be essentially evaluated at the small eigenvalue part of the polynomial. a is complex valued and $a \sim x^{-1}$, while $\lambda_\pm \approx x^{-\frac{\alpha_1}{2}}$. Thus, as in [14, 34], one uses a block transformation, which only acts on the small eigenvalue block to transform the system into Levinson's form, before proceeding with other transformations. The transformation $S = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ turns A into

$$S^{-1}AS = \frac{1}{2} \begin{pmatrix} \lambda_+ + \lambda_- - 2ia & -i(\lambda_+ - \lambda_-) \\ i(\lambda_+ - \lambda_-) & \lambda_+ + \lambda_- + 2ia \end{pmatrix}.$$

Since $\lambda_\pm \in \mathcal{L}^1$, the small eigenfunctions are essentially given by (2.12). They are independent of z though, which by itself implies the discreteness of the spectrum. Thus, one sees as in [14] that $\sigma(H)_1$ is discrete while $(\mathrm{def}T)_1 = (1,1)((2,2))$ if $p_1 \to \infty$ ($p_1 \to -\infty$). The case where $p_1 \sim x^2$ is described in [14, Theorem 4.2]. In this case, absolutely continuous spectrum may arise.

The extension of the above example to more general coefficients or more of the low lying clusters should be obvious and will be pursued in Section 2.6.

Remark 2.1.8. The restriction in Theorem 2.1.6 (e) and (f) to the complement of $[\underline{c}, \overline{c}]$ is necessary. As an example, consider a differential operator T generated by $\tau y = -y'' + A(\cos x^\alpha)y$ with $0 < \alpha < 1$. This operator could lead to singular continuous spectrum in $[-A, A]$. These methods, however, only show $\sigma_{ess}(H) = [-A, \infty)$ and $\sigma_{ac}(H, 1) \supset [A, \infty)$ in this case.

2.2 Roots of Polynomials

In this section and the subsequent ones, the previous analysis will be pursued for systems for which the eigenvalues have distinct sizes and fall into clusters of arbitrary length. To some extent, this requires that the coefficients satisfy some rather stringent growth conditions. Actually, one will assume that the coefficients form bands leading to eigenvalues of distinct sizes. The endpoint of these bands will be called pivot coefficients, since their growth determines the magnitude of the eigenvalues. Needless to say that this set up extends and unifies approaches that have been used in connection with the analysis of the deficiency index [33] and Section 2.1. In particular, it will be an extension of the results of Section 2.1 to clusters of general length, that is, three or more. So the starting point are general characteristic Fourier polynomials (1.24) and (1.25) that are associated to differential operators generated by (1.1) and (1.2) respectively.

To unify the analysis of (1.24) and (1.25), one introduces pivot coefficients $r_{k_j}(x)$, $0 \leq j \leq l = n$ in case of even order operator or $0 \leq j \leq n+1$ for odd order operators. It will be assumed that $r_{2k} = p_k$ and $r_{2k-1} = 2q_k$. The introduction of pivot coefficients will make the analysis of the roots of the characteristic Fourier polynomial independent of the order of the differential operator unless stated otherwise. Now assume the pivot indices to be $n = k_l > k_{l-1} \cdots k_j > k_{j-1} > \cdots > k_1 > k_0 = 0$ and let r_{k_j} be pivot coefficients associated with p_k or q_j. The pivot coefficients r_{k_j} $(1 \leq j \leq n)$ are assumed to be nonzero for $x \geq a > 0$ and will limit the growth of the in-between coefficients. Absorbing zw into p_0 as mentioned earlier and replacing ν by λ, (1.24) and (1.25) can be written as

$$\mathcal{P}(\lambda, x, z) = \sum_{k=0}^{n} r_k \lambda^k. \tag{2.16}$$

Although the odd order coefficients have a factor 2, for the simplicity of the analysis, these odd order coefficients will also be normalised correspondingly to r_k. This type of notation is chosen because in the remaining analysis of the roots, it is irrelevant, whether a coefficient is even or odd. The jth segmental polynomial will be controlled mainly by the pivot coefficients r_{k_j} and $r_{k_{j-1}}$. One then defines the jth segmental polynomial as

$$\mathcal{P}_j(\lambda, x, z) = \sum_{k=k_{j-1}}^{k_j} r_k \lambda^{k-k_{j-1}}. \tag{2.17}$$

This interval can be of odd or even length depending on which coefficients of odd or even order the end pivot coefficients are associated to. To this interval one then associates

$$m_j = \mid \frac{r_{k_{j-1}}}{r_{k_j}} \mid^{\frac{1}{k_j - k_{j-1}}} \tag{2.18}$$

and demands that $m_l \gg m_{l-1} \gg \cdots \gg m_1$. This means that $m_k = o(m_j)$ for $k = 1, \ldots, j-1$. Hence the condition

$$\mid r_{k_{j-2}} \mid^{k_j - k_{j-1}} \cdot \mid r_{k_j} \mid^{k_{j-1} - k_{j-2}} = o(\mid r_{k_{j-1}} \mid^{k_j - k_{j-2}}) \tag{2.19}$$

will be necessary.

The m_j will be a measure of the size of the eigenvalues in any cluster j if for the in-between coefficients r_k, $k_{j-1} < k < k_j$, one has

$$\mid r_k \frac{m_j^{(k-k_j)}}{r_{k_j}} \mid \approx \mid r_k \frac{m_j^{(k-k_{j-1})}}{r_{k_{j-1}}} \mid = O(1). \tag{2.20}$$

In order to see this, apply to the cluster polynomial $\mathcal{P}_j$ a normalising transform by dividing by $r_{k_{j-1}}$ and factoring $\lambda = m_j \cdot \nu$. Thus

$$\begin{aligned} \widetilde{\mathcal{P}_j}(x, \lambda, z) = \frac{\mathcal{P}_j(x, \lambda, z)}{r_{k_{j-1}}} &= \sum_{k=k_{j-1}}^{k_j} \frac{1}{r_{k_{j-1}}} (r_k m_j^{k-k_{j-1}}) \nu^{k-k_{j-1}} \\ &= \sum_{k=k_{j-1}}^{k_j} \widetilde{r}_k \nu^{k-k_{j-1}}, \end{aligned}$$

where $\widetilde{r}_k = \frac{r_k}{r_{k_{j-1}}} m_j^{k-k_{j-1}}$. One then demands that $\widetilde{r}_k = O(1)$ which is just (2.20). Throughout, $\widetilde{\mathcal{P}}_j(x, \nu, z)$ will be called the jth reduced cluster polynomial.

Lemma 2.2.1. *Consider a polynomial $P(\gamma) = a_n\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0$ such that $\mid a_0 \mid, \mid a_n \mid = 1$ and a_i $(1 \leq i \leq n-1)$ are bounded. Then there exists a constant K so that for any root γ of $P(\gamma)$, one has $K^{-1} \leq \mid \gamma \mid \leq K$.*

Proof. Assume that $| a_i |\leq M$ with $M \geq 1$ for all $i = 1, \ldots, n-1$. Then each root γ of the polynomial $P(\gamma)$ satisfies $| \gamma |\geq \frac{1}{2M(n+1)}$. The proof will be by contradiction. Assume that γ with $| \gamma |< \frac{1}{2M(n+1)}$ is a root of $P(\gamma)$, then it follows that

$$\begin{aligned} 1 =| a_0 | = | \sum_i^n a_i \gamma^i |\leq \sum_i^n | a_i || \gamma |^i \\ < M \sum_{i=1}^n \left(\frac{1}{2M(n+1)} \right)^i < \frac{1}{2}, \end{aligned}$$

which leads to the desired contradiction. The proof in the other direction is now similar. Just replace γ by γ^{-1}.

□

The uniform dichotomy condition as demanded in the book of Eastham [33] is stronger but in our case we will only need a weaker version even though for our analysis we need a z-uniform dichotomy condition. The uniform dichotomy condition that will suffice in this study simply states that sign $\mathrm{Re}(\lambda_j(x,z) - \lambda_i(x,z))$ is constant modulo $\mathcal{L}^1$ for any unequal pair of indices i and j. The dichotomy condition thus gives an x-uniform control of different solutions of the unperturbed equation. It says that the ratio of two given solutions does not oscillate between zero and infinity for large values of x but the change of behaviour should take place in a controlled fashion. Since the analysis is carried out for Fourier characteristic polynomials, throughout, the following assumptions will suffice as a dichotomy condition and should hold uniformly in x and z.

- D(i) The ν_s, $(s = 1, \ldots, k_j - k_{j-1})$, roots of the reduced cluster polynomial $\widetilde{\mathcal{P}}_j$ are distinct for all $x \geq a$ and for all $z \in \mathcal{K}$.
- D(ii) The sign $\mathrm{Im}(\nu_i(x,z) - \nu_k(x,z))$ is constant modulo $\mathcal{L}^1$ for all $x \geq a$, $\forall z \in \mathcal{K}$, for each unequal pair of indices k and i of the same cluster and also
  $| \mathrm{Im}\nu_i(x,z) |\geq c > 0$ and $\mathrm{Im}\nu_i(x,z)$ has constant sign for all $x \geq a$ and $\forall z \in \mathcal{K}$, for the nonreal eigenvalues of $\widetilde{\mathcal{P}}_j(x,\nu,z)$, $j = 1, \ldots, n$.
- D(iii) $| \partial_\lambda \widetilde{\mathcal{P}}_j(\lambda_k(x,z)) |> 0$ and $\partial_\lambda \widetilde{\mathcal{P}}_j(\lambda_k(x,z))$ has constant sign for all $x \geq a$ and $\forall z \in \mathcal{K}$, $k = 1, \ldots, k_j - k_{j-1}$, $j = 1, \ldots, n$ for all real eigenvalues.

D(iv) For real eigenvalues $\lambda_i(x,z)$ and $\lambda_j(x,z)$ of the same cluster k, $k > 1$, such that $\lambda_i(x,z) \neq \lambda_j(x,z)$, and $\frac{w}{M_k}$ is nonintegrable, we will assume that $\partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_j(x,z)) \neq \partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_i(x,z))$ and $\partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_j(x,z)) - \partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_i(x,z))$ has a constant sign for all $x \geq a$ and for all $z \in \mathcal{K}$.

The roots of the reduced Fourier cluster polynomial ν now determine the roots of the cluster polynomial $\mathcal{P}_j$ which lie in circular bands of almost equal magnitude. In turn, the roots of the $\mathcal{P}_j$ are almost the roots of the Fourier polynomial $\mathcal{P}_F$. In the remaining part of the study, the analysis will be carried out based on the Fourier polynomial $\mathcal{P}_F(x,\lambda,z)$. So the subscript F will be dropped temporarily unless there is confusion arising, then it will be stated clearly.

Now consider the Fourier characteristic polynomial $\mathcal{P}$ with reduced cluster polynomials $\widetilde{\mathcal{P}}_j$, $j = 1,\ldots,n$ where j is fixed. Then denote the corresponding reduced Fourier characteristic polynomial by $\widetilde{\mathcal{P}}$ and write

$$\widetilde{\mathcal{P}} = \widetilde{R}_1 + \widetilde{\mathcal{P}}_j + \widetilde{R}_2, \tag{2.21}$$

where $\widetilde{R}_1 = \sum_{k=0}^{k_{j-1}-1} \frac{r_k}{r_{k_{j-1}}} \lambda_j^{k-k_{j-1}}$ and $\widetilde{R}_2 = \sum_{k=k_j+1}^{n} \frac{r_k}{r_{k_{j-1}}} \lambda_j^{k-k_{j-1}}$ with
$\widetilde{R}_1, \widetilde{R}_2 \to 0$ uniformly in λ_j.

In the remaining part of the analysis, the term $O(\max_j(\frac{m_{j-1}}{m_j}))$ will appear quite often thus we define a function $c(x,z) = O(\max_j(\frac{m_{j-1}}{m_j}))$. In most cases, we will denote this function as c but one has to remember that it is a function of x and z. Of course, $c(x,z) = o(1)$.

Theorem 2.2.2. *a) Let r_k be any coefficient of (2.16) such that k is an index outside the jth cluster interval $[k_{j-1}, k_j]$. Then $r_k(m_j)^k \ll r_{k_j} m_j^{k_j}$.*

b) The functions $\widetilde{R}_i(\lambda)$, $i = 1,2$ defined by (2.21) satisfy $\widetilde{R}_i(\lambda) \approx c(x,z)$.

Proof. It suffices to show that for any k outside the jth cluster

$$\frac{r_k}{r_{k_j}} m_j^{k-k_j} \approx c(x,z). \tag{2.22}$$

First consider the pivot coefficients r_{k_l} outside the jth cluster. By equation (2.19) and (2.20), one has

$$r_{k_{j-2}} m_{j-1}^{k_{j-2}} \approx r_{k_{j-1}} m_{j-1}^{k_{j-1}} = r_{k_{j-1}} m_j^{k_{j-1}} (\frac{m_{j-1}}{m_j})^{k_{j-1}} \ll r_{k_{j-1}} m_j^{k_{j-1}}.$$

This is true because $m_{j-1} \ll m_j$. Continuing this way, one easily obtains $r_{k_l} m_j^{k_l} \ll r_{k_j} m_j^{k_j}$ for $k_l < k_j$. Now consider $k_l > k_j$ and begin with $k_l = k_{j+1}$, then one obtains $r_{k_{j+1}} m_{j+1}^{k_{j+1}} \approx r_{k_j} m_{j+1}^{k_j}$ which implies that $\frac{r_{k_{j+1}}}{r_{k_j}} \approx \frac{m_{j+1}^{k_j}}{m_{j+1}^{k_{j+1}}}$. Hence, it follows that

$$\frac{r_{k_{j+1}}}{r_{k_j}} \cdot \frac{m_j^{k_{j+1}}}{m_j^{k_j}} = m_{j+1}^{(k_j - k_{j+1})} \cdot \frac{m_j^{k_{j+1}}}{m_j^{k_j}} \approx (\frac{m_j}{m_{j+1}})^{(k_{j+1} - k_j)} \approx o(1).$$

By induction, the claim is now true for all pivot coefficients outside the $[k_{j-1}, k_j]$ interval.

Assume now that k is an index from the ith cluster such that $i < j$. With (2.20), one has $r_k m_i^k \approx r_{k_{i-1}} m_i^{k_{i-1}} \approx r_{k_i} m_i^{k_i}$. It now follows that

$$r_k \approx O(r_{k_i} m_i^{(k_i - k)}).$$

So that

$$\begin{aligned} r_k m_j^k &\approx O(r_{k_i} m_j^{k_i} m_i^{(k_i - k)} m_j^{(k - k_i)}) \\ &\approx O(r_{k_i} m_j^{k_i} \cdot \frac{m_i^{(k_i - k)}}{m_j^{(k_i - k)}}) \\ &\approx o(r_{k_i} m_j^{k_i}). \end{aligned}$$

This is true because $m_i \ll m_j$ and $k_i - k > 0$. The claim now follows since for all pivot coefficients it has been shown that $r_{k_i} m_j^{k_i} = o(r_{k_j} m_j^{k_j})$ for k_i an index outside the jth cluster.

Now consider the ith-interval with $k_i > k_j$ ($j < i$) and let again k be from the $[k_{i-1}, k_i]$ interval. Then one has $r_k \approx O(r_{k_{i-1}} m_i^{(k_{i-1} - k)})$ so that

$$\begin{aligned} r_k m_j^k &\approx O(r_{k_{i-1}} m_i^{(k_{i-1} - k)} m_j^k) \\ &\approx O(r_{k_{i-1}} m_j^{k_{i-1}} m_i^{(k_{i-1} - k)} m_j^{(k - k_{i-1})}) \approx o(r_{k_{i-1}} m_j^{k_{i-1}}). \end{aligned}$$

The claim again follows at once since $m_j \ll m_i$ and $k - k_{i-1} > 0$. This proves (a).

In order to prove (b), the same arguments in (a) apply and by induction one can easily show that

$$\frac{r_{k_l}}{r_{k_{j-1}}} \approx m_j^{k_{j-1}-k_l} (\frac{m_l}{m_j})^{k_{j-1}-k_l}, \qquad l < j-1.$$

It follows that $\widetilde{R}_1$ can be expressed as

$$\widetilde{R}_1 = \sum_{l=0}^{j-1} m_j^{k_{j-1}-k_l} (m_j \nu)^{k_l - k_{j-1}} (\frac{m_l}{m_j})^{k_{j-1}-k_l}, \qquad l < j-1. \tag{2.23}$$

(2.23) holds also for arbitrary indices k, such that $k < j-1$. In that case, simply replace k_l by k and note that $k < j-1$. Since ν is finite and bounded away from zero by results of Lemma 2.2.1, and for the lower clusters $m_l \ll m_j$, it now follows that $| \widetilde{R}_1 |= o(1)$. In the case of upper clusters, one can equally show by induction that

$$\frac{r_{k_l}}{r_{k_{j-1}}} \approx m_j^{k_{j-1}-k_l} (\frac{m_j}{m_l})^{k_l - k_j}, \qquad j < l,$$

and hence $\widetilde{R}_2$ can be expressed as

$$\widetilde{R}_2 = \sum_{l=j+1}^{n} m_j^{k_{j-1}-k_l} (m_j \nu)^{k_l - k_{j-1}} (\frac{m_j}{m_l})^{k_l - k_j}, \qquad j < l. \tag{2.24}$$

This expression again holds for arbitrary indices k, $k > j$. In that case, simply replace k_l by k and note again that $k > j$. Since ν is finite and bounded away from zero, and for upper clusters $m_l \gg m_j$. It now follows that $| \widetilde{R}_2 |= o(1)$.

□

It should be noted, however, that the further one moves away from the cluster polynomial, the smaller the terms $\frac{m_l}{m_j}$ $(l < j-1)$ and $\frac{m_j}{m_l}$ $(j < l)$ become. Since $\widetilde{R} = \widetilde{R}_1 + \widetilde{R}_2$ is a meromorphic function of λ outside the set of eigenvalues, $\widetilde{R}_i(\lambda)$ is analytic even. Its first derivative exists and by previous arguments can be estimated as

$$\begin{aligned} \widetilde{R}' = & \sum_{k < k_{j-1}} m_j^{k_{j-1}-k} (m_j \nu)^{k - k_{j-1} - 1} (\frac{m_k}{m_j})^{k_{j-1}-k} \\ & + \sum_{k > k_j} m_j^{k_{j-1}-k} (m_j \nu)^{k - k_{j-1} - 1} (\frac{m_j}{m_k})^{k - k_j}. \end{aligned}$$

Similarly, one can show that $\widetilde{R}_i'(\lambda, x) \to 0$ as $x \to \infty$.

The M-factors as defined in chapter one are given by $M(\lambda_j) = \partial_\lambda \mathcal{P}(\lambda_j)$. Here, $\mid M_j \mid = \mid \partial_{\lambda_j} \mathcal{P}_j(\lambda_j) \mid \approx \mid \partial_{\lambda_j}(\sum_{l=0}^{k_j-k_{j-1}} r_l \lambda_j^l) \mid$ will be the measure of the size of the M-factor corresponding to a particular eigenvalue λ_j.

Corollary 2.2.3. *The size of the M-factors can be estimated by* $\mid M_j \mid \approx \mid r_{k_j} \mid m_j^{k_j-1}$ *and* $\mid M_l \mid \gg \mid M_{l-1} \mid \gg \cdots \gg \mid M_1 \mid$.

Proof. It suffices to show that $\sum_{k=1}^n \mid r_k \mid m_j^{k-1}$ for k outside the jth cluster interval is small. This is similar to showing that $\mid r_k \mid m_j^{k-1} \approx o(r_{k_j} m_j^{k_j-1})$ for every k outside the jth cluster interval $[k_{j-1}, k_j]$. The procedure of doing this is similar to the proof in Theorem 2.2.2 and only the outline will be included here with details omitted. Thus for an index k from the ith cluster with $i < j$ and with (2.20), one has $\mid r_k \mid m_i^{k-1} \approx \mid r_{k_{i-1}} \mid m_i^{k_{i-1}-1} \approx \mid r_{k_i} \mid m_i^{k_i-1}$. It now follows that

$$\mid r_k \mid \approx O(\mid r_{k_i} \mid m_i^{(k_i-k)}).$$

So that

$$\begin{aligned} \mid r_k \mid m_j^{k-1} &\approx O(\mid r_{k_i} \mid m_j^{k_i-1} m_i^{(k_i-k)} m_j^{(k-k_i)}) \\ &\approx O(\mid r_{k_i} \mid m_j^{k_i-1} \cdot \frac{m_i^{(k_i-k)}}{m_j^{(k_i-k)}}) \\ &\approx o(\mid r_{k_i} \mid m_j^{k_i-1}). \end{aligned}$$

This is true because $m_i \ll m_j$ and $k_i - k > 0$.
Now consider ith interval with $k_i > k_j$ $(j < i)$ and let again k be from the $[k_{i-1}, k_i]$ interval. Then one has $\mid r_k \mid \approx O(\mid r_{k_{i-1}} \mid m_i^{(k_{i-1}-k)})$ so that

$$\begin{aligned} \mid r_k \mid m_j^{k-1} &\approx O(\mid r_{k_{i-1}} \mid m_i^{(k_{i-1}-k)} m_j^{k-1}) \\ &\approx O(\mid r_{k_{i-1}} \mid m_j^{k_{i-1}-1} m_i^{(k_{i-1}-k)} m_j^{(k-k_{i-1})}) \\ &\approx o(\mid r_{k_{i-1}} \mid m_j^{k_{i-1}-1}). \end{aligned}$$

The claim again follows at once since $m_j \ll m_i$ and $k - k_{i-1} > 0$.
Assume that $\mid M_j \mid \approx \mid r_{k_{j-1}} \mid (m_j \mid \nu \mid)^{(k_{j-1}-1)}$ and M_{j-1} is from $(j-1)$th cluster.

It then follows that $| M_{j-1} | \approx | r_{k_{j-1}} | (m_{j-1} | \nu |)^{(k_{j-1}-1)}$. But $m_j \gg m_{j-1}$ implies that

$$\begin{aligned} | M_j | &\approx | r_{k_{j-1}} | (m_j | \nu |)^{(k_{j-1}-1)} \gg | r_{k_{j-1}} | (m_{j-1} | \nu |)^{(k_{j-1}-1)} \\ &\approx | M_{j-1} | . \end{aligned}$$

The proof is then completed by induction since

$$m_l \gg m_{l-1} \gg \cdots \gg m_1.$$

□

Theorem 2.2.4. *a) To each root $\lambda(x,z)$ of $\mathcal{P}_F(x,\lambda,z)$, there exists a unique interval (segment) j and a root $\lambda_j(x,z)$ of $\mathcal{P}_j(x,\lambda,z)$, the jth segmental shadow eigenvalue of λ, such that $| \lambda(x,z) - \lambda_j(x,z) | \leq c(x,z)$ where $c(x,z) = O(\max_j(\frac{m_{j-1}}{m_j}))$, $j = 1,\ldots,n$.*

b) $\lambda(x,z)$ is real if and only if its segmental shadow eigenvalue $\lambda_j(x,z)$ is real. Similarly, $\lambda(x,z)$ is non-real if and only if its segmental shadow eigenvalue $\lambda_j(x,z)$ is non-real.

Here, a is choosen so large so that all the eigenvalues of the Fourier polynomial $\mathcal{P}_F(x,\lambda,z)$ are distinct and satisfy

$$| \lambda(x,z) - \lambda_j(x,z) | \leq c(x,z).$$

Proof. Let $\mathcal{P}_F$ be the Fourier polynomial in (2.16). Now write $\mathcal{P}_F$ as $\mathcal{P}_F = R_1 + \mathcal{P}_j + R_2$. Normalise $\mathcal{P}_F$ by dividing it by $r_{k_{j-1}}\lambda_j^{k_{j-1}}$ ($\lambda_j = m_j\nu$) to obtain $\widetilde{\mathcal{P}}_F$. It thus follows that $\widetilde{\mathcal{P}}_F(\nu,x,z) = \widetilde{\mathcal{P}}_j(\nu,x,z) + \widetilde{\mathcal{R}}(\nu,x,z)$ which is similar to (2.21) with $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_1 + \widetilde{\mathcal{R}}_2$. Let ν_0 be a fixed root of $\widetilde{\mathcal{P}}_j$ and write $\widetilde{\mathcal{P}}_j = (\nu - \nu_0)\widetilde{Q}_j(\nu)$. Then the equation $\widetilde{\mathcal{P}}_F = 0$ amounts to: $0 = \nu - \nu_0 + \frac{\widetilde{R}}{\widetilde{Q}_j}(\nu)$. Without loss of generality, restrict this equation to a bounded neighbourhood of ν_0 which may be taken as a disc of radius $\frac{c}{4}$, where $c = O(\max_j(\frac{m_{j-1}}{m_j}))$, $j = 1,\ldots,n$. It now remains to find a root $\widetilde{\nu}_0$ within this neighbourhood that satisfies the equation $0 = (\widetilde{\nu}_0 - \nu_0) + \frac{\widetilde{R}}{\widetilde{Q}}(\widetilde{\nu}_0)$. Now let $\frac{\widetilde{R}(\nu)}{\widetilde{Q}(\nu)} = f(x,\nu)$ for a large x and use the estimates $f', f \approx \max(\frac{m_{j-1}}{m_j})$, $j = 2,\ldots,n$. The proof now follows by an obvious iteration scheme. By applying the method of successive iterations to the equation $0 = \nu - \nu_0 + f(\nu,x)$, define the sequence $\{\nu_i\}_{i=0}^n$ where $\nu_{i+1} = \nu_0 - f(x,\nu_i)$. The function $f(x,\nu)$ caters for the remainder term $\widetilde{\mathcal{R}}(\nu)$. Because of this, it follows that in approximating the λ roots of the Fourier polynomial by their corresponding segmental shadow eigenvalues λ_j, the correction terms as a result of this approximation are built into the iteration procedure through this function $f(x,\nu)$. One now has to show that the sequence $\{\nu_i\}_{i=0}^n$ converges uniformly on the disc of radius $\frac{c}{4}$ to a limiting root $\widetilde{\nu}_0$.

By the mean value theorem, one has

$$\begin{aligned}\mid \nu_{n+1}-\nu_n \mid &= \mid f(x,\nu_n)-f(x,\nu_{n-1}) \mid \\ &\leq \max_{\nu_\omega\in(\nu_{n-1},\nu_n)} \mid (f(x,\nu_\omega))' \mid\mid \nu_n-\nu_{n-1} \mid < \frac{c}{4} \mid \nu_n-\nu_{n-1} \mid .\end{aligned}$$

Thus one obtains

$$\mid \nu_{n+1}-\nu_n \mid < (\frac{c}{4})^n \mid \nu_1-\nu_0 \mid .$$

If one further assumes that $\mid \nu_1-\nu_0 \mid = \mid f(x,\nu_0) \mid \leq \frac{c}{8}$, one easily gets

$$\begin{aligned}\mid \nu_n-\nu_0 \mid &\leq \mid \nu_n-\nu_{n-1} \mid + \mid \nu_{n-1}-\nu_{n-2} \mid + \cdots + \mid \nu_1-\nu_0 \mid \\ &< \sum_{k=0}^{n-1}(\frac{c}{4})^k \mid \nu_1-\nu_0 \mid \leq \frac{c}{8}\sum_{k=0}^{n-1}(\frac{c}{4})^k.\end{aligned}$$

As $n\to\infty$, $\nu_n\to\nu_\infty$, it follows that $\mid \nu_\infty-\nu_0 \mid < \frac{c}{4}$ and one has $\nu_\infty=\nu_0-f(x,\nu_\infty)$ with $f(x,\nu_\infty)\to 0$ uniformly as $x\to\infty$. One thus concludes that the sequence $\{\nu_i\}_{i=0}^\infty$ converges uniformly in the bounded neighbourhood of ν_0 as $x\to\infty$. Let the limit of this convergence be $\widetilde{\nu}_0$. The uniqueness of this convergence now follows by the Banach fixed point theorem. Since for a real spectral parameter z the remainder term $\widetilde{\mathcal{R}}(\nu)$ is real, the correction term to the segmental shadow eigenvalue is real and thus a real segmental shadow eigenvalue leads only to a real root of the Fourier polynomial. The proof for a complex root is similar. One only notes that if ν_0 is a root then $\overline{\nu}_0$ is a root too.

□

Since all the eigenvalues of the reduced cluster polynomials are distinct by the assumption $D(i)$, thus by iteration, we obtain all the eigenvalues of the Fourier polynomial which will be distinct too. Therefore, the Fourier polynomial has $2n$ or $2n+1$ distinct roots for even or odd order differential operators respectively.

For a non-real valued spectral parameter $z = z_0 + i\eta$, η small and $z_0 \in \mathcal{K}_\epsilon(z_0)$, it follows that for each root λ of the Fourier polynomial $\mathcal{P}(x, \lambda, z)$ with $\lambda(z) = \lambda(z_0) + i\eta w(\partial_\lambda \mathcal{P}(\lambda(z_0))^{-1}$, there exists a unique jth segmental shadow root λ_j of $\mathcal{P}_j(x, \lambda, z)$ with $\lambda_j(z) = \lambda_j(z_0) + i\eta w(\partial_\lambda \mathcal{P}_j(\lambda_j(z_0))^{-1}$ such that $\mid \lambda(z_0) - \lambda_j(z_0) \mid \leq c(x, z)$. $i\eta w(\partial_\lambda \mathcal{P}(\lambda(z_0))^{-1}$ and $i\eta w(\partial_\lambda \mathcal{P}_j(\lambda_j(z_0))^{-1}$ are the perturbation terms of the imaginary part of the spectral parameter z for λ and λ_j respectively. By results of Corollary 2.2.3, they can be estimated by $\frac{i\eta w}{M_j}$ and $\frac{i\eta w}{M_j m_j^{-k_j - 1}}$ respectively. In general, up to a multiplication by a constant, the two values can be estimated by $O(\frac{i\eta w}{M_j})$.

One could also use Kantorovic theorem [4] to show the existence of the roots of $\mathcal{P}$ near the roots of $\mathcal{P}_j$.

2.3 The Dichotomy Condition

Once the approximate values for the roots of the Fourier polynomial $\mathcal{P}_F(x, \lambda, z)$ and the M-factors are known, one has enough ingredients to establish the uniform dichotomy condition for the eigenvalues of the differential operator. As stated earlier, the uniform dichotomy condition in Levinson's Theorem guarantees an x-uniform control of the unperturbed equation $u' = \Lambda u$ which in some sense is also uniform in Imz, $0 \leq \text{Im}z = \eta \leq \epsilon$. Since the roots of $\mathcal{P}(x, \lambda, z)$ are calculated from characteristic Fourier polynomial, the uniform dichotomy condition needed is equivalent to sign $\text{Im}(\lambda_j(x, z) - \lambda_i(x, z))$ being constant modulo $\mathcal{L}^1$ for all unequal pair of indices i and j. In spectral theory, a z-uniform dichotomy condition is needed in general but this will only be relevant for the first cluster if $p_0 \approx w$. In this study, slightly stronger conditions will suffice. Before the proof of dichotomy condition, one needs the following results.

Theorem 2.3.1. *Consider the system $u' = (\Lambda + R)u$ and assume $\lambda_i(x) = \lambda_{i0} + \lambda_{i1}(x) + \lambda_{i2}(x)$ with $\lambda_{i1} = o(1)$ and $\lambda_{i2}(x)$ conditionally integrable, $i = 1, \ldots, 2n$. Sort the eigenvalues into classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ so that*

i) $\lambda_i \in \mathcal{C}_l$ then $Re\lambda_{i0} = \alpha_l$, where α_l is a constant.

ii) $\lambda_i \in \mathcal{C}_l, \lambda_j \in \mathcal{C}_m, l \neq m$ then $|Re(\lambda_{i0} - \lambda_{j0})| \geq \delta > 0, \quad l, m = l, \ldots, k$.

Now let $m_{l\pm} = max_{\lambda_i \in \mathcal{C}_l}(Re\lambda_{i1}(x))_\pm$*, where* $f_{(\pm)}$ *denotes the positive (negative) part of* $f, f = f_+ - f_-$*. Let* $|\mathcal{C}_l|$ *denote the number of elements in* $\mathcal{C}_l$*. Then the system has* $|\mathcal{C}_l|$ *independent solutions* u *associated to* $\mathcal{C}_l$ *satisfying*

$$(2.25) K_1 \exp(\alpha_l x - \int_a^x m_{l-}(t)\,dt) \le ||u(x)|| \le K_2 \exp(\alpha_l x + \int_a^x m_{l+}(t)\,dt),$$

where K_1 *and* K_2 *are constants.*

Since $\lambda_{i2}(x)$ is conditionally integrable, a simple transformation $\exp(\int_0^x \Lambda_{i2}(t)dt)$ eliminates these terms while preserving the $\mathcal{L}^1$ nature of the off-diagonal terms. The rest of the proof follows directly from that of [12, Theorem 2.1].

Assume the roots of the Fourier polynomial $\mathcal{P}(x, \lambda, z)$ for $z_0 \in \mathbb{R}$ are $\alpha_1 \pm i\beta_1, \ldots, \alpha_r \pm i\beta_r, \gamma_{2r+1}, \ldots, \gamma_{2n}(\gamma_{2n+1})$ where α_j, β_j and γ_l are functions of x and z, with $\alpha_j(x,z), \beta_j(x,z), \gamma_l(x,z) \in \mathbb{R}$, then Theorem 2.3.1 implies that the nonreal eigenvalues lead to r square integrable solutions, which decay exponentially and a corresponding set of exponentially increasing solutions. This holds regardless of the dichotomy conditions. For the real eigenvalues, one has by the implicit function theorem

$$\gamma_l(x,z) = \gamma_l(x,z_0) + (\partial_\lambda \mathcal{P}(x,\gamma_l,z))^{-1}(z - z_0) \quad \text{for small} \quad | z - z_0 | \,.$$

Thus, the dichotomy condition holds if

$$(\partial_\lambda \mathcal{P}(\gamma_l)) \ne (\partial_\lambda \mathcal{P}(\gamma_m)), \qquad l \ne m,$$

because $(\partial_\lambda \mathcal{P}(\lambda))$ is real. $\gamma_l = \gamma_l(x, z_0)$ will then contribute to the deficiency index if $(\partial_\lambda \mathcal{P}(\gamma_l)) < 0$, because the corresponding exponent is $-i\gamma_l(x, z_0 + i\eta) \approx -i\gamma_l(x, z_0) + \eta(\partial_\lambda \mathcal{P}(\gamma_l(x, z_0)))^{-1}$. The associated eigenfuctions, however, will lose their square integrability as $\eta \to 0_+$ if the coefficients are bounded. Since the signs of $(\partial_\lambda \mathcal{P}(\gamma_l))$ are evenly distributed, half of the γ's will lead to square integrable solutions for $\eta = \text{Im}z > 0$ and the complementary γ's will lead to square integrable solutions in the lower half plane. This shows that for bounded coefficients with an even number of eigenvalues say $2n$, T is limit point and def$T = (n, n)$. If the coefficients are unbounded, some of the γ-eigenfunctions may stay square integrable as $\text{Im}z \searrow 0$ and T may be non-limit-point. But in all cases, it suffices to check the dichotomy condition only for the real roots of $\mathcal{P}(x,\lambda,z)$)–see [26].

Theorem 2.3.2. *[39, Bezout's Theorem, Theorem 7.5] Suppose two plane algebraic curves C_1 and C_2 have no common curve components (that is, the polynomials defining C_1 and C_2 have no common factor). Then their intersection number is given by*

$$(C_1 \cdot C_2) = \sum_{p \in C_1 \cap C_2} (C_1 \cdot C_2)_p = degC_1 \cdot degC_2.$$

Bezout's theorem implies that two irreducible algebraic curves intersecting in infinitely many points are identical. That is, two distinct irreducible algebraic curves intersect at most in finitely many points.

Lemma 2.3.3. *Assume D((i)-(iv)) hold uniformly in x and z for reduced cluster polynomials $\widetilde{\mathcal{P}}_j(x,\lambda,z)$, $j = 1,\ldots,n$. If $p_0 \approx w$, then assume the coefficients of the lowest cluster to be almost constant. Then the roots of the Fourier polynomial satisfy the uniform dichotomy condition.*

Proof. Since D(i) holds uniformly in x and z for all the reduced cluster polynomials, it implies that all the segmental polynomials $\mathcal{P}_j(x,\lambda,z)$ have distinct roots. By results of Theorem 2.2.4, for each root λ of $\mathcal{P}(x,\lambda,z)$, there is a unique segmental shadow root $\widetilde{\lambda}$ of $\mathcal{P}_j(x,\lambda,z)$, thus the Fourier polynomial $\mathcal{P}(x,\lambda,z)$ has $2n$ distinct roots. The analysis will be restricted to even order operators since the case for odd order operators is similar. Assume now that D((ii)-(iv)) hold uniformly in x and z for all clusters j such that $j > 1$. Then the proof of uniform dichotomy condition is divided into two parts, namely: the dichotomy condition between the eigenvalues of different clusters and the dichotomy condition between the eigenvalues of the same cluster.

a) Assume that the roots $\lambda_j(x,z)$ and $\lambda_l(x,z)$ are from clusters j and l respectively such that $l < j$. Then, one has three cases to consider.

   (i) Assume that $\text{Im}\lambda_j(x,z_0) \approx m_j$, then for any arbitrary $\text{Im}\lambda_l(x,z)$ the dichotomy condition is satisfied since $m_j \gg m_l$.

   (ii) Assume also that $\text{Im}\lambda_j(x,z_0) = 0$ and $\text{Im}\lambda_l(x,z_0) \approx m_l \gg \eta w \mid M_l^{-1} \mid \gg \eta w \mid M_j^{-1} \mid$, then the dichotomy condition is satisfied again.

(iii) Finally, assume $\text{Im}\lambda_j(x, z_0) = \text{Im}\lambda_l(x, z_0)$ or that the two roots are essentially real. Then in both cases, one would consider the z-influence on the zeros of the higher clusters. The dichotomy condition would still be satisfied since D(iii) holds and $| M_l^{-1} | \gg | M_j^{-1} |$.

b) By results of Theorem 2.3.1, for the roots of the same cluster, one only needs to check the dichotomy condition for real roots. Now assume $\lambda_i(x, z)$ and $\lambda_j(x, z)$ to be real eigenvalues of $\mathcal{P}(x, \lambda, z)$ coming from the same cluster k, $k > 1$. If $\frac{w}{M_k}$ is integrable, then the eigenfunctions determined by $\lambda_i(x, z)$ and $\lambda_j(x, z)$ will be square integrable irrespective of the uniform dichotomy condition. Assume that $\frac{w}{M_k}$ is nonintegrable, then the assumption that $\partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_i(x, z)) \neq \partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_j(x, z))$ in D(iv) implies that $\partial_\lambda \mathcal{P}_k(\lambda_i(x, z)) \neq \partial_\lambda \mathcal{P}_k(\lambda_j(x, z))$ if $\lambda_i(x, z) \neq \lambda_j(x, z)$. We need to show that $\partial_\lambda \mathcal{P}(\lambda_i(x, z)) \neq \partial_\lambda \mathcal{P}(\lambda_j(x, z))$.

Now write $\mathcal{P}$ as $\mathcal{P} = \mathcal{P}_k + \mathcal{R}$. It then follows that $\partial_\lambda \mathcal{P}_k(\lambda_i(x, z)) = \partial_\lambda \mathcal{P}(\lambda_i(x, z)) - \partial_\lambda \mathcal{R}(\lambda_i(x, z))$. $\partial_\lambda \mathcal{P}_k(\lambda_j(x, z))$ can be expressed in a similar way. By the triangle inequality

$$\begin{aligned} 0 &< | \partial_\lambda \mathcal{P}_k(\lambda_j(x, z)) - \partial_\lambda \mathcal{P}_k(\lambda_i(x, z)) | \\ &\leq | \partial_\lambda \mathcal{P}(\lambda_j(x, z)) - \partial_\lambda \mathcal{P}(\lambda_i(x, z)) | \\ &+ | \partial_\lambda \mathcal{R}(\lambda_j(x, z)) - \partial_\lambda \mathcal{R}(\lambda_i(x, z)) | \, . \end{aligned}$$

The second term of the last inequality above can be approximated by $c(x, z)$ which tends to zero uniformly both in $\lambda_i(x, z)$ and $\lambda_j(x, z)$ as $x \to \infty$ by results of Theorem 2.2.2 (ii). This is the required result at the cluster level. If $w = o(p_0)$, then the above reasoning applies to the lowest cluster too.

It now remains to establish the z-uniform dichotomy condition within the eigenvalues of the lowest cluster when $p_0 \approx w$. Here also, one follows the same steps as above. Assume $\lambda_k(x, z)$ and $\lambda_1(x, z)$ are roots from clusters k, $k > 1$, and the lowest cluster respectively. In order to show the dichotomy condition between these two eigenvalues, we need to compare $\text{Im}\lambda_k(x, z)$ and $\text{Im}\lambda_1(x, z)$. In this case, one applies similar argument as in a((i)-(iii)) above and the uniform dichotomy condition will follow immediately since $m_k \gg m_1$ and $\eta w \mid M_1^{-1} | \gg \eta w \mid M_k^{-1} |$.

Now we only need to prove z-uniform dichotomy condition for real roots of the lowest cluster because if $p_0 \approx w$, then the dichotomy condition is needed uniformly in z. The proof now follow closely the techniques used in [26]. There exists a countable exceptional set $\mathcal{E}_0$ such that the lowest cluster Fourier polynomial $\mathcal{P}_1(x, \lambda, z)$ has multiple roots. we will exclude multiple roots of the lowest Fourier polynomial $\mathcal{P}_1(x, \lambda, z)$ from our study. Thus let $-\infty < \omega_1 < \omega_2 < \cdots < \omega_k < \infty$ be the enumeration of all the spectral values which lead to double roots of the Fourier cluster polynomial $\mathcal{P}_1(x, \lambda, z)$. Because the coefficients of lowest cluster in this case are assumed to be almost constant, and since the eigenvalues are analytic functions of $z \in \mathcal{K}_\epsilon(z_0)$ and the coefficients, the eigenvalues $\lambda(x, z)$ of the lowest cluster Fourier polynomial $\mathcal{P}_1(x, \lambda, z)$ will converge to the eigenvalues $\lambda(z)$ of the limiting lowest cluster Fourier polynomial $\mathcal{P}_{1,0}(x, \lambda, z)$. Thus with $z \in \mathcal{K} \cap \mathbb{R}$, one defines a set

$$U_{1,2} = \left\{ z \in (\omega_1, \omega_{i+1}) \mid \quad \frac{d\lambda_1(z)}{dz} = \frac{d\lambda_2(z)}{dz} \right\}$$

for two distinct real roots of the first cluster $\lambda_1(z)$ and $\lambda_2(z)$. We show that uniform dichotomy condition is satisfied for z values off this set $U_{1,2}$. If $U_{1,2}$ is countable with accumulation points (ω_i, ω_{i+1}), then one has uniform dichotomy condition satisfied for a real z outside $U_{1,2}$, because $\lambda_1(z+i\eta) = \lambda_1(z)+iw\eta\frac{d\lambda_1(z)}{dz}$ with $\frac{d\lambda_1(z)}{dz}$ real by implicit function theorem. Since the eigenvalues of the first cluster depend analytically on the pivot coefficients r_{k_1} and $p_0 - z$, this extends immediately to $\lambda_1(z + i\eta)$ by continuity. This, however, is the required uniform dichotomy condition for $\lambda_1(z)$ and $\lambda_2(z)$ for the first cluster. If $U_{1,2}$ has accumulation point inside (ω_i, ω_{i+1}), analyticity gives

$$\frac{d\lambda_1(z)}{dz} = \frac{d\lambda_2(z)}{dz} \quad \text{or} \quad \lambda_1(z) = \lambda_2(z) + K$$

for $z \in (\omega_i, \omega_{i+1})$. Finally, one has to show that the above relation is possible only if $K = 0$. To see this, consider the algebraic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ generated by the zeros of the polynomials $\mathcal{P}_1(\lambda_1, z)$ ($\mathcal{P}_1(\lambda_2 + K, z)$) and $\mathcal{P}_1(\lambda_2, z)$ respectively. It is easy to see that $\mathcal{C}_1$ and $\mathcal{C}_2$ are irreducible because $\lambda_1(z)$ and $\lambda_2(z)$ appears linearly and the leading coefficient $r_{k_1} \neq 0$. The Riemann surfaces generated by $\mathcal{C}_1$ respectively $\mathcal{C}_2$ have a common segment since $\lambda_1(z) = \lambda_2(z) + K$ and hence by Bezout's theorem, $\mathcal{C}_1$ and $\mathcal{C}_2$ must agree and this can only be true if $\lambda_1(z) = \lambda_2(z)$ which again proves the uniform dichotomy condition for the first cluster.

□

Remark 2.3.4. (i) The condition $\partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_i(x,z)) \neq \partial_\lambda \widetilde{\mathcal{P}}_k(\lambda_j(x,z))$ if $\frac{w}{M_k}$ is nonintegrable is not optimal even, because the difference then will arise in the rest terms $\widetilde{\mathcal{R}}(\lambda)$.

(ii) The dichotomy condition for eigenvalues of different clusters will certainly be satisfied if $\mathrm{Im}(\lambda_j(x,z) - \lambda_l(x,z)) \lesseqgtr 0$ for all $x \geq a$. This can be made so by choosing appropriate a. Thus, the only remaining case will be to determine the uniform dichotomy condition if $\mathrm{Im}\lambda_j(x,z) = \mathrm{Im}\lambda_l(x,z)$ or if the two eigenvalues are essentially real. The uniform dichotomy condition will also be achieved by the roots of intermediate clusters if the reduced cluster polynomials have almost contant coefficients.

2.4 Deficieny Indices and Spectra

With the dichotomy condition settled, one can now apply Levinson's Theorem to obtain the eigenfunctions of the system (1.5) respectively (1.8). In this case, the system has to be in Levinson's form. This requires that (1.8) be transformed into Levinson's form by applying standard K.L-transformation, two diagonalisations and possibly $(I+Q)$-transformations. In such a case, a condition similar to (2.9) and (2.11) will thus be important in controlling the remainder terms and hence is necessary for the application of second diagonalisation and further $(I+Q)$-transformations. Thus we demand that

$$(2.26) \qquad \frac{f'}{f} m_1^{-1} = o(1), \quad \frac{f'}{f} m_1^{-\frac{1}{2}} \in \mathcal{L}^2, \quad (\frac{f'}{f})^2 m_1^{-1}, \frac{f''}{f} m_1^{-1} \in \mathcal{L}^1,$$

$$f = p_k, q_j.$$

Equation (2.26) is true for all m_j, $j > 1$, because $m_1^{-1} \gg m_j^{-1}$. If the coefficients are of approximate power type, then (2.26) amounts to: $\mid r_{k_1} \mid \approx o(wx^{k_1})$. After the first diagonalisation, the system is then diagonalised again using the matrix $(I+B)$, with B constructed as in (1.34). By estimating B_{ij} using the techniques in Section 2.1, where $(\mathcal{T}^{-1}\mathcal{T}')_{ij}$ is estimated by (2.7) and (2.8), and because of (2.26), it follows that $B = o(1)$. The perturbing term after the second diagonalisation is given by R_1 (2.10). If the remainder term R_1 is not a Levinson's term, then a chain of $(I+Q)$-transformations can be applied to the system after the second diagonalisation to make it a Levinson's term.

As mentioned in Section 2.1, the corrections to the diagonal terms as a result of the second transformation can be estimated by

$$\sum_j (\mathcal{T}^{-1}\mathcal{T}')_{ij} B_{ji} = (\mathcal{T}^{-1}\mathcal{T}')_{ij}(\lambda_j - \lambda_i)^{-1}(\mathcal{T}^{-1}\mathcal{T}')_{ji} = \Lambda_{2ii}$$

because $B_{ii} = 0$. By (2.26), these terms are integrable and therefore irrelevant for the square integrability of the eigenfunctions. As in chapter one and also Section 2.1, the eigenfunctions associated to an eigenvalue $\lambda(x,z)$ of $\mathcal{P}(x,\lambda,z)$ within the jth cluster is approximately given by

$$y(x,z) \approx (M_j^{-\frac{1}{2}} + \Sigma_k M_j^{-\frac{1}{2}} B_{jk})(e_j + r_j(x,z)) \cdot \exp \int_a^x \widetilde{\lambda}(t,z)dt, \tag{2.27}$$

where $\widetilde{\lambda}(x,z)$ is $\lambda(x,z)$ plus correctional terms arising from the various transformations. As a consequence of (2.26), note that one has conditions similar to those in (1.46), that is, $B = o(1)$, $r_j(x,z) = o(1)$ and $Q_j = o(B)$. As noted above, the assumption (2.26) implies that the first and higher order correction terms to the diagonals as a result of the transformation are all integrable, so that we may take even $\widetilde{\lambda}(x,z) = \lambda(x,z)$. In general, the growth of the eigenfunctions are mainly determined by the exponential factors. However, for the general coefficients, the regularity conditions on the form factors as stated in Section 2.1 are necessary. If the coefficients are of approximate power type, in order to exclude the occurence of logarithmic terms in the exponentials, one demands that the coefficients be power bounded, that is,

$$r_k(x) = O(x^N), \quad \left| \frac{w(x)}{M_j}(x) \right| \leq Cx^{-1-\epsilon} \quad \text{or} \tag{2.28}$$

$$\left| \frac{w(x)}{M_j}(x) \right| \geq Cx^{1+\epsilon}.$$

These are just technical constraints to simplify the representation. The assumptions avoid the interference of the form factors $G_j(x)$ with the exponents.

Since $| M_n | \gg | M_{n-1} | \gg \cdots \gg | M_1 |$, there exists an index s, $(1 \leq s \leq n)$, such that $wM_n^{-1}, \ldots, wM_s^{-1}$ are integrable, while $wM_{s-1}^{-1}, \ldots, wM_1^{-1}$ are not integrable, see [32]. The following lemma, Lemma 2.4.1, is an extension of Lemma 2.1.5 to clusters of arbitrary length and the proof will be done by applying similar techniques.

Lemma 2.4.1. *Assume that the pivot coefficients* r_{k_j} *are nowhere zero in* $[a, \infty)$. *If* wM_k^{-1} *is integrable, then* $(\mathcal{T}B)_k$ *(form factor) is square integrable.*

Proof. Assume that $\frac{w}{M_k}$ is integrable, then it suffices to analyse the expression $(\mathcal{T}B)_k$ for $k > j$ because $\mid M_j^{-1} \mid \ll \mid M_k^{-1} \mid$ if $j > k$ and thus the higher ones will be integrable too. Since the correction term to the M_k-factors and the eigenvalues can be estimated by $O(\max_k(\frac{m_{k-1}}{m_k}))$, which is bounded and tends to zero uniformly as $x \to \infty$, this correction term does not affect the integrability and square integrability of $\frac{w}{M_k}$ and $(\mathcal{T}B)_k$ respectively. As we have seen above in the formula of $(\mathcal{T}^{-1}\mathcal{T}')_{k,j}$, $(\mathcal{T}B)_{k,j}$ is given by

$$(\mathcal{T}B)_{k,j} = \sum_j \sum_l M_j^{-1} M_k^{-\frac{1}{2}} (\lambda_j - \lambda_k)^{-2} ((-1)^l p_l' \lambda_k^l \lambda_j^l \\ - i q_l' (\lambda_j^l \lambda_k^{l-1} + \lambda_k^l \lambda_j^{l-1})) = \sum A_{k,j,l} + B_{k,j,l}.$$

It suffices to show that each summand $A_{k,j,l}$ and $B_{k,j,l}$ in the above expression is square integrable. As in Lemma 2.1.5, the result will be shown only for $A_{k,j,l}$ since the result for $B_{k,j,l}$ is shown in a similar way. The proof is by induction on n. If also $\frac{w}{M_{k-1}}$ is integrable, the problem reduces to $k-1$ because $\mid M_k \mid \gg \mid M_{k-1} \mid$ and $\mid \frac{A_{k,j,l}}{A_{k-1,j,l}} \mid = o(1)$ for $k - 1 \geq l \geq j$. Assume that $k = n$ is the only term for which $\frac{w}{M_n}$ is integrable. By using a K.L-transform if necessary, one may also assume that $\mid r_n \mid = \mid p_0 \mid = 1$. Then $w \mid A_{k,j,l} \mid^2$ can be estimated by

$$w \mid M_k^{-1} \mid m_k^{-4} \mid M_j^{-2} \mid\mid \frac{p_l'}{p_l} \mid^2 \mid p_l \lambda_k^{2l} \mid\mid p_l \lambda_j^{2l} \mid \leq$$

$$\leq \mid M_j^{-1} \mid m_k^{-2} \mid \frac{p_l'}{p_l} \mid^2 w (m_k^{-1} \mid M_j^{-1} \mid\mid p_l \lambda_j^{2l} \mid)(m_k^{-1} \mid M_k^{-1} \mid\mid p_l \lambda_k^{2l} \mid).$$

Since the terms in brackets are bounded and since $\mid \frac{p_l'}{p_l} \mid$ is w-square integrable, it remains to show that $m_k^{-2} \mid M_j^{-1} \mid \leq m_k^{-2} \mid M_1^{-1} \mid$ is bounded.

Let r_{k_l} and $r_{k_{l+1}}$ be the two pivot coefficients associated with p_k's such that $k_l < k_{l+1} < j < k$. Then it follows that $m_k^{(k_{l+1}-k_l)} \gg m_{l+1}^{(k_{l+1}-k_l)} \approx \mid \frac{r_{k_l}}{r_{k_{l+1}}} \mid$ from which one has $\mid r_{k_{l+1}} m_k^{k_{l+1}} \mid \gg \mid r_{k_l} m_k^{k_l} \mid$. By induction, this will be true for all pivot coefficients r_{k_s} associated with p_k's such that

$k_s < j < k$. Similarly, if $l > k$, then $\mid \frac{r_{k_l}}{r_{k_{l+1}}} \mid \approx m_{l+1}^{(k_{l+1}-k_l)} \gg m_k^{(k_{l+1}-k_l)}$ and $\mid r_{k_l} m_k^{k_l} \mid \gg \mid r_{k_{l+1}} m_k^{k_{l+1}} \mid$ and by induction again, it will be true for all pivot coefficients r_{k_s} such that $k_s > k$. It remains to show that the same inequalities hold for the in-between coefficients that are associated with the p_k's. Assume that r_s is a coefficient from the $[k_l, k_{l+1}]$ cluster such that $k_{l+1} < j < k$, then by (2.20), $r_s m_{l+1}^s \approx r_{k_{l+1}} m_{l+1}^{k_{l+1}}$ so that $\mid \frac{r_s}{r_{k_{l+1}}} \mid \approx m_{l+1}^{k_{l+1}-s}$. But $m_{l+1}^{k_{l+1}-s} \ll m_k^{k_{l+1}-s}$. This implies that $\mid r_{k_{l+1}} m_k^{k_{l+1}} \mid \gg \mid r_s m_k^s \mid$. By induction, this will be true for all the in-between coefficients r_s associated with p_k's that are from cluster l, $l < k$. Similarly, if r_s is from a higher cluster l such that $l > k$, we can assume r_s to be an in-between coefficient of the cluster $[k_l, k_{l+1}]$ and is associated with p_k. In this case, $r_s m_{l+1}^s \approx r_{k_l} m_{l+1}^{k_l}$. This implies that $\mid \frac{r_{k_l}}{r_s} \mid \approx m_{l+1}^{s-k_l}$. But $m_{l+1}^{s-k_l} \gg m_k^{s-k_l}$ since $s - k_l > 0$ and $m_{l+1} \gg m_k$. Then it follows that $\mid r_{k_l} m_k^{k_l} \mid \gg \mid r_s m_k^s \mid$. By induction, this will be true again for all the in-between coefficients r_s associated with p_k's that are from cluster l, $l > k$. These inequalities imply that the value of $\mid r_l m_k^l \mid$ can be estimated by its value within the cluster k. Since $M_1 = O(1)$, $m_1 \ll m_j$, $\mid M_j^{-1} \mid \ll \mid M_1^{-1} \mid$, $j > 1$ and since $w \mid M_k^{-1} \mid = w(\mid p_k \mid m_k^{2k-1})^{-1} = w m_k^{-(2k-1)}$ is integrable, while w is not, it now follows that $m_k^{-2} \mid M_j^{-1} \mid$ is bounded.

□

One needs also the following lemma before establishing the segmental deficiency indices and spectral results.

Lemma 2.4.2. *Let $\lambda(x, z)$ be an eigenvalue of the operator T and $\lambda_j(x, z)$ its jth segmental shadow eigenvalue and assume D((i)-(iv)) hold. Then the eigenfunctions determined by $\lambda(x, z)$ are in a one to one correspondence with the segmental shadow eigenfunctions determined by $\lambda_j(x, z)$. If $p_0 \approx w$ and if the coefficients of the lowest cluster are bounded, then the segmental deficiency index of T are determined by the segmental shadow eigenfunctions and the deficiency index of the operator T is the sum of all the segmental deficiency indices.*

Proof. Let $\mathcal{P}(x, \lambda, z)$ be the Fourier polynomial (2.16) and $\mathcal{P}_j(x, \lambda, z)$ be the jth segmental (cluster) polynomial (2.17). By the results of Theorem 2.2.4, for every root $\lambda(x, z)$ of $\mathcal{P}(x, \lambda, z)$, there exists a shadow root $\lambda_j(x, z)$ of $\mathcal{P}_j(x, \lambda, z)$ such that $\mid \lambda(x, z) - \lambda_j(x, z) \mid \leq c(x, z)$. Since the cluster polynomial is a real valued polynomial, its roots can be written as $\alpha_1 \pm i\beta_1, \ldots, \alpha_r \pm i\beta_r, \gamma_{2r+1}, \ldots, \gamma_l$, where α_k, β_k and γ_s are functions of x and z with $\alpha_k, \beta_k, \gamma_s \in \mathbb{R}$, $k = 1, \ldots, r$, $s = 2r+1, \ldots, l$, $l = k_j - k_{j-1}$. The uniform dichotomy condition follows from Lemma 2.3.3. Assume the eigenfunctions and their segmental shadow eigenfunctions are given by

$$y(x,z) \approx (\check{M}_j^{-\frac{1}{2}} + \sum_s \check{M}_j^{-\frac{1}{2}} B_{js})(e_k + \check{r}_k(x,z)) \cdot \exp \int_a^x \lambda(t,z)dt$$

and

$$y_j(x,z) \approx (M_j^{-\frac{1}{2}} + \sum_s M_j^{-\frac{1}{2}}(B_{js})(e_k + \check{r}_k(x,z)) \cdot \exp \int_a^x \lambda_j(t,z)dt$$

respectively with $\check{r}(x,z) = o(1)$, $\check{r}(x,z)$ is the diagonal element of the remainder matrix R in (1.21). Here, $\check{M}_j = M_j + c(x,z)$. It also follows that $\sum_s M_j^{-\frac{1}{2}} B_{js} = o(M_j^{-\frac{1}{2}})$. There are now two cases to consider in determining the square integrability of the eigenfunctions and their corresponding segmental shadow eigenfunctions.

Case I: Nonreal Eigenvalues
Assume the root $\lambda(x,z)$ of the Fourier polynomial $\mathcal{P}(x,\lambda,z)$ is nonreal, then its jth segmental shadow root $\lambda_j(x,z)$, is nonreal too by Theorem 2.2.4 and the square integrability of the eigenfunction determined by $\lambda(x,z)$ and of its segmental shadow eigenfunction determined by $\lambda_j(x,z)$ are determined from the formula

$$\|y\|^2 \approx \int w \mid G_j \mid^2 \cdot \exp(-2i \int_a^x \lambda(t,z)dt)dx$$

and

$$\|y_j\|^2 \approx \int w \mid G_j \mid^2 \exp(-2i \int_a^x \lambda_j(t,z)dt)dx$$

respectively. If $\text{Im}\lambda(x,z) < 0$, then $\text{Im}\lambda_j(x,z) < 0$ too since the correction term to $\lambda(x,z)$ is real and does not affect the square integrability of the eigenfunctions. In this case, both the eigenfunction and its segmental shadow eigenfunction are square integrable. If $\text{Im}\lambda(x,z) > 0$, then $\text{Im}\lambda_j(x,z) > 0$ too and the eigenfunction and its shadow segmental eigenfunction are nonsquare integrable. It thus implies that if $y(x,z)$ (the eigenfunction) is square integrable, then $y_j(x,z)$ (the segmental shadow eigenfunction) is also square integrable and vice versa. Here, the eigenfunctions are determined by their corresponding eigenvalues.

Case II: Real Eigenvalues
Assume $\lambda(x,z)$ is a real root of the Fourier polynomial $\mathcal{P}(x,\lambda,z)$, then its corresponding jth segmental shadow eigenvalue $\lambda_j(x,z)$ is real too since first and higher order correction terms of a real eigenvalue are real too. In this case, the square integrability of the eigenfunctions and their corresponding segmental shadow eigenfunctions are determined off the real axis, that is, by a nonreal spectral parameter z. In particular, choose the spectral parameter z so that $z = z_0 + i\eta$ where $\text{Im}z = \eta \neq 0$ and $z \in \mathcal{K}_\epsilon(z_0)$ and write $\lambda(x,z) = \lambda(x,z_0) + iw\eta(\partial_\lambda \mathcal{P}(\lambda(x,z_0)))^{-1}$ and $\lambda_j(x,z) = \lambda_j(x,z_0) + iw\eta(\partial_\lambda \mathcal{P}_j(\lambda_j(x,z_0)))^{-1}$. Thus the square integrability of the eigenfunctions and their segmental shadow eigenfunctions are determined by the terms $iw\eta(\partial_\lambda \mathcal{P}(\lambda(x,z_0)))^{-1}$ and $iw\eta(\partial_\lambda \mathcal{P}_j(\lambda_j(x,z_0)))^{-1}$ respectively. Note that up to a multiplication by a constant, the two terms can be estimated as $O(\frac{iw\eta}{M_j})$. The square integrability in both cases is thus determined by the formula

$$\|y_j\|^2 \approx \int w \mid G_j \mid^2 \exp(-2i \int_a^x \eta w(\partial_\lambda \mathcal{P}(t,\lambda_j,z))^{-1} dt) dx.$$

Assume wM_j^{-1} is integrable, then $w(M_j + c(x,z))^{-1}$ is integrable too. By results of Lemma 2.4.1, the form factors and the segmental shadow form factors are both square integrable. The eigenfunction and its corresponding segmental shadow eigenfunction are both square integrable.

If wM_j^{-1} is nonintegrable, then $w(M_j + c(x,z))^{-1}$ is nonintegrable too. Assume the jth segment has even length, then the signs of $\partial_\lambda \mathcal{P}(\lambda(x,z_0))$ within the jth segment and the signs of $\partial_\lambda \mathcal{P}_j(\lambda_j(x,z_0))$ are evenly distributed. Thus if the sign of $iw\eta(\partial_\lambda \mathcal{P}(\lambda(x,z_0)))^{-1} \approx \frac{iw\eta}{M_j+c}$ leads to a square integrable eigenfunction, then the sign of $iw\eta(\partial_\lambda \mathcal{P}_j(\lambda_j(x,z_0)))^{-1} \approx \frac{iw\eta}{M_j}$ will also lead to a square integrable eigenfunction and vice versa. The same behaviour will be exhibited for odd length clusters but with one more eigenfunction and its segmental shadow eigenfunction either both square integrable in the lower half plane or upper half plane because odd length clusters have odd number of real eigenvalues. Again in this case, the square or nonsquare integrability of an eigenfunction implies the square or nonsquare integrability of its segmental shadow eigenfunction.

Thus, the square or non-square integrability of the eigenfunctions of T within the jth cluster will imply the square or non-square integrability of their jth segmental shadow eigenfunctions. This implies that the deficiency indices of the jth segments, $(\mathrm{def}T)_j$, can be read off from the jth segmental shadow eigenfunctions. If $w = o(p_0)$, then the deficiency indices of the lowest cluster are determined by the above reasoning. But if $p_0 \approx w$ and if the coefficients of the lowest cluster are bounded, then the deficiency indices results of [10] or [18] apply when the lowest cluster has even or odd length respectively. In general, def$T$ will be the sum of all these segmental deficiency indices, that is,

$$\mathrm{def}T = \sum_j (\mathrm{def}T)_j.$$

This also gives the standard inequalities for $(\mathrm{def}T)_j$.

□

We now introduce the term USI to mean uniformly square integrable since the z-uniformly square integrable eigenfunctions will never contribute to absolutely continuous spectrum.

Theorem 2.4.3. *Consider the differential operator T generated by (1.1) or (1.2) with the coefficients p_k, q_j twice differentiable, the pivot coefficients r_{k_j} nowhere zero in $[a,\infty)$ and D((i)-(iv)) hold. Assume that all the pivot coefficients r_{k_j} satisfy (2.18)-(2.20) and (2.26). Moreover, assume the regularity conditions on the form factors to be satisfied for the general coefficients and (2.28) to be satisfied for approximate power type coefficients. Let s be the critical index, $1 \leq s \leq n$ for even order differential operator or $1 \leq s \leq n+1$ for odd order differential operator, so that wM_j^{-1} is integrable for $j \geq s$ and nonintegrable for $1 \leq j < s$. Then the following results hold.*

(i) Assume the cluster j, $j > 1$, has even length, that is, $k_j - k_{j-1} = 2k$ and assume the roots of the cluster Fourier polynomial $\mathcal{P}_j(x, \lambda, z)$ are given by

$$\alpha_1 \pm i\beta_1, \ldots, \alpha_r \pm i\beta_r, \gamma_{2r+1}, \ldots, \gamma_{2k}$$

with α, β and γ as functions of x and z are real valued. If $j \geq s$ so that wM_j^{-1} is integrable, then $(\mathcal{T}B)_j$ is square integrable and $2k - r$ solutions are z-uniformly square integrable, $(defT)_j = (2k-r, 2k-r)$ and USI holds. If $j < s$ so that wM_j^{-1} is not integrable, then $(\mathcal{T}B)_j$ is not square integrable and $(defT)_j = (k,k)$.

(ii) *Assume cluster j, $j > 1$, has odd length, that is, $k_j - k_{j-1} = 2k+1$ and assume the roots of the cluster polynomial $\mathcal{P}_j(x, \lambda, z)$ are given by*

$$\alpha_1 \pm i\beta_1, \ldots, \alpha_r \pm i\beta_r, \gamma_{2r+1}, \ldots, \gamma_{2k+1}$$

with α, β and γ real valued functions. If $j \geq s$ so that wM_j^{-1} is integrable, then $(\mathcal{T}B)_j$ is square integrable and $2k-r+1$ solutions are z-uniformly square integrable, $(defT)_j = (2k-r+1, 2k-r+1)$ and again USI holds. If $j < s$ so that wM_j^{-1} is not integrable, then $(\mathcal{T}B)_j$ is not square integrable and $(defT)_j = (k, k+1)$ if $\frac{r_{k_{j-1}}}{r_{k_j}} > 0$ and $(defT)_j = (k+1, k)$ whenever $\frac{r_{k_{j-1}}}{r_{k_j}} < 0$.

(iii) *If $w = o(p_0)$, then the results in (i) and (ii) remains valid for cluster 1.*

(iv) *If $p_0 \approx w$, assume the coefficients of the lowest cluster to be almost constant so that they converge to their limiting values as $x \to \infty$. Thus if cluster 1 has even length with the eigenvalues assumed to be given as in part (i) above, then $(defT)_1 = (k, k)$ by the results of [10]. If cluster 1 has odd length with eigenvalues assumed to be given as in part (ii) above, then by results of [18], $(defT)_1 = (k, k+1)$ for $r_{k_1} > 0$ and $(defT)_1 = (k+1, k)$ for $r_{k_1} < 0$.*

Proof. By the results of Lemma 2.4.2, there is a one to one correspondence between the eigenfunctions of the operator T and their corresponding segmental shadow eigenfunctions. Thus the segmental deficiency indices can be read off directly from the segmental shadow eigenfunctions that are of the form (2.27). The square integrability of the form factors $(\mathcal{T}B)_j$ follows immediately from Lemma 2.4.1 whenever wM_j^{-1} is integrable. In case wM_j^{-1} is integrable, then all the segmental shadow eigenfunctions determined by real segmental shadow eigenvalues are uniformly square integrable and can at most contribute only to discrete spectrum, that is, USI holds.

□

Theorem 2.4.4. *Let T be a differential operator generated by (1.1) or (1.2) with equal deficiency index and defined on $[0, \infty)$. Assume the coefficients of T to be twice differentiable, the pivot coefficients r_{k_j} associated to its coefficients p_k and q_j are nowhere zero in $[a, \infty)$ and* *satisfy (2.18)-(2.20) and (2.26), and D((i)-(iv)) hold. For the general coefficients, assume that the regularity condition on the form factors are satisfied and that the approximate power type coefficients satisfy (2.28). Moreover, if* $p_0 \approx w$*, then assume the coefficients of the lowest cluster to be almost constant. Then the singular continuous spectrum is absent,* $\sigma_{sc}(H) = \emptyset$*, and the eigenfunctions that lose their square integrability as Imz* $\to 0$ *contribute to absolutely continuous spectrum with the multiplicity of the absolutely continuous spectrum equal to the number of such eigenfunctions. In particular, if all the clusters have equal deficiency indices then* $\sigma_{ac}(H) = \cup_j \sigma_{ac}(H)_j$ *with the spectral multiplicities of the clusters regarded as additive.*

Proof. Let T be the differential operator generated by (1.1) or (1.2). Then by using quasiderivatives, (1.1) or (1.2) can be written in its first order system (1.8). An application of standard K.L-transformation simplifies the leading and p_0 coefficients as well as the weight function $w(x)$ to 1. In order to transform the first order system into Levinson's form, one applies two diagonalisations since the coefficients are assumed to be twice differentiable. This needs the eigenvalues of $\mathcal{C}$, which can be approximated from the polynomial (2.16) by employing pivot coefficients technique. By the assumption D(i), all the cluster polynomials have distinct eigenvalues. It now follows that the Fourier polynomial $\mathcal{P}(x, \lambda, z)$ has distinct eigenvalues which can be written as

$$\alpha_1 \pm i\beta_1, \ldots, \alpha_r \pm i\beta_r, \gamma_{2r+1}, \ldots, \gamma_{2n}$$

for even order operator or the real eigenvalues γ are $\gamma_{2r+1}, \ldots, \gamma_{2n+1}$ if the operator is of odd order. Here, α_j, β_j and γ_s as functions of x and z are real valued functions. Since D((i)-(iv)) hold, the uniform dichotomy condition follows from Lemma 2.3.3. Thus, one uses the eigenvectors to diagonalise the first order system (1.8) to transform it into Levinson's form. After two diagonalisations, the system will almost be in diagonal form and if not, then the off-diagonal elements can be made smaller by application of more $(I + Q)$-transformations. Once Levinson's form is achieved, one applies Levinson Theorem to the system and thus obtains the eigenfunctions of the form (2.27). The deficiency indices of T can therefore be read off directly from these eigenfunctions.

By assumption, T has equal deficiency indices. In this case, one has the following two cases to consider. First, assume the operator T is limit point at infinity, then all the eigenfunctions that are square integrable are z-uniformly square integrable as $\mathrm{Im}z \to 0^+$. In this case, only the self-adjoint extension H of T generated by (1.1) can be obtained and this is defined by $D(H) = \{y \in D(T^*) \mid (\alpha_1, \alpha_2)y(a) = 0\}$, where α_1 and α_2 satisfy (1.10) and (1.11). One now uses the results of [40, 43] to construct the M-matrix of the self-adjoint operator H. Since all eigenfunctions that are square integrable are z-uniformly square integrable, that is, USI holds and one only obtains discrete spectrum at most.

Secondly, assume that the operator T is non-limit-point at infinity, that is, $\mathrm{def}T = (n + r, n + r)$, $1 \leq r \leq n$. Then one constructs a subspace from the space of all square integrable eigenfunctions by imposing further boundary conditions. In this case, we choose the functions $w_j(x) \in D(T^*)$ such that $\lim_{x\to\infty} w_j^*(x)\mathcal{J}y(x) = 0$, $j = 1, \ldots, r$ thereby imposing r-further boundary conditions at infinity. The functions $w_j(x)$ are linearly independent modulo $D(T)$ at infinity and may be chosen as eigenfunctions of $T^*w_j = zw_j$, z is nonreal, and they satisfy also $w_k^*(x)\mathcal{J}w_j(x) = 0$, $k, j = 1, \ldots, r$. By introducing these boundary conditions at infinity, we construct a subspace of $D(T^*)$ with co-dimension r. This subspace consists of only square integrable eigenfunctions which satisfy further boundary conditions at infinity and they are a linear combination of those square integrable eigenfunctions of T. Now using the results of Lemma 1.1.5, we construct a limit point operator $\widetilde{T}$ from this subspace such that the operator T^* is an r-dimensional extension of $\widetilde{T}^*$. For further details on the operator $\widetilde{T}$, see Lemma 1.1.5.

By results of Theorem 1.1.6, one then constructs the self-adjoint extension H_α of operator $\widetilde{T}$. The operator H_α is constructed from the domain of the operator $\widetilde{T}^*$, $D(\widetilde{T}^*)$, and its domain is defined by (1.12). Now choose $z \in \mathcal{K}$, such that $z = z_0 + i\eta$, $\mathrm{Im}z = \eta \neq 0$ and from Theorem 1.1.7, it is possible to construct a $2n$ by n system $V_\alpha(x, z)$ of square integrable eigenfunctions which stay square integrable as $\eta = \mathrm{Im}z \searrow 0^+$ and also satisfy the separated boundary conditions. With (1.14), one obtains the corresponding M-matrix for the self-adjoint extension H_α. Note that the absolutely continuous spectrum of H_α and its spectral multiplicity is identical to that of H. By (1.14), one has

$$\mathrm{Im}M_\alpha(z_+) = \lim_{\eta\to 0^+} \eta\langle \chi_a(x, z + i\eta), \chi_a(x, z + i\eta)\rangle.$$

Thus $\lim_{\eta\to 0+} \mathrm{Im}M_\alpha(z_0+i\eta) = \mathrm{Im}M_\alpha(z_0+)$, $z = z_0+i\eta$ exists boundedly for all $z \in \mathcal{K}\cap\mathbb{R}$ and is even continuous within each $\mathcal{K}\cap\mathbb{R}$. To show this claim, it suffices to show that the limit is finite. Let the elements of $\langle \chi_a(x, z+i\eta), \chi_a(x, z+i\eta)\rangle$ be denoted by $I_{ij}(x, z)$. In evaluating the elements of the scalar product, we will consider two cases namely: the scalar product of eigenfunctions determined by nonreal eigenvalues and the scalar product of eigenfunctions determined by real eigenvalues. To simplify the argument, we will assume that $\mid G_j(x,z) \mid \approx \mid M_j^{-\frac{1}{2}}(x,z) \mid$. In the estimation below, we will assume that C_i are positive constants that are independent of $z = z_0 + i\eta$.

Assume the eigenfunctions are determined by nonreal eigenvalues $\alpha_j(x,z)-i\beta_j(x,z)$ with $\beta_j(x,z) > 0$ a real valued function. Then these eigenfunctions are z-uniformly decreasing and therefore their integrals exist boundedly. This implies that the limit of the product of their integral and η as $\eta \to 0^+$ is zero. By using Cauchy-Schwarz inequality and similar argument, one obtains the same limit for any scalar product of two eigenfunctions with one of the eigenfunctions determined by nonreal eigenvalue. It now remains to show that the limit exists if the two eigenfunctions are determined by real eigenvalues. In this case, one has

$$\begin{aligned}
\lim_{\eta\to 0^+} \eta I_{kk}(a,z) &= \lim_{\eta\to 0^+} \eta \int_a^\infty \frac{w(x)}{\mid M_k(x) \mid} \mid e_k + r_{kk}(x,z) \mid^2 \\
&\quad \cdot \exp(-2\int_a^x \frac{\eta w(t)}{\mid M_k \mid} dt)dx \\
&\le \lim_{\eta\to 0^+} \eta C_1 \int_a^\infty \frac{w(x)}{\mid M_k(x) \mid} \exp(-2\int_a^x \frac{\eta w(t)}{\mid M_k \mid} dt)dx \\
&\le \lim_{\eta\to 0^+} \eta C_2 \int_0^\infty \exp(-\eta s)ds = C_3.
\end{aligned}$$

Here, of course, a is the left regular endpoint. To obtain the inequalities, we have used the fact that $r_{kk}(x,z) = o(1)$ and can be estimated uniformly in x and z, assumption D(iii) which implies that $\mid \partial_\lambda \widetilde{\mathcal{P}}_k(x,\lambda,z) \mid > 0$ and so $\frac{w(x)}{|M_k(x)|}$ is bounded by a certain constant. Even if $\int_a^\infty \frac{w(x)}{|M_k(x)|}dx = \infty$, note that the exponential function is determined by the same function $\frac{w(x)}{|M_k(x)|}$ and thus decreases rapidly as $x \to \infty$. Finally, we have also applied transformation of the variable. It follows that $\lim_{\eta\to 0^+} \eta I_{kk}(a,z) \le C_3$. With a similar argument, one can show that $\lim_{\eta\to 0^+} \eta I_{kk}(a,z) \ge C_4$ for some positive constant C_4. Thus we may take even $\lim_{\eta\to 0^+}\eta I_{kk}(a,z)\approx 1$. This implies that $\mathrm{Im}M_\alpha(z_+)$ is finite and consists of the elements 0 and $O(1)$.

By Theorem 1.1.15, H_α has no singular continuous spectrum and apart from the isolated poles, the density of the spectral measures of H_α is continuous. From the resolvent calculus of differential operators, it follows that the M-function is the Borel transform of the spectral measure. Thus, the existence of a finite limit implies the absolute continuity of the corresponding spectral measure ρ_α and in fact, $\frac{1}{\pi}\mathrm{Im}M_\alpha(z_0+)$ gives the density of the absolutely continuous spectrum, see [52]. The rank of $\mathrm{Im}M_\alpha(z_0+)$ is thus the multiplicity of $\sigma_{ac}(H_\alpha)$. Thus z_0 belongs to the associated absolutely continuous spectrum if and only if there is at least one eigenvalue $\lambda(x,z_0)$, which for $x\to\infty$, its eigenfunction loses square integrability as $\mathrm{Im}z\searrow 0$. The multiplicity of the absolutely continuous spectrum is then given by the number of these eigenvalues. If all the clusters have equal deficiency indices, then the eigenfunctions that lose their square integrability as $\mathrm{Im}z\to 0$ are those that are determined by the real eigenvalues with segmental shadow eigenvalues that are real too. Since there is a one to one correspondence between the eigenfunctions and their corresponding segmental shadow eigenfunctions, it implies that the absolutely continuous spectrum of H and its spectral multiplicity is determined by all the segments. Thus $\sigma_{ac}(H)=\cup_j\sigma_{ac}(H)_j$ and the spectral multiplicity is the sum of all cluster spectral multiplicities. Now Remling's results [52] can be used to extend this result to the interval $[0,\infty)$.

□

Remark 2.4.5. It is worth noting that even order differential operators with clusters of mixed lengths, that is, odd and even lengths, may have unequal deficiency indices. As an example, consider a fourth order differential operator generated by

$$\tau y = y^{iv} - \frac{1}{2}i\{(qy')'' + (qy'')'\} + (p - zw)y,$$

where z is a spectral parameter. Assume that $w(x)=1$ and absorb z into p. Moreover, assume that the coefficients p and q are twice differentiable, satisfy all the smoothness and decay conditions (2.26), the regularity conditions on the form factors if they are general coefficients and (2.28) if they are approximate power type coefficients that are needed for asymptotic integration. Assume also that $|q|^4 \gg |p|$. Then, the measure of the magnitude of the eigenvalues of the operator

T generated by τy are $m_2 \approx \mid q \mid$, $m_1 \approx \mid \frac{p}{q} \mid^{\frac{1}{3}}$. It follows that $m_2 \gg m_1$. Assume further that the critical index s in Theorem 2.4.3 is $s = 2$, so that $\mid q \mid^{-3}$ is integrable while $\mid q \mid^{-\frac{1}{3}} \mid p \mid^{-\frac{2}{3}}$ is not integrable. Then, the eigenfunctions of τy are given by (2.27). The deficiency index of the operator T can therefore be read off from the eigenfunctions. It follows that cluster 2 contributes $(1, 1)$ to the deficiency index while cluster 1 contributes $(1, 2)$ if $q > 0$ or $(2, 1)$ if $q < 0$ to the deficiency index. Thus def$T = (2, 3)$ if $q > 0$ or def$T = (3, 2)$ whenever $q < 0$.

Similarly, one can construct an example of an odd order operator with equal deficiency indices on half line.

2.5 Operators on $\mathbb{R}$

As noted in the previous section, the differential operators studied in Sections 2.2-2.4, might have unequal deficiency indices on the positive half line irrespective of their order. Furthermore, odd order differential operators on half line tend to have unequal deficiency indices in general and therefore, no proper spectral results can be obtained apart from the abstract theory of the M-matrix of these operators as developed in [41, 42]. For operators on $\mathbb{R}$, however, the situation is quite different. In this case, the decomposition method studied by [18, 19] can be used. The methods of [18, 19] show that the classical decomposition method can be applied not only in the computation of the deficiency indices but can also be extended to derive spectral results.

Assume the differential operator T generated by (1.1) or (1.2) is defined on $\mathbb{R}$. Let $T_\pm$ be the 'partial' operators defined on $\mathbb{R}_\pm$ respectively. To define T_-, set $y(x)$ to be $y(-x)$ in (1.1) or (1.2) [18, 19] so that T_- is defined on $\mathbb{R}_-$. Then $y(-x)$ satisfies an equation like (1.1) or (1.2) where all qs are replaced by $-q$ and then the new equation is multiplied by -1. Thus one obtains an operator that is defined on $[0, \infty)$ and which is unitarily equivalent to T_-. Similarly T_+ is defined as the differential operator generated by (1.1) or (1.2) on $\mathbb{R}_+$. The basic idea of reflecting T_- on $[0, \infty)$ is to obtain a combined system of either $2(2n)$ or $2(2n+1)$ dimensional equation on $[0, \infty)$ to which one can even apply even-order theory with one singular endpoint. Thus, the point $x = 0$ becomes an intermediate regular point. Now by means of two separate standard K.L-transformations, one can achieve $r^{\pm}_{k_n} = w_\pm = 1$. Then define the operator T_0 by

$$D(T_0) = \Big\{ y \in \mathcal{L}_w^2 \mid y^{[0]}, \cdots, y^{[2n-1]} \text{ are absolutely continuous,}$$
$$Ty \in \mathcal{L}_w^2, \quad y^{[0]}(0) = \ldots y^{[2n-1]}(0) = 0 \Big\}$$

for an even order differential operator T. Similarly, one defines $D(T_0)$ for an odd order differential operator T though with $(2n+1)$ conditions. Then

$$T_0 = T_- \oplus T_+.$$

Thus by definition

$$\text{def}T_0 = \text{def}T_- + \text{def}T_+.$$

By results of [19], the operator T_0 has a dimension twice that of T. The Hamiltonian and first order systems of T_0 as well as the resolvent can be constructed and for more details on how to do this, see [18, 19].

For asymptotic integration and spectral results, we will demand that the following conditions are satisfied.

(a) Assume the coefficients $r_k^{\pm}$ satisfy the smoothness and decay conditions (2.18)-(2.20) and (2.26) needed for asymptotic integration with $r_{k_j}^{\pm}$ being the pivot coefficients. In addition, assume the regularity conditions on the form factors as stated in Section 2.1 are satisfied for the general coefficients and for approximate power type coefficients, assume (2.28) holds.

There are various conditions which guarantee that T_0 has equal deficiency indices. In our case, we will have to assume, in addition to condition (a) above, that one of the following conditions (b)-(d) is satisfied.

(b) For the operators T_- and T_+, assume that $\text{def}T_- + \text{def}T_+ = (l, l)$ where $2n \leq l \leq 4n$ or $2n+1 \leq l \leq 2(2n+1)$ for even or odd order differential operator respectively.

(c) Assume the deficiency index reflection property on the deficiency indices of T_- and T_+, that is, $\text{def}T_- = (n+s_1, n+s_2)$ and $\text{def}T_+ = (n+s_2, n+s_1)$, $0 \leq s_1, s_2 \leq n$ or $0 \leq s_1, s_2 \leq n+1$.

(d) (i) Assume the same cluster ordering for the pivot coefficients both on $\mathbb{R}_-$ and $\mathbb{R}_+$ so that

$$(2.29) \qquad |\frac{r^{\pm}_{k_{j-1}}}{r^{\pm}_{k_j}}|^{\frac{1}{k_j-k_{j-1}}} \to m^{\pm}_j.$$

Here, m^-_j and m^+_j need not necessarily be of the same size but $m^-_1 \ll m^-_2 \ll \cdots \ll m^-_n$ and $m^+_1 \ll m^+_2 \ll \cdots \ll m^+_n$.

(ii) Assume $\frac{w_+}{M_j}$ is integrable if and only if $\frac{w_-}{M_j}$ is integrable so that the square integrability of the eigenfunctions are preserved.

(iii) Assume the reflection property on the segmental deficiency indices of T_- and T_+, that is, $(\text{def}T_-)_j = (n_1, n_2)$ and $(\text{def}T_+)_j = (n_2, n_1)$.

If the condition (a) in addition to one of the conditions (b)-(d) above are satisfied, then the operator T_0 has equal deficiency indices and thus the self-adjoint extensions of T_0 exist. Denote these self-adjoint extensions of T_0 by $\widetilde{H}$. If T is limit point at $\pm\infty$, then T is essentially self-adjoint and either a $2n$- or $2n+1$-dimensional extension of T. In fact, T is the extension which gives the continuity of $y^{[k]}$, $0 \leq k \leq 2n-1$ or $0 \leq k \leq 2n$ at $x = 0$ for even or odd order operators respectively.

The construction of the resolvent and the M-matrix of the operator T_0 can be obtained from [18, 19]. The construction of the M-matrix as done in [19] is so general that the results apply in our case too. Since the coefficients satisfy smoothness and decay conditions necessary for asymptotic integration, the self-adjoint extension of T has no singular continuous spectrum and its absolutely continuous spectrum can be computed in all cases from the Fourier polynomial [19]. One then has the following result.

Theorem 2.5.1. *Consider the differential operator T generated by (1.1) or (1.2) on $\mathbb{R}$ with coefficients that are twice differentiable. Assume the pivot coefficients $r^{\pm}_{k_j}$ to be nowhere zero in $\mathbb{R}$ and condition (a) in addition to one of the conditions (b)-(d) to be satisfied. Moreover, assume D((i)-(iv)) hold. If $p_0 \approx w$, assume the coefficients of the lowest cluster to be almost constant . Then T_0 has equal deficiency indices and self-adjoint extensions $\widetilde{H}$. The singular continuous spectrum of $\widetilde{H}$ is absent and the absolutely continuous spectrum of $\widetilde{H}$ is contributed by the eigenfunctions of both T_- and T_+ that lose their square integrability as Im$z \to 0^+$. In particular, the spectral multiplicity is the number of the eigenfunctions of both T_- and T_+ with this property.*

Proof. Since the coefficients of each operator T_- and T_+ satisfy the necessary conditions needed for asymptotic integration, one can apply method of asymptotic integration to the operator T_0. The dimension of the operator T_0 on $[0,\infty)$ is twice that of T and one can apply even-order theory on half lines with one singular endpoint in order to obtain the deficiency indices and spectral results. The Hamiltonian and the first order systems of the operator T_0 follow immediately from [19]. Assume after the two diagonalisations and some further $(I+Q)$-transformations, that the system is in Levinson's form, then one can apply Levinson Theorem so as to obatin the corresponding eigenfunctions. These eigenfunctions will be of the form (2.27). The deficiency index can be read off from the eigenfunctions. Because one of the conditions (b)-(d) is satisfied, T_0 has equal deficiency indices and thus the self-adjoint extensions of T_0 exist. For spectral results, one has the following two cases. If T is limit point on $[0,\infty)$, then T defined on $\mathbb{R}$ is self-adjoint and is also one of the self-adjoint extensions of T_0. In this case, the spectral theory of self-adjoint operators applies. But if T is non-limit-point on $[0,\infty)$, then T_0 is non-limit-point and one can construct a limit point operator $\widetilde{T_0}$ from a subspace of $D(T_0^*)$. For more details on this, see Lemma 1.1.5 and Theorems 1.1.6 and 2.4.4. One then uses the results of [19] to construct the corresponding M-matrix. With the knowledge of the M-matrix, all other spectral information can be obtained.

□

2.6 Power Coefficients and Extensions

2.6.1 Power Coefficients

In [32, 34], Eastham and others considered mainly power coefficients. For approximate power type coefficients, the conditions for asymptotic integration can be weakened in several ways. In this Section, we show that for the approximate power type coefficients, condition (2.26) is not necessary for the application of the second diagonalisation. As in Section 1.1.2, a coefficient $f(x)$, $f(x) = p_k, q_j$, of the operator T is said to be of approximate power type if $f(x) = (1 + h_f(x))x^{\alpha_f}$, $h_f(x) \in \mathcal{F}_l$ such that

$$\mathcal{F}_l = \{h(x) \mid h^{(k)}(x) = o(x^{-k}), \quad 0 \le k \le l\}.$$

Before the generalised result, consider the following example.

Example 2.6.1. Consider a fourth order differential operator T defined on $[0, \infty)$ and generated by

$$\tau y = w^{-1}\left\{ y^{(iv)} - \frac{1}{2}i((q_2 y'')' + (q_2 y')'') \right.$$
$$\left. - (p_1 y')' + \frac{1}{2}i((q_1 y') + (q_1 y)') + p_0 \right\},$$

where $p_0 = (1+h_f)$, $q_1 = (1+h_f)x^{\gamma}$, $p_1 = (1+h_f)x^{\beta}$, $q_2 = (1+h_f)x^{\alpha}$ and the following conditions are satisfied:

$$x^{\alpha} = O(x^{\frac{\beta}{2}}), \quad x^{\gamma} = O(x^{\frac{\beta}{2}}), \quad x^{-\beta} = o(x^{\beta}) \quad \text{and} \quad \beta > 2.$$

It then follows that there are two sets of eigenvalues with different magnitudes, namely; $\lambda_{2\pm} \sim x^{\frac{\beta}{2}}$ and $\lambda_{1\pm} \sim x^{-\frac{\beta}{2}}$. Assume that the measures of magnitude of $\lambda_{2\pm}$ and $\lambda_{1\pm}$ are denoted by m_2 and m_1 respectively, then $m_1 \ll m_2$, $M_1^{-1} \approx x^{-\frac{\beta}{2}}$, $M_2^{-1} = x^{-\frac{3\beta}{2}}$ and wM_1^{-1}, wM_2^{-1} are all integrable. Since $\lambda_{1\pm} \in \mathcal{L}^1$ and $\beta > 2$, (2.26) is definitely violated and hence one cannot proceed with the second diagonalisation without transforming the in-block matrix elements of the first cluster after the first diagonalisation into integrable terms. This can be done with a similar procedure outlined in Example 2.1.7. One then has the following results for $\beta > 2$.

If $\mathrm{Im}\lambda_2 \approx m_2$ and $\mathrm{Im}\lambda_1 \approx m_1$, then in each cluster, only one of the eigenfunctions is square integrable and $\mathrm{def}(T)_1 = \mathrm{def}(T)_2 = (1, 1)$ and $\mathrm{def}T = (2, 2)$ and the solutions that are square integrable are z-uniformly square integrable hence $\sigma_{ac}(H) = \emptyset$. On the other hand, if $\mathrm{Im}\lambda_2 \approx \eta w M_2^{-1}$ and $\mathrm{Im}\lambda_1 \approx \eta w M_1^{-1}$, for $\mathrm{Im}z = \eta \neq 0$, then $\mathrm{def}(T)_1 = \mathrm{def}(T)_2 = (2, 2)$ and $\mathrm{def}T = (4, 4)$ and $\sigma_{ac}(H) = \emptyset$. One equally obtains $\mathrm{def}T = (3, 3)$ and $\sigma_{ac}(H) = \emptyset$ if either $\mathrm{Im}\lambda_2 \approx m_2$ and $\mathrm{Im}\lambda_1 \approx \eta w M_1^{-1}$ or if $\mathrm{Im}\lambda_2 \approx \eta w M_2^{-1}$ and $\mathrm{Im}\lambda_1 \approx m_1$.
Whenever $\beta < 2$, the condition (2.26) for this example is not violated and hence the theory developed in Sections 2.1-2.4 applies. There is no need to perform the in-block diagonalisation but one demands that (2.26) is satisfied before the second diagonalisation is carried out.

Theorem 2.6.2. *Let T be a differential operator generated by (1.1) or (1.2) having approximate power type coefficients and D((i)-(iv)) hold. Assume the coefficients r_k satisfy (2.18)-(2.20) and (2.28) and the pivot coefficients r_{k_j} are nowhere zero in $[0, \infty)$. Moreover, assume*

(2.26) holds for some $l+1 \leq k \leq n$ *and* $\frac{r_k'}{r_k} m_k^{-1} \notin \mathcal{L}^2$, $1 \leq k \leq l$. *Here,* r_k *are associated with* p_k *and* q_j. *Then*

(i) $defT = \sum_j (defT)_j$ *and if* T *has equal deficiency indices then* T *has self-adjoint extensions,* $\sigma_{sc}(H) = \emptyset$ *and the absolutely continuous spectrum of* H *are contributed by eigenfunctions that lose their square integrability as* $Imz \to 0^+$. *In particular, if all segments have equal deficiency indices, then* $\sigma_{ac}(H) = \cup_j \sigma_{ac}(H)_j$ *with the spectral multiplicities of the clusters regarded as additive.*

(ii) *If* $\mid \frac{p_0}{r_{k_1}} \mid = o(x^{-k_1})$ *and* $\alpha_0 > 0$, *then the square integrable eigenfunctions of clusters* j, $j > 1$, *are* z*-uniformly square integrable and contribute at most to discrete spectrum only. But if the lowest cluster has even length,* $p_0 \approx w$, $r_{k_1} = O(x^{k_1})$ *and the coefficients of the lowest cluster are almost constant then absolutely continuous spectrum can arise from the lowest cluster.*

Proof. By performing a standard K.L-transformation if necessary, one can achieve even $\alpha_n = \alpha_0 = 0$ and $w(x) = 1$. In order to diagonalise the system so as to transform it into Levinson's form, one needs the eigenvalues of T. These can be calculated from the roots of the characteristic polynomial $\det(\mathcal{C} - \lambda \cdot I) = 0$ where $\mathcal{C}$ is the $2n$ by $2n$ or $2n+1$ by $2n+1$ matrix in (1.8). Since the coefficients r_k satisfy (2.18)-(2.20), the roots of the Fourier polynomial $\mathcal{P}(x, \lambda, z)$ can be estimated by techniques of Theorem 2.2.4 and because D(i) holds, these roots will be distinct. Thus $\mathcal{P}(x, \lambda, z)$ has either $2n$ or $2n+1$ distinct roots. The uniform dichotomy condition follows immediately from Lemma 2.3.3 because D((i)-(iv)) hold. It is well-known that the matrix formed by the eigenvectors of these eigenvalues diagonalises the matrix $\mathcal{C}$.

After the first diagonalisation, assume the system is of the form $v' = (\Lambda - \mathcal{T}^{-1}\mathcal{T}')v$ where $\Lambda = \mathrm{diag}(\lambda_i(x, z))$. Provided the eigenvalues of the system are suitably ordered, one can adjoin the diagonals of $-(\mathcal{T}^{-1}\mathcal{T}')$ into Λ and write the system in the form

$$v' = (\widetilde{\Lambda} + S + R_1)v,$$

where $\widetilde{\Lambda} = \Lambda + \mathrm{diag}(-\mathcal{T}^{-1}\mathcal{T}')$,

$$S = \begin{pmatrix} 0 & O(x^{-1}) \\ O(x^{-1}) & 0 \end{pmatrix}, \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & O(x^{-1-\epsilon}) \\ O(x^{-1-\epsilon}) & 0 \end{pmatrix}$$

for some $\epsilon > 0$. Here, $R_1 \in \mathcal{L}^1[a,\infty)$ by assumption. The terms $O(x^{-1})$ and $O(x^{-1-\epsilon})$ are complex valued. Because (2.26) is not satisfied for k, $1 \leq k \leq l$, one cannot diagonalise $(\widetilde{\Lambda}+S)$ as a system. We therefore require block transformations in order to transform the off-diagonal elements of $(\widetilde{\Lambda}+S)$ into integrable terms. Now write the block elements of $(\widetilde{\Lambda}+S)$ as $(\widetilde{\Lambda}_k+S_k)$, $k=1,\dots,n$. Then each kth- block matrix is a skew symmetric matrix which has a diagonalising matrix. Let this diagonalising matrix be P_k. Each kth-block matrix is then diagonalised using the corresponding P_k matrix. The kth-blocks are then substituted by the new block matrices $P_k^{-1}(\widetilde{\Lambda}_k+S_k)P_k$. Since the coefficients are assumed to be twice differentiable, the diagonal elements of $P_k^{-1}(\widetilde{\Lambda}_k+S_k)P_k$ can be approximated by $\widetilde{\lambda}_k+O(x^{-1-\epsilon})$ where $O(x^{-1-\epsilon})$ are the correction terms to the diagonals as a result of block transformation and $\widetilde{\lambda}_k=\lambda_k+(-\mathcal{T}^{-1}\mathcal{T}')_{kk}$. The terms $O(x^{-1-\epsilon})$ are, however, integrable and thus will not affect the square integrability of the eigenfunctions. The off-diagonal elements of the new block matrices are determined by $P_k^{-1}P_k'$ and can be approximated by $O(x^{-1-\epsilon})$. The off-diagonal elements are now integrable by assumption but if they are not then the above procedure can be repeated again. The system is now in Levinson's form to which one can apply Levinson Theorem.

The eigenfunctions and their segmental shadow eigenfunctions are of the form (2.27) and the segmental deficiency indices can be read off from the segmental shadow eigenfunctions. Thus by the results of Lemma 2.4.2, $\text{def}T=\sum_j(\text{def}T)_j$. If T has equal deficiency indices, then T has self-adjoint extensions. By constructing the appropriate M-matrix, all spectral information can be obtained and if all the segments have equal deficiency indices, then by results of Theorem 2.4.4, $\sigma_{ac}(H)=\cup_j\sigma_{ac}(H)_j$ and the spectral multiplicity is the sum of all cluster spectral multiplicities.

To prove the second claim, assume that $\mid \frac{p_0}{r_{k_1}} \mid = o(x^{-k_1})$ and $\alpha_0>0$. It suffices to show that $\frac{w}{M_j}$, $j=2,\dots,n$ are integrable. Since the coefficients are of approximate power type, by assumption, it follows that $-\frac{\alpha_{k_1}}{k_1} < -1-\frac{\alpha_0}{k_1}$. Now $M_1^{-1} \approx x^{-\alpha_0+\frac{\alpha_0}{k_1}-\frac{\alpha_{k_1}}{k_1}}$ and by analysing the exponent of x and substituting the previous inequality into the exponent of x, one can show that $M_1^{-1}=o(x^{-1})$. From the results of Corollary 2.2.3, $\mid M_n^{-1} \mid \ll \mid M_{n-1}^{-1} \mid \ll \cdots \mid M_2^{-1} \mid \ll \mid M_1^{-1} \mid$, and it follows that $\frac{w}{M_j}$, $j=2,\dots,n$ are all integrable. Therefore, by Lemma 2.4.1, all the form factors of the clusters j, $j>1$, are all square integrable. Thus the eigenfunctions of the clusters j, $j > 1$, that are square integrable are z-uniformly square integrable and contribute at most to discrete spectrum only.

The lowest cluster behaves like an operator itself such that if the lowest cluster has even length, $p_0 \approx w$, $r_{k_1} \approx O(x^{k_1})$ and the coefficients of the lowest cluster are almost constant, then the absolutely continuous spectrum can arise from the lowest cluster. The proof of this follows from results of [10].

□

The following example shows that the theory developed above in Sections 2.1-2.5 works also when the operators T_- and T_+ are defined by coefficients with different exponents.

Example 2.6.3. Assume that the operators T_- and T_+ are generated by the following differential equations

$$\tau_- v(x) = -\frac{1}{2} i\{(q_-(x)v'''(x))'' + (q_-(x)v''(x))'''\} - (p_-(x)v'(x))' + v(x)$$

and

$$\tau_+ y(x) = -\frac{1}{2} i\{(q_+(x)y'''(x))'' + (q_+(x)y''(x))'''\} - (p_+(x)y'(x))' + y(x)$$

respectively. Here, $x \in [0, \infty)$, $v(x) = y(-x)$, $p_- = p_0(1 + h_p)x^\gamma$, $p_+ = p_0(1 + h_p)x^\alpha$, $q_- = -q_0(1 + h_q)x^\omega$, $q_+ = q_0(1 + h_q)x^\beta$ and p_0, q_0 are constants. The mode in which the coefficients are defined is to make them be consistent with the theory developed in Section 2.5, see also [18, 19]. Now T_- and T_+ are defined on $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively. Here, of course, $\gamma \neq \alpha$ and likewise $\beta \neq \omega$. In order to have eigenvalues of different magnitudes, we demand that

$$5\gamma > 2\omega \quad \text{and} \quad 5\alpha > 2\beta.$$

One then employs the techniques in Section 2.2 to estimate the roots of the corresponding Fourier polynomials. A straight forward calculation gives the following estimates $m_1^- \sim x^{-\frac{\gamma}{2}}$, $m_1^+ \sim x^{-\frac{\alpha}{2}}$, $m_2^- \sim x^{\frac{\gamma-\omega}{3}}$ and $m_2^+ \sim x^{\frac{\alpha-\beta}{3}}$ so that $m_1^- \ll m_2^-$ and $m_1^+ \ll m_2^+$. Similarly, the M-factors can be estimated by $M_1^- \sim x^{\frac{\gamma}{2}}$, $M_1^+ \sim x^{\frac{\alpha}{2}}$, $M_2^- \sim x^{\frac{1}{3}(4\gamma-\omega)}$ and $M_2^+ \sim x^{\frac{1}{3}(4\alpha-\beta)}$ so that $| M_1^- | \ll | M_2^- |$ and $| M_1^+ | \ll | M_2^+ |$.

In order to preserve the square integrability of the eigenfunctions, we demand that $\gamma > 0$ if and only if $\alpha > 0$ and also that $\omega < 4\gamma$ if and only if $\beta < 4\alpha$ so that $w(M_i^-)^{-1}$ is integrable if and only if $w(M_i^+)^{-1}$ is integrable. After two diagonalisations and possibly $(I+Q)$-transformations, the system is transformed into Levinson's form. Thus the deficiency index and spectra results are as follows.

Assume $\beta + 3 < 4\alpha$ then $\omega + 3 < 4\gamma$ and $w(M_2^-)^{-1}$ and $w(M_2^+)^{-1}$ are all integrable and $(\text{def}T_-)_2 = (\text{def}T_+)_2 = (2,2)$. Thus $(\text{def}T_- \oplus T_+)_2 = (4,4)$ and USI. Therefore, $\sigma_{ac}(\widetilde{H})_2 = \emptyset$. If $\beta > 4\alpha$, it implies that $\omega > 4\gamma$ and $w(M_2^-)^{-1}$ and $w(M_2^+)^{-1}$ are all nonintegrable. Then $(\text{def}T_-)_2 = (2,1)$ and $(\text{def}T_+)_2 = (1,2)$ whenever $\frac{p_0}{q_0} > 0$ and $(\text{def}T_-)_2 = (1,2)$ and $(\text{def}T_+)_2 = (2,1)$ for $\frac{p_0}{q_0} < 0$. In either case, $(\text{def}T_- \oplus T_+)_2 = (3,3)$.

Finally, if $p_0 > 0$ then $(\text{def}T_-)_1 = (\text{def}T_+)_1 = (1,1)$, $\sigma_{ac}(\widetilde{H})_1 = \emptyset$ and USI. If $p_0 < 0$ and $\alpha > 0$ implying that $\gamma > 0$ then $(\text{def}T_-)_1 = (\text{def}T_+)_1 = (2,2)$ and USI. But if $p_0 < 0$ and $\alpha < 0$ implying that $\gamma < 0$ then $(\text{def}T_-)_1 = (\text{def}T_+)_1 = (1,1)$. Now using the results of Lemma 2.4.2, Theorem 2.4.4 and Theorem 2.5.1, one can obtain $\text{def}(T_- \oplus T_+)$ and $\sigma_{ac}(\widetilde{H})$ for the various conditions imposed on the coefficients and the exponents.

As a remark, note that one can define the operator T on $\mathbb{R}$ by taking T_- to be generated by differential equation $(-1)^n(p_n y^{(n)})^{(n)} + p_0 y = zw$ on $\mathbb{R}_-$ and T_+ to be generated by (1.1) on $\mathbb{R}_+$. One can show that the eigenvalues of T_- satisfy the uniform dichotomy condition. The methods of [19] can then be used to obtain the spectral results. This is akin to Example 6.6 of [18].

2.6.2 Extensions

Let the coefficients p_k and q_j in (1.1) and (1.2) have a decomposition of the form

$$f = f_1 + f_2 + f_3, \qquad f = p_k, q_j \quad f_2 = o(f_1), \tag{2.30}$$

$$w(x) = 1 \quad \text{and} \quad q_{n,3} = p_{n,3} = 0,$$

where f_1 is twice differentiable, f_2 is once differentiable and f_3 integrable. In addition, we require decay conditions like $f_1' = o(1)$, $f_1'', f_2', f_3 \in \mathcal{L}^1$. These decay conditions are of course adopted to asymptotic integration. Note that the matrix $\mathcal{C}$ in (1.8) will also assume a decomposition similar to that of the coefficients in (2.30). Here, $\mathcal{C}_1+\mathcal{C}_2$ will be called the smooth part of $\mathcal{C}$. Thus after the first diagonalisation, we demand that

$$\mathcal{T}^{-1}\mathcal{C}_3\mathcal{T}, \quad (\lambda_k-\lambda_j)^{-1}M_k^{-\frac{1}{2}}M_j^{-\frac{1}{2}}[p'_{l,2}\lambda_k^l\lambda_j^l, q'_{l,2}\lambda_k^l\lambda_j^{l-1}, q'_{l,2}\lambda_k^{l-1}\lambda_j^l] \in \mathcal{L}^1$$

so that these terms become Levinson's terms and thus are irrelevant for the asymptotics. Because we needed (2.9), (2.11) and (2.26) in Section 2.1 and Section 2.4 respectively for application of the second diagonalisation, we will need also similar conditions here, too. Thus, the above conditions amount to;

$$\frac{f_1'}{f}m_1^{-1}=o(1), \quad \frac{f_1'}{f}m_1^{-\frac{1}{2}}\in\mathcal{L}^2, \tag{2.31}$$

$$((\frac{f_1'}{f})^2, \frac{f_1''}{f}, \frac{f_2'}{f}, f_3)m_1^{-1}\in\mathcal{L}^1, \quad f=p_k, q_j.$$

One has

Theorem 2.6.4. *Let T be a differential operator generated by (1.1) or (1.2), defined on $[0,\infty)$ and D((i)-(iv)) hold. Assume the coefficients p_k, q_j to have a decomposition of the form (2.30) and the pivot coefficients r_{k_j} associated to p_k and q_j are nowhere zero in $[0,\infty)$. Moreover, assume (2.18)-(2.20), (2.31) and regularity conditions on the form factors to be satisfied. If T has equal deficiency indices, then $\sigma_{sc}(H)=\emptyset$ and the absolutely continuous spectrum is contributed by the eigenfunctions that lose their square integrability as $Imz\to 0^+$.*

Proof. The proof follows closely that of Theorem 2.4.4 and only a sketch of the proof will be given with many details omitted. The procedure of asymptotic integration by now is standard, that is, standard K.L-transformation if necessary, two diagonalisations, $(I+Q)$-transformations, deficiency indices and the M-matrix. After the first diagonalisation, one obtains a system of the form

$$v'=(\Lambda+V_1+R_2+R_3)v,$$

where R_2, R_3 are integrable and consist of $p'_{k,2}, q'_{j,2}$ and $p_{k,3}, q_{j,3}$ respectively. The matrix $V_1\in\mathcal{L}^2$ consists of the elements $p'_{k,1}, q'_{j,1}$. This implies that the system must be diagonalised again before application of Levinson Theorem. Since (2.31) is satisfied, it is possible to diagonalise the system for the second time. With these smoothness and decay conditions, the matrix $\Lambda + V_1 + R_2 + R_3$ can be diagonalised again using (1.34). The resulting system will be in Levinson's form and if not, then $(I + Q)$-transformations can be used to make the off-diagonal elements smaller.

□

The next example shows that the techniques developed in the previous sections can also be applied to differential operators with conditionally integrable terms.

Example 2.6.5. Consider a $2n$th order differential operator generated by

$$(-1)^n y^{(2n)}(x) + (-1)^k (p(x)y^{(k)}(x))^{(k)} + (q(x) - zw(x)) = 0.$$

z will be absorbed into q. Assume $w(x) = 1$, $\mid p \mid^n \gg \mid q \mid^{n-k}$ and D((i)-(iv)) hold. Let the coefficients p and q admit a decomposition of the form $p = p_1 + f_1 \sin g_1$ and $q = q_1 + f_2 \sin g_2$ with p_1 and q_1 twice differentiable. In this case, the standard K.L-transformation is not necessary. Let the term $f_i \sin g_i$, $i = 1, 2$ be rapidly oscillating with $g_i' \nearrow \infty$. Then we need a $(I + Q)$-transformation to eliminate these terms before we perform diagonalisations. In this case, the elements of $Q(x, z)$ will be of the form $-\int_x^\infty f_i(t, z) \sin g_i(t, z) dt$. Thus one needs that $\frac{f_i}{g_i'}, (\frac{f_i}{g_i'})' \in \mathcal{L}^1$. Here, f_i can also be unbounded. Then the oscillating terms do not alter the spectrum and this can be shown by repeated $(I + Q)$-transformations. But if $\frac{f_i}{g_i'} \notin \mathcal{L}^1$, that is, $\frac{f_i}{g_i'}$ does not tend to zero as $x \to \infty$, then the spectrum might be altered. As an example for this, consider $\tau y = y^{(iv)} + (py')'$ with $p = ae^x \sin e^x$ where a is a constant. In this case, $\sigma_{ac}(H) = [-\frac{a^4}{16}, \infty)$ with a subzero part being of multiplicity 2, see [8]. By asymptotic integration, one can achieve the Levinson form of this differential operator. Note that we have two groups of eigenvalues with different magnitudes because of the assumptions we have made on the coefficients p and q. The uniform dichotomy condition follows from Lemma 2.3.3 and the eigenfunctions are of the form (2.27). Thus if $p < 0$, $q > 0$, $\mid p \mid^{\frac{1-2n}{2(n-k)}}$ and $\mid p \mid^{-\frac{1}{2k}} \cdot \mid q \mid^{\frac{1-2k}{2k}}$ are nonintegrable, and $w = o(q)$ then $\sigma_{ac}(H, 2) = \mathbb{R}$. But if $q \approx w$, $p > 0$, $q < 0$ and $\mid p \mid^{\frac{1-2n}{2(n-k)}}$ and $\mid p \mid^{-\frac{1}{2k}} \cdot \mid q \mid^{\frac{1-2k}{2k}}$ are nonintegrable, then $[c, \infty) \subset \sigma_{ac}(H, 1)$ where $c = \limsup q(x)$.

Chapter 3

DIFFERENCE OPERATORS

This chapter is concerned with the spectral theory of $2n$th order difference operator L, defined on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$, generated by (1.49). It will be assumed generally that the coefficients p_k and q_j are almost constant, real valued and admit a decomposition of the form (1.50). For the highest order coefficients and the weight function, one will assume (1.51). Here, the actual proofs will be carried out with $f = f_0 + f_1$ only, $f = p_k, q_j, w$, $k = 0, 1, \ldots, n$, $j = 1, \ldots, n$, since the extension of these results to the more general case is rather routine now. This chapter, therefore, extends the spectral results of differential operators with almost constant coefficients to difference setting. The results are thus considered as discrete counterparts to those in [10, 26, 52].

The chapter is divided into three sections namely: the constant coefficients, even order operators and examples.

3.1 The Constant Coefficient

Before the study of the almost constant coefficient case, some results on the constant coefficient situation will suffice. This means that it is natural to begin the analysis with the unperturbed problem, that is, with operators having constant coefficients. One thus considers an operator L generated by (1.49) whose coefficients are constants. The advantage of this is that the domain of the self-adjoint realisation H_α of such an operator is still easy to describe after taking Fourier transforms. Here, H_α refers to the self-adjoint realisation of L (with constant coefficients), whereas the index α refers to the boundary conditions at $t = 0$. In the differential equation situation, the cosine- or sine- transformation which are derived from the Fourier transform and an extension from $[0, \infty)$ to $\mathbb{R}$, can be used to diagonalise the operator, that is, obtain a multiplication representation of the operator. Here, one proceeds in the same fashion and extends the operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{Z})$. Because of the λ and λ^{-1} symmetry, this means essentially doubling the operator. On $\ell^2(\mathbb{Z})$ the operator, however, becomes $\mathcal{F}(U)$ with U the bilateral shift and $\mathcal{F}$ as in (1.63). So apart from the boundary conditions which amounts to a finite rank perturbation, the operator is unitarily equivalent to multiplication with $\mathcal{F}(e^{i\alpha})$, $0 \leq \alpha \leq \pi$, because the spectrum of U is absolutely continuous of multiplicity 1 and because a finite rank perturbation leaves the absolutely continuous spectrum and its multiplicity invariant. Note that the reduction from $\mathbb{R}$ to $[0, \infty)$ corresponds to the restriction to $[0, \pi]$ here. The λ versus the λ^{-1} symmetry of $\mathcal{F}$ also shows that $\mathcal{F}$ takes on the same values in the upper semicircle as in the lower. Thus, the range of $\mathcal{F}(e^{i\alpha})$, $0 \leq \alpha \leq \pi$, determines the absolutely continuous spectrum. In particular, μ belongs to the absolutely continuous spectrum of multiplicity k if $\mathcal{F}(e^{i\alpha}) = \mu$ has k solutions in $[0, \pi]$.

While the above method is tailored parallel to the situation with the differential equations, see [26, Sect. 3], one can also employ the method of the M-matrix to obtain the result on the spectrum of H_α in the constant coefficient case. This will also indicate how to proceed in the nonconstant coefficient case.

In [52], Remling had proposed two methods to determine $\mathrm{Im}M(z)$. The first is based on [52, equation (8)]

$$\langle F_\alpha(., z), F_\alpha(., z')\rangle(\overline{z} - z') = M^*_\alpha(z) - M_\alpha(z'). \tag{3.1}$$

Here, F_α is the n by $2n$ system of square integrable solutions satisfying the boundary conditions α at 0 and the (i, j) matrix element is formed with the ith and jth vector. The method of proof shows that it is valid for discrete systems as well. Indeed, the discrete analogue is Theorem 6.3 in [63]. The second method is stated in [53] and has been used extensively in [26]. Nonetheless, (3.1) is preferable since it yields $\mathrm{Im}M(z)$ for $z = \mu \in \mathbb{R}$ directly as

$$\text{(3.2)} \qquad \begin{aligned} \mathrm{Im}M(\mu^+) &= \lim_{\epsilon\to 0^+} \mathrm{Im}M(\mu^+ + i\epsilon) \\ &= \lim_{\epsilon\to 0^+} \epsilon\langle F_\alpha(.,\mu+i\epsilon), F_\alpha(.,\mu+i\epsilon)\rangle. \end{aligned}$$

Now let $F_\alpha(.,z) = VC$ where C is a constant matrix and V is a $2n\times n$ matrix. Decompose V into $n\times n$ matrices, that is, $V = \begin{pmatrix} V_1(.,z) \\ V_2(.,z) \end{pmatrix}$. By considering the difference operator on $[a,\infty)$ for a large a, one can impose α-boundary conditions at a, and in particular, one chooses the Dirichlet boundary conditions with $\alpha_1 = I_n$ and $\alpha_2 = 0_n$. It then follows that the square summable solution matrix $F_\alpha(.,z)$ has the initial values $F_\alpha(a,z) = \begin{pmatrix} I_n \\ M_\alpha(z) \end{pmatrix}$. If one computes

$$\mathrm{Im}M(\mu_+) = \lim_{\epsilon\to 0^+} \epsilon C^*\langle V(.,z), V(.,z)\rangle C,$$

at the initial value $t = a$, then C is a regular matrix and

$$M_\alpha(z) = V_2(a,z)V_1^{-1}(a,z).$$

This second method works only in the even case. So it can be used if fairly complete knowledge of the eigenfunctions is obtained.

If the boundary condition α does not give rise to a bound state, the limit on the right hand side of (3.2) exists boundedly and defines a continuous function of μ. To see this, note that the eigenfunctions are proportional to (λ^k), where λ is a root of $\mathcal{F}$ (1.63). Now let $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ be two roots of $\mathcal{F} = 0$ for the spectral value $z = \mu + i\epsilon$ defining square summable solutions, that is, $\mid \lambda_j(\epsilon) \mid < 1$ for $\epsilon > 0$, $j = 1,2$. Then one has in first order for $z = \mu + i\epsilon$

$$\text{(3.3)} \qquad \lambda_j(\epsilon) = \exp(i(\alpha_j + \epsilon\beta_j)).$$

For this, one has to consider $\mathcal{F} = \mathcal{F}(t,z,\zeta)$, as a function of $\zeta = \lambda + \lambda^{-1}$ first and determine

$$\zeta(\epsilon) \approx \zeta(0) + (\partial_\zeta\mathcal{F})^{-1} i\epsilon.$$

Then use (3.3) to determine the β_j. With (3.3), the expression in (3.2) becomes a geometric series, for which the existence of the limit can be shown easily. To see this, note that the eigenfunctions are $c\cdot\lambda^t$ where λ is a root of $\mathcal{F}(t,\lambda,z)$ and c is a finite constant. Now let $\lambda_i(\epsilon)$ and $\lambda_j(\epsilon)$

be two roots of $\mathcal{F}(t,\lambda,z)$ defining square summable solutions, that is, $\mid \lambda_i(\epsilon) \mid, \mid \lambda_j(\epsilon) \mid < 1$, for $\epsilon > 0$. Assume that $y_i(t,\epsilon)$ and $y_j(t,\epsilon)$ are two square summable solutions determined by $\lambda_i(\epsilon)$ and $\lambda_j(\epsilon)$ respectively, then by Cauchy-schwarz and Hölders (Cauchy-Buniakowski) inequalities, (3.2) gives

$$\lim_{\epsilon\to 0^+} \epsilon \mid \langle y_i(t), y_j(t)\rangle \mid$$

$$\leq \lim_{\epsilon\to 0^+} \epsilon\Big(\sum_{k=1}^{2n} \mid c_{i,k} \mid^2 \mid \lambda_i^t(\epsilon) \mid^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{2n} \mid c_{j,k} \mid^2 \mid \lambda_j^t(\epsilon) \mid^2\Big)^{\frac{1}{2}},$$

where $c_{i,k}$ and $c_{j,k}$ are the appropriate eigenvectors. The term on the right hand side is bounded and converges absolutely as $t \to \infty$ since $\mid \lambda_i(\epsilon) \mid\mid \lambda_j(\epsilon) \mid < 1$. This implies that its limit exists and consequently $\mathrm{Im}M(\mu_+)$ is nontrivial. This shows that the spectrum of H_α has no singular continuous part.

Here also, the analysis will follow that one of [26]. There exists an exceptional set $\mathcal{E} \subset \mathbb{R}$ so that for $\mu \notin \mathcal{E}$, $M(\mu_+) = \lim_{\epsilon\to 0_+} M(\mu + i\epsilon)$ exists finitely. This set $\mathcal{E}$ has only finitely many accumulation points and is thus irrelevant for the continuous spectrum. By results of [26, Proposition 3.2 and Lemma 3.3], off the critical set $\mathcal{E}$, it can be shown that the point spectrum of H_α has no accumulation points on extended $\mathbb{R}$. On the other hand, a non-real λ cannot be an eigenvalue of H_α, since H_α is a self-adjoint operator and thus its spectrum is real. So non-real eigenvalues cannot be accumulation points. Thus the point spectrum of H_α is finite. Moreover, assume that the first k solutions of $\mathcal{F}_\alpha$ are uniformly square summable, then the corresponding limit matrix elements of $\mathrm{Im}M_\alpha(\mu_+)$ vanish so that $\mathrm{Im}M_\alpha$ has nontrivial matrix elements only in the lower $(n-k)$ by $(n-k)$ right hand corner. Thus, rank $M_\alpha(\mu_+) \leq n-k$.

This result implies that whenever a solution loses its square summability as $\mathrm{Im}z \searrow 0$, then it contributes only to absolutely continuous spectrum. In order to see that rank $M_\alpha(\mu_+) = n-k$, we state a general Lemma from abstract algebra.

Lemma 3.1.1. *Let $\mathcal{H}$ be a Hilbert space and let $v_1,\ldots,v_n \in \mathcal{H}$. Then the matrix A with $A_{ij} = \langle v_i, v_j\rangle$ is nonnegative definite and rank$A$ =* dim *linspan$\{v_1,\ldots,v_n\} = \mathcal{H}_0$.*

Proof. Let $w_1,\ldots,w_k$ be an orthornormal basis of $\mathcal{H}_0$ with $v_i = \sum b_{il}w_l$. Then $A = BB^*$ and the result follows. □

The results are formulated suggestively as:

"*Only eigenfunctions, which lose their square summability as Imz* $\to 0$ *contribute to the absolutely continuous spectrum.*"

This implies that the multiplicity of the absolutely continuous spectrum is k, if $\mathcal{F}$ has $2k$ roots of absolute value 1. Needless to say that this result is the analogue of the differential equation situation. There, one has $2k$ real roots of the Fourier polynomial [26].

3.2 Even Order Difference Operators

In this section, the spectral analysis of the difference operator L generated by (1.49) will be pursued. Thus consider the difference operator L generated by (1.49), defined on $\ell^2(\mathbb{N})$. Assume the coefficients p_k and q_j to be real valued and admit a decomposition of the form (1.50) with smoothness conditions in (1.50) satisfied. For the highest order coefficient and the weight measure, p_n and w, assume that

$$p_n, w > 0 \quad \text{and normalise} \quad p_{n,0} = w_0 = 1. \tag{3.4}$$

The underlying Hilbert space will be $\ell^2(\mathbb{N})$ with the scalar product defined as in Section 1.2.2. For the eigenvalues of the propagator matrix $S(t, z)$, one considers the characteristic polynomial (1.63). In order to obtain the eigenfunctions of (1.55) and the spectral results of H by asymptotic summation, one requires that z-uniform dichotomy condition be satisfied by the eigenvalues of $S(t, z)$.

One, therefore, needs the following results and in particular, Theorem 3.2.2 which is the discrete analogue of Theorem 2.3.1, in order to simplify the discussion of the uniform dichotomy condition.

Lemma 3.2.1. *A pair of eigenvalues* λ, λ^{-1} *with* $\zeta = \lambda + \lambda^{-1}$ *satisfies* $|\lambda| > 1$ *and* $|\lambda^{-1}| < 1$ *unless* ζ *is real with* $|\zeta| < 2$.

The analysis is based mainly on the limiting constant coefficient polynomial

$$\begin{aligned}\mathcal{F}_0(\lambda, z, t) &= \frac{p_{n,0} - iq_{n0}}{\lambda^n}\mathcal{P}_0(\lambda, t, z) \\ &= \sum_{k=0}^{n} p_{k,0}(1-\lambda)^k(1-\lambda^{-1})^k \\ &+ \sum_{j=1}^{n} q_{j,0}(1-\lambda)^{j-1}(1-\lambda^{-1})^{j-1}(i\lambda + (i\lambda)^{-1}).\end{aligned} \tag{3.5}$$

Here, $p_{n,0}$, $p_{k,0}$ and $q_{j,0}$ are the limiting constant values of the coefficients p_n, p_k and q_j respectively, $k = 0, \ldots, n-1$, $j = 1, \ldots, n$. Also in this case, a finite exceptional set $\mathcal{E}_0$ of real z values which leads to double roots will be excluded. Now fix a small set

$$\mathcal{K} = \{z \mid \ \mid z - z_0 \mid \leq \epsilon, \quad \text{Im} z \geq 0\}$$

of spectral values $z_0 \in \mathbb{R}$ off $\mathcal{E}_0$. In (3.5), if λ is a root of $\mathcal{F}_0(t, \lambda, z)$, then λ^{-1} is also a root. Now let the roots of $\mathcal{F}_0 = \mathcal{F}(\lambda, z_0)$ be $\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}$. Since the roots are analytic functions of the coefficients, this extends to $\mathcal{F}(t, \lambda, z)$. Thus, for a suitable ϵ and $t \geq t_0$, $\mathcal{F}(t, \lambda, z)$ has the λ-roots, $\lambda_1(t, z), \lambda_1^{-1}(t, z), \ldots, \lambda_n(t, z), \lambda_n^{-1}(t, z)$ which for $t \to \infty$ converge to the appropriate limits. Assume that $\zeta = \lambda + \lambda^{-1}$, then ζ-values such that $\mid \zeta \mid > 2$ will lead to λ-values with $\mid \lambda \mid > 1$ and $\mid \lambda^{-1} \mid < 1$. All this will extend from $\mathcal{F}_0(\lambda, z_0)$ to $\mathcal{F}(t, \lambda, z)$, $t \geq t_0$, $\mid z - z_0 \mid < \epsilon$, $z \in \mathbb{R}$, for t_0 sufficiently large and $\epsilon > 0$ sufficiently small by analyticity.

Now one forms eigenvalue classes based on $\mid \lambda \mid < 1$, because one will be looking for square summable solutions.

Theorem 3.2.2. *Let*

$$u(t+1) = [\Lambda(t) + R(t)]u(t), \quad t \geq t_0, \tag{3.6}$$

$$\Lambda(t) = diag(\lambda_1(t), \ldots, \lambda_{2n}(t)),$$

be asymptotically constant difference equation such that $\sum_{t=t_0}^{t-1} \|R(t)\| \mid \lambda_i^{-1}(t) \mid < \infty$. *Assume the eigenvalues* $\lambda_i(t)$ *for* $i = 1, \ldots, 2n$ *satisfy*

$$\lambda_i(t) = \lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2}, \quad \textit{with} \quad \lambda_{i,0} \quad \textit{constant}, \tag{3.7}$$

$\lambda_{i,1}(t) \to 0$ *as* $t \to \infty$, $\lambda_{i,2}$ *is conditionally summable and* λ_{i0} *distinct. Let* $h(t) > 0$ *be a nonsummable, monotonic function in* $\mathbb{N}$ *and assume* *the eigenvalues* $\lambda(t)$ *can be sorted into classes* $\mathcal{C}_1, \ldots, \mathcal{C}_n$ *so that* *if* $\lambda_i(t)$, $\lambda_j(t) \in \mathcal{C}_k$, *then* $(\frac{|\lambda_i(t)|}{|\lambda_j(t)|} - 1) = o(h(t))$; *if* $\lambda_i(t) \in \mathcal{C}_k$, $\lambda_j(t) \in \mathcal{C}_l$, $k \neq l$, *then* $\frac{|\lambda_i(t)|}{|\lambda_j(t)|} \leq 1 - h(t)$ *or* $\frac{|\lambda_i(t)|}{|\lambda_j(t)|} \geq 1 + h(t)$. *For each* $\lambda(t)$ *write now* $\mid \lambda(t) \mid = 1 + \mu(t)$ *with* $\mu_+ = \max(0, \mu)$ *and* $\mu_- = \min(0, \mu)$ *and define for each class* k

$$a_k(t) = \max_{\lambda \in \mathcal{C}_k} \mu(t)_+ \quad \textit{and} \quad b_k(t) = \max_{\lambda \in \mathcal{C}_k} \mu(t)_-.$$

Then associated to each $\mathcal{C}_k$ *there are* $\mid \mathcal{C}_k \mid$ *(*$\mid \mathcal{C}_k \mid$ *is the number of elements in the* k*th-class) solutions* $u(t)$ *of (3.6) satisfying*

$$K_1 \prod_{t=t_0}^{t-1} (1 - b_k(t)) \leq \|u(t)\| \leq K_2 \prod_{t=t_0}^{t-1} (1 + a_k(t)). \tag{3.8}$$

The conditionally summable terms can be removed by a simple transformation $\prod_{t_0}^{t-1} \Lambda_{i2}(s)$. The rest of the proof follow by iteration and is identical to the proof of Theorem 5.1 in [12].

When λ is a root of the polynomial $\mathcal{F}(t, \lambda, z)$, then λ^{-1} is also a root. Thus let

$$\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}$$

be roots of the polynomial $\mathcal{F}(t, \lambda, z)$ which for $t \to \infty$ converge to the appropriate limits. One can arrange these roots into two groups, that is, $\lambda_1, \lambda_1^{-1}, \ldots, \lambda_m, \lambda_m^{-1}$ and $\lambda_{2m+1}, \ldots, \lambda_{2n}$ such that $\mid \lambda_l \mid > 1$, $\mid \lambda_l^{-1} \mid < 1$, $l = 1, \ldots, m$ and $\mid \lambda_j \mid = 1$, $j = 2m+1, \ldots, 2n$. Theorem 3.2.2 implies that the first $2m$ λ-roots lead to m square summable solutions and m solutions which are not square summable. This holds regardless of the uniform dichotomy condition. The other $2(n-m)$ λ-roots with magnitude 1 can be written with their first order correction terms as

$$\lambda_j(z) = \lambda_j(z_0) + (\partial_\lambda \mathcal{F}(t, \lambda_j, z))^{-1}(z - z_0) \quad \text{for small} \quad \mid z - z_0 \mid .$$

Thus, the dichotomy condition holds if

$$\partial_\lambda \mathcal{F}(t, \lambda_j, z) \neq \partial_\lambda \mathcal{F}(t, \lambda_i, z), \quad i \neq j.$$

By symmetry of λ and λ^{-1},

$$\lambda_j(z_0) + (\partial_\lambda \mathcal{F}(t, \lambda_j, z))^{-1}(z - z_0),$$

$j = 2m+1, \ldots, 2n$ will contribute $(n-m)$ to the deficiency index since $(n-m)$ solutions off the real axis will be square summable for Im$z > 0$. The other complementary $(n-m)$ solutions will be square summable in the lower half plane. This shows that for limiting coefficients, L is limit point and def$L = (n, n)$. It thus suffices to check uniform dichotomy condition only for the λ-roots of $\mathcal{F}(t, \lambda, z)$ with $\mid \lambda \mid = 1$.

Lemma 3.2.3. *Assume the λ-roots of the polynomial $\mathcal{F}(t, \lambda, z)$ to be distinct. Moreover, assume that $\partial_\lambda \mathcal{F}(t, \lambda_i, z) \neq \partial_\lambda \mathcal{F}(t, \lambda_j, z)$ for $\mid \lambda_i \mid = \mid \lambda_j \mid = 1$, $i \neq j$. Then the eigenvalues of $\mathcal{F}(t, \lambda, z)$ satisfy the uniform dichotomy condition.*

Proof. In order to sum (1.55) asymptotically by employing the Levinson-Benzaid-Lutz Theorem, Theorem 1.2.2, one needs the uniform dichotomy condition for the roots of $\mathcal{P}$ respectively $\mathcal{F}$. By continuity, it suffices to prove it only for $\mathcal{P}_0$ or $\mathcal{F}_0$, because then it can be extended to $t \geq a$. The dichotomy condition for discrete systems requires a t-uniform control of $\mid \lambda_i \mid$ and $\mid \frac{\lambda_i(t)}{\lambda_j(t)} \mid$ for different eigenvalues. For this one will, to begin with, exclude all double roots of $\mathcal{F}$ from the study. Thus let $\omega_1 < \ldots < \omega_l$ be the set of all real z for which $\mathcal{F}$ has double roots. Moreover, let $\omega_0 = -\infty$ and $\omega_{l+1} = \infty$. If one stays with Rez strictly in (ω_i, ω_{i+1}), $i = 0, \ldots, l$, all relevant quantities like eigenvalues, eigenvectors,.. will be analytic functions of the matrix elements of $S(t, z, \lambda)$, in particular z. If possible, one requires $\mid \lambda_i \mid \neq \mid \lambda_j \mid$ for $i \neq j$. This cannot be achieved always because with λ_i also $\overline{\lambda_i}$ is an eigenvalue and one has $\mid \lambda_i \mid = \mid \lambda_j \mid$ at least as long as z is real. Off the real axis, however, this will change.

Because of Theorem 3.2.2, it suffices to show the uniform dichotomy condition only for eigenvalues λ with $\mid \lambda \mid = 1$. To do this, one employs the z-transform in the analysis of $\mathcal{F}_0 = \mathcal{F}_0(\lambda, z)$ [35]. It is defined by

$$\lambda = \frac{(s+1)}{(s-1)}. \tag{3.9}$$

(3.9) leads to $s = \frac{(\lambda+1)}{(\lambda-1)}$. The z-transform maps the interior of the unit circle onto the left hand plane and the unit circle onto the imaginary axis. With

$$(1-\lambda)(1-\lambda^{-1}) = -4(s^2-1)^{-1} \quad \text{and} \quad i(\lambda - \lambda^{-1}) = 4is(s^2-1)^{-1},$$

one gets

$$(3.10)\quad (s^2-1)^n\mathcal{F}_0\left(\frac{s+1}{s-1},z\right) = \sum_{k=0}^{n}(-1)^k p_{k0}2^{2k}(s^2-1)^{n-k} + \sum_{j=1}^{n}(-1)^{j-1}2^{2j}q_{j0}(s^2-1)^{n-j}is - z(s^2-1)^n.$$

As in the study of differential operators, however, one should switch to the Fourier variant Q of this polynomial, by replacing s by is. Then

$$(3.11)\qquad Q(s,z) = (s^2+1)^n\mathcal{F}_0(\frac{is+1}{is-1},z) = \sum_{k=0}^{n} p_{k0}2^{2k}(s^2+1)^{n-k} + \sum_{j=1}^{n} 2^{2j}q_{j0}(s^2+1)^{n-j}s - z(s^2+1)^n.$$

Q is a polynomial with real coefficients. Thus, its roots will be of the form $\alpha_1 \pm i\beta_1, \ldots, \alpha_r \pm i\beta_r, \gamma_{2r+1}, \ldots, \gamma_{2n}$ with $\alpha_1, \beta_1, \ldots, \gamma_{2n}$ real and they are functions of t and z. It follows that the solutions corresponding to the eigenvalues $\alpha_j - i\beta_j$ will be z-uniformly square summable for $z \in \mathcal{K}$. In order to see this, multiply $\alpha_j - i\beta_j$ by $-i$ since s was replaced by is. The eigenfunctions are given by

$$v_j(t,z) = (\varrho_j(t,z) + r_j(t,z))\prod_{l=t_0}^{t-1} s_j(l,z)$$

with $r_j(t,z) \to 0$ as $t \to \infty$ uniformly for $z \in \mathcal{K}$.

In order to evaluate $\prod_{l=t_0}^{t-1} s_j(l,z)$, take logarithms and use Euler summation formula to get $\ln\prod_{l=t_0}^{t-1} \mid s_j(l,z) \mid^2 \approx -2\sum_{t_0}^{t-1}\beta_j(l,z)$. Thus one has modulo some constant factor that

$$\|v_j(.,z)\|^2 \approx \int_{t_0}^{\infty} \varrho_j(t,z)\exp(-2\int_{t_0}^{t}\beta_j(l,z)dl)dt,$$

where $\varrho_j(t,z)$ is a suitable eigenvector of the eigenvalue $\alpha_j - i\beta_j$. This norm is bounded and converges to a unique limit as $t \to \infty$. It follows that $v_j(.,z)$ is square summable. As a consequence of Theorem 3.2.2 even, square summability has to be checked for eigenvalues λ with

$|\lambda| = 1$ only. In this case, the z-uniform dichotomy condition for two roots λ_1 and λ_2 with $|\lambda_1(z_0)| = |\lambda_2(z_0)| = 1$ amounts to (cf. [26])

$$|\frac{\lambda_1(z_0+i\delta)}{\lambda_2(z_0+i\delta)}| \geq 1+c\delta \quad \text{or} \quad |\frac{\lambda_1(z_0+i\delta)}{\lambda_2(z_0+i\delta)}| \leq 1-c\delta \tag{3.12}$$

with $c > 0$ and all $0 \leq \delta \leq \epsilon > 0$.
If (3.12) holds for the pair λ_1, λ_2 of eigenvalues, this inequality can be extended by continuity to $\{\lambda_i(t,z)\}$, $i = 1,2$ for $t \geq a$ and $z \in \mathcal{K}$, if ϵ, c and a are adjusted if necessary. In order to analyse this condition, write

$$\lambda_j = \exp(i\alpha_j) = \frac{(is_j+1)}{(is_j-1)} = \exp(i\beta_j)\cdot\exp(-i(\pi-\beta_j)),$$

where $\beta_j = \arctan s_j(z)$. Thus,

$$\frac{d\beta}{dz} = (1+s^2)^{-1}\frac{ds}{dz}. \tag{3.13}$$

Note that $\frac{ds}{dz}$ is real for $z \in \mathbb{R}\cap\mathcal{K}$. Thus, one has for $z \in \mathbb{R}\cap\mathcal{K}$ by analyticity

$$\lambda(z+i\delta) = -\exp(2i\beta(z))\cdot\exp(-2\frac{d\beta}{dz}\delta).$$

(3.12) will hold for the pair (λ_1, λ_2) out of $\gamma_{2r+1}, \ldots, \gamma_{2n}$ if

$$\frac{d\beta_1}{dz}(z_0) \neq \frac{d\beta_2}{dz}(z_0).$$

Somewhat, more generally, it suffices that the set

$$U_{12} = \{z \mid \quad \frac{d\beta_1}{dz}(z) = \frac{d\beta_2}{dz}(z)\} \tag{3.14}$$

is countable, with accumulation points at the ω_j only. More generally, it suffices that U_{12} is spectrally irrelevant, that is, it cannot carry a singular continuous measure. Now define the sets U_{12} for eigenvalue pairs λ_1, λ_2 with $|\lambda_1(z)| = |\lambda_2(z)| = 1$. Assume that U_{12} has accumulation point z inside (ω_i, ω_{i+1}). Then the analyticity of λ_1 and λ_2 implies

$$\frac{d\beta_1}{dz}(z) = \frac{d\beta_2}{dz}(z), \qquad z \in (\omega_i, \omega_{i+1}) \tag{3.15}$$

and in fact this relation can be extended to all of $\mathbb{C}$ excluding the singular points where Q has multiple roots. (3.15) implies

$$\arctan s_1(z) = \arctan s_2(z) + C.$$

Then the addition theorem of the tangent gives

$$s_1(z) = \frac{s_2(z) + C'}{1 - s_2(z)C'}. \tag{3.16}$$

It remains to show that (3.16) is possible only for $C' = 0$ or $s_1(z) = s_2(z)$. To see this, consider the polynomial

$$\widetilde{Q}(s,z) = (1 - sC'^n)Q\left(\frac{s + C'}{1 - sC'}, z\right). \tag{3.17}$$

It is easy to see that Q as well as $\widetilde{Q}$ are irreducible, because z appears linearly and because $p_{n,0} \neq 0$. The Riemann surfaces generated by Q, respectively $\widetilde{Q}$, have a common segment by (3.16), hence by Bezout's Theorem, Theorem 2.3.2, Q and $\widetilde{Q}$ must agree, that is, $s_1(z) = s_2(z)$. This shows that the set U_{12} is spectrally irrelevant.
Similarly, this can be shown for all other sets U_{ij} for pairs λ_i and λ_j with $|\lambda_i(z)| = |\lambda_j(z)| = 1$. □

Theorem 3.2.4. *Consider the difference operator L generated by (1.49) on $\ell^2_w(\mathbb{N})$ with real coefficients satisfying (1.50) and (3.4). Let $\mathcal{P}_0$ be the characteristic limiting polynomial*

$$\begin{aligned}\mathcal{P}_0(\lambda, z) = &\sum_{k=0}^{n} p_{k,0}(1-\lambda)^k(1-\lambda^{-1})^k \\ &+ \sum_{j=1}^{n} q_{j,0}(1-\lambda)^{j-1}(1-\lambda^{-1})^{j-1}(i\lambda + (i\lambda)^{-1}) - zw.\end{aligned}$$

Here $p_{k,0}$, $q_{j,0}$ are the limiting constant values of the coefficients. Then, the associated minimal operator L is limit point and for any boundary condition α at 0, the corresponding self-adjoint extension H_α has no singular continuous spectrum, $\sigma_{sc}(H_\alpha) = \emptyset$. The absolutely continuous spectrum, $\sigma_{ac}(H_\alpha)$ of H_α, agrees with that of constant coefficient limiting operator H_0. In particular, μ belongs to absolutely continuous spectrum of multiplicity k, $\mu \in \sigma_{ac}(H_\alpha, k)$ if the characteristic limiting polynomial has $2k$ zeros of absolute value one. So the spectrum is given by the image of the characteristic polynomial on the torus.

Proof. By considering the resultant or the discriminant of $\mathcal{P}_0$ and $\partial_\lambda \mathcal{P}_0$, one can show that there are only finitely many spectral values z for which $\mathcal{P}_0(\lambda, z)$ has multiple roots. Let $\omega_1 < \omega_2 < \cdots < \omega_k$ denote all of the real spectral values z leading to multiple roots. The analysis will be restricted to small complex neighbourhoods of $z_0 \in (\omega_i, \omega_{i+1})$, $i = 0, \ldots, k$, where $\omega_0 = -\infty$ and $\omega_{k+1} = \infty$. For a given $z_0 \in (\omega_i, \omega_{i+1})$, one can now choose $\epsilon > 0$ and $a > 0$ so that $\mathcal{P}(t, \lambda, z) = 0$ has no multiple roots for any $z \in \mathcal{K}_\epsilon(z_0)$ and $t \geq a$. This is possible because the roots of $\mathcal{P}_0$ depend analytically on the coefficients. Throughout the proof, it may be necessary to adjust a and ϵ repeatedly. This will be done without further comment. The results of Remling [[52], Sect.6] can be used to extend the results from $[a, \infty)$ to $[0, \infty)$. For $z_0 \in (\omega_i, \omega_{i+1})$ chosen as above, one can find a constant coefficient transformation D, which diagonalises the limiting matrix S_0, $D^{-1} S_0 D = \operatorname{diag}(\lambda_{1,0}, \ldots, \lambda_{2n,0})$. Thus

$$v = D \begin{pmatrix} x \\ u \end{pmatrix}$$

satisfies

$$v(t+1) = (\Lambda(t) + R(t))v(t) \quad \text{with} \quad \Lambda = \operatorname{diag}(\lambda_i(t)) \quad \text{and} \tag{3.18}$$

$$\lambda_i(t) = \lambda_{i0} + (R_1 + R_2)_{ii}(t).$$

Here, R_1, R_2 arise from the f_1, f_2 parts of the coefficients. One has, in particular, $R_{ii}(t) = 0$ and

$$\Delta^2 R_1, \;\; (\Delta R_1(t))^2, \;\; \Delta R_2, \;\; R_3 \in \ell^1 \quad \text{and} \tag{3.19}$$

$$R_1(t), R_2(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Since the eigenvalues of Λ are distinct and since $R_1, R_2 = o(1)$, the matrix $\Lambda + R_1 + R_2$ can be diagonalised again with a diagonalising transformation of the form $(I + B)$ with $B_{ii} = 0$ and $B_{ij} = (\lambda_j - \lambda_i)^{-1}(R_1 + R_2)_{ij}$. Such a diagonalisation can be repeated and leads to a system in Levinson-Benzaid-Lutz form (3.18) with $R \in \ell^1$. It should be noted that the $\lambda_i(t)$ may be taken as the roots of $\mathcal{P}(t, \lambda, z)$, because the final two diagonalisations create at most summable perturbations of the diagonal.

The dichotomy condition now follows from Lemma 3.2.3. In (3.13), note that $\frac{ds}{dz}$ is real for $z \in \mathbb{R} \cap \mathcal{K}$. Thus, one has for $z \in \mathbb{R} \cap \mathcal{K}$ by analyticity

$$\lambda(z + i\delta) = -\exp(2i\beta(z)) \cdot \exp(-2\frac{d\beta}{dz}\delta).$$

The contribution of eigenvalues $\gamma_{2r+1}, \dots, \gamma_{2n}$, to the deficiency index will depend on the sign of $\frac{ds_j}{dz}$ where $\gamma_j(z) = -\exp(2i\beta_j(z))$. However, because Q is an even polynomial, the signs are evenly distributed. Thus, the deficiency index of L is (n, n) and half of the eigenfunctions of L corresponding to the eigenvalues $\gamma_{2r+1}(z_0 + i\delta), \dots, \gamma_{2n}(z_0 + i\delta)$ are square summable as long as $\delta > 0$. But these solutions lose their square summability as $\delta \to 0$.

The proof will now be completed by first determining the eigenfunctions of L for $z \in \mathcal{K}$. They are of the form

$$v_i(t, z) = (\varrho_i(t, z) + r_i(t, z)) \prod_{l=t_0}^{t-1} \lambda_i(l, z) \quad \text{with} \quad r_i(t, z) = o(1).$$

Here, $\varrho_i(t, z)$ is a suitable eigenvector of $S(t, z)$ for the eigenvalue $\lambda_i(t, z)$. Let $F_\alpha(t, z + i\delta)$, $z \in \mathcal{K} \cap \mathbb{R}$, $\delta > 0$, denote the n by $2n$ system of n square summable solutions of (1.55) satisfying the α boundary condition at 0. Then by (3.1) and (3.2),

$$\lim_{\delta \to 0^+} \epsilon \langle F_\alpha(., z + i\delta), F_\alpha(., z + i\delta) \rangle = \mathrm{Im} M(z_+) \tag{3.20}$$

exists boundedly for $z \in \mathcal{K} \cap \mathbb{R}$. Thus, the continuous spectrum of H_α is absolutely continuous with multiplicity $(n - r)$. This proof also shows that the density of the absolutely continuous measure $\frac{1}{\pi}\mathrm{Im} M(z_+)$ is continuous. □

Remark 3.2.5. Let $\widetilde{L}$ be a difference operator generated by (1.49) with p_k, $k = 0, 1, \dots, n$, satisfying (1.50) and $q_j = 0$ for all $j = 1, \dots, n$. Then one obtains a limiting Fourier polynomial of the form

$$\mathcal{F}_0(\lambda, z) = \sum_{k=0}^{n} c_k (2 - \zeta)^k - z, \qquad p_k(t) \to c_k, \quad t \to \infty,$$

where $\zeta = \lambda + \lambda^{-1}$. In this case, the analysis will be based on ζ as a function of t and z. Thus the dichotomy condition is only proved for the ζ-roots such that $| \zeta | \leq 2$. The uniform dichotomy condition will hold whenever $\partial_\zeta \mathcal{F}(t, \zeta_i, z) \neq \partial_\zeta \mathcal{F}(t, \zeta_j, z)$, for $| \zeta_i | \leq 2$, $| \zeta_j | \leq 2$ and $i \neq j$. One can then show that $\widetilde{L}$ is limit point and for any boundary condition α at 0, the self-adjoint extension of $\widetilde{L}$ has identical spectral results to those of H_α in Theorem 3.2.4.

By the decomposition method, this result can readily be extended to L on $\ell^2_w(\mathbb{Z})$. For this, one considers the minimal operators L_+ and L_- corresponding to the two half lines $\mathbb{Z}_+ = \mathbb{N}$ and $\mathbb{Z}_- = \{-1, -2, \ldots\}$. Then $L_+ \oplus L_-$ has deficiency index $(2n, 2n)$ and the original self-adjoint operator L on $\mathbb{Z}$ is a finite rank perturbation of $H_+ \oplus H_-$. Thus, $\sigma_{sc}(H) = \emptyset$ and

$$\sigma_{ac}(H, k) = \cup_{k=k_1+k_2}(\sigma_{ac}(H_+, k_1) \cap \sigma_{ac}(H_-, k_2)).$$

3.3 Examples

The following examples illustrate the results of Sections 3.1-3.2. Example 3.3.1 and 3.3.2 are two term difference operators, defined on $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N})$ respectively, showing that one can obtain absolutely continuous spectrum of multiplicity one. In particular, Example 3.3.1 is a step potential operator.

Example 3.3.1. Let $p_n = 1$,

$$p_0(t) = \begin{cases} 0; & t < 0 \\ a; & t \geq 0 \end{cases}$$

and $p_k = q_j = 0$, $k = 1, \ldots, n-1$, $j = 1, \ldots, n$. Now consider a $2nth$ order step potential difference operator L defined by these coefficients. In this case, $\sigma_{ac}(H_-, 1) = [0, 4^n]$ and $\sigma_{ac}(H_+, 1) = [a, a + 4^n]$. Thus, for $0 < a < 4^n$, $\sigma_{ac}(H, 1) = [0, a] \cup [4^n, a + 4^n]$, $\sigma_{ac}(H, 2) = [a, 4^n]$. While for $a \geq 4^n$, $\sigma_{ac}(H, 1) = [0, 4^n] \cup [a, a + 4^n]$.

The following example shows how absolutely continuous spectrum can easily change in the case of a two term difference operator depending on the p_0 term. The example shows that if $p_0 \to \infty$, as $t \to \infty$, that is, dominant p_0, then the spectrum is discrete but if $p_0 \to 0$ as $t \to \infty$, then absolutely continuous spectrum is obtained of multiplicity 1.

Example 3.3.2. Consider a $2n$th order difference operator generated by the difference expression

$$(-1)^n \Delta^{2n} y(t-n) + t^\alpha y(t) = zw(t)y(t),$$

where $t \in \mathbb{N}$, $w(t) = 1$, α is a nonzero real-valued constant and z the spectral parameter. Thus, the ζ-charateristic polynomial from this equation is of the form

$$\mathcal{F}(t,\zeta,z) = (2-\zeta)^n + t^\alpha - z.$$

By adjusting a and ϵ as mentioned in Section 1.2.2 and choosing $z \in \mathcal{K}_\epsilon(z_0)$ appropriately, one can obtain n distinct ζ-roots of the above polynomial.

Thus, by determining the values of z that satisfy the inequality $\mid \zeta \mid = \mid 2-(z-t^\alpha)^{\frac{1}{n}} \mid \leq 2$, one obtains $t^\alpha \leq z \leq 4^n + t^\alpha$. Now assume that $\alpha > 0$, then as $t \to \infty$, the spectrum is discrete and $\sigma_{ac}(H) = \emptyset$. On the other hand, if $\alpha < 0$, then one obtains absolutely continuous spectrum of multiplicity 1, that is, $(0, 4^n] \subset \sigma_{ac}(H,1)$. Thus, a dominant $p_0(t)$ term, $p_0(t) \to \infty$ as $t \to \infty$, leads to discrete spectrum.

The next example shows that in contrast to the continuous square well of diameter one which has large number of bound states for deep wells, the discrete analogue has at most one bound state.

Example 3.3.3 (The square well of diameter 1)**.**

$$p_n = 1, \quad p_0(t) = -a\delta_{0t}, \quad t \in \mathbb{Z}, \quad a > 0.$$

This is a rank 1 perturbation of the zero potential operator. So there can at most be one bounded state, in contrast to the differential equation situation, where one has a large number of bound states for deep wells.

The next example illustrates the results of Theorem 3.2.4.

Example 3.3.4 (The case of the vanishing weight function)**.** Assume the weight function $w = w(t)$ satisfies $w(t) = o(1)$ and assume $p_n = 1 = \mid p_0 \mid$ and

$$p_k(t) - c_k, \quad q_j(t) - d_j \in \ell^1 \quad k = 0, \ldots, n-1, \quad j = 1, \ldots, n.$$

Moreover, assume that the limiting Fourier polynomial

$$\mathcal{F}(t,\lambda,z) = (1-\lambda)^n(1-\lambda^{-1})^n + \sum_{k=0}^{n-1} c_k(1-\lambda)^k(1-\lambda^{-1})^k + \sum_{j=1}^{n} d_j(1-\lambda)^{j-1}(1-\lambda^{-1})^{j-1}(i\lambda + (i\lambda)^{-1}) - zw,$$

has $2n$ distinct roots. Let $\mu_1, \ldots, \mu_{2k}$ denote the roots of magnitude 1 and $\lambda_1, \lambda_1^{-1}, \ldots, \lambda_r, \lambda_r^{-1}$, $r = 1, \ldots, n-k$ denote the roots with either

$| \lambda_r | < 1$ or $| \lambda_r | > 1$. Also assume that $w(t)$ is not summable and that $| \partial_\lambda \mathcal{F}(\mu, z) | \neq | \partial_\lambda \mathcal{F}(\widetilde{\mu}, z) |$ for roots μ and $\widetilde{\mu}$ with $| \mu | = | \widetilde{\mu} | = 1$. The z-uniform dichotomy condition follows immediately from Theorem 3.2.2 and Lemma 3.2.3. Then $\mathrm{def}L = (n, n)$ and $\sigma_{ac}(H, k) = \mathbb{R}$, because k square summable eigenfunctions lose their square summability as $\mathrm{Im}z \to 0_+$. To see this, write the polynomial $\mathcal{F}(t, \lambda, z)$ above in terms of $Q(s, z)$ (3.11) and note that the sign of $\frac{d\mu_j}{dz}$, $j = 1, \ldots, 2k$, are evenly distributed since $Q(s, z)$ is an even polynomial. This leads to a limit point situation whenever $w(t)$ is not summable. For the spectral result, since $w(t)$ vanishes as $t \to \infty$, the absolutely continuous spectrum is determined by evaluating the range of z values for which the eigenvalues μ_j, the roots of $Q(s, z)$, has $| \mu_j | = 1$. Thus, the absolutely continuous spectrum covers the whole of $\mathbb{R}$ and this can easily be verified for even operators of lower order. Besides, k eigenfunctions lose their square summability as $\mathrm{Im}z \to 0^+$. If $w(t)$ is summable then $\mathrm{def}L = (n + k, n + k)$ and $\sigma(H)$ is discrete.

Chapter 4

SPECTRAL THEORY OF FOURTH ORDER DIFFERENCE OPERATORS

The spectral analysis of difference operators generated by difference equations discussed in Chapter three has been extended here to fourth order difference operators with unbounded coefficients. In particular, this chapter has deficiency indices and spectral results of fourth order difference operators when certain smoothness and decay conditions are imposed on the coefficients. It can be considered as the extension of some known spectral results of fourth order differential operators to difference operator setting. These results illustrate the problems arising with unbounded coefficients but also extend some results which are known so far only for Jacobi matrices, see [30]. Concretely, the deficiency indices and spectral results of fourth order difference operator defined on $\ell^2(\mathbb{N})$ are obtained through asymptotic summation with the help of Levinson-Benzaid-Lutz theorem, Theorem 1.2.2.

The chapter is divided into three sections: System formulation, Unbounded Coefficients and Examples. In most cases, the analysis follows closely that in Chapter three.

4.1 System Formulation

Consider a fourth order difference operator L defined on $\ell^2(\mathbb{N})$ and generated by a difference equation of the form

$$\Delta^2[\Delta^2 y(t-2)] - \Delta[p(t)\Delta y(t-1)] + (q(t) - zw(t))y(t) = 0. \tag{4.1}$$

The corresponding quasi-differences can be obtained by application of (1.52). By absorbing zw into q, (4.1) can be written into its propagator form which is equivalent to (1.55).

$$\begin{pmatrix} x(t+1) \\ u(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ q & q & 1 & q \\ 0 & p & -1 & 1+p \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \tag{4.2}$$

For application of asymptotic summation, one needs the eigenvalues of the four by four propagator matrix in (4.2). This requires the expansion of $\det(S(t,z) - I_4 \cdot \lambda) = \mathcal{P}(t,\lambda,z)$ where $S(t,z)$ is the first term on the right hand side of (4.2) while I_4 is a four by four identity matrix. Dividing the characteristic polynomial $\mathcal{P}(t,\lambda,z)$ by λ^2, one obtains a Fourier polynomial $\mathcal{F}(t,\lambda,z)$ of order four with the term $q(t)$ independent of λ, that is

$$\begin{aligned} \mathcal{F}(t,\lambda,z) &= \frac{\mathcal{P}(t,\lambda,z)}{\lambda^2} \\ &= (1-\lambda)^2(1-\lambda^{-1})^2 + p(1-\lambda)(1-\lambda^{-1}) + q - zw. \end{aligned} \tag{4.3}$$

The eigenvalues can be determined explicitly, because (4.3) is the analogue of a biquadratic, well known from differential equations. Concretely, one has with $\zeta = \lambda + \lambda^{-1}$

$$\zeta_\pm = \left(2 + \frac{p}{2}\right) \pm \left[\frac{p^2}{4} - (q - z)\right]^{\frac{1}{2}} \tag{4.4}$$

if $w(t)$ is assumed to be 1. So the roots are $\lambda_\pm$, $\lambda_\pm^{-1}$.

4.2 Unbounded Coefficients

4.2.1 $p(t) \nearrow \infty$, $q(t)$ bounded

Assume that as $t \to \infty$

$$p(t) \nearrow \infty, \qquad q(t) \text{ bounded and } w(t) = 1, \tag{4.5}$$

$p(t)$, $q(t)$ are real valued functions and have second order difference. In order to apply asymptotic summation to determine the solutions of (4.2), the following smoothness and decay assumptions are necessary.

$$\frac{\Delta p}{p^{\frac{1}{2}}},\ p^{\frac{1}{2}}\Delta q \in \ell^2, \qquad \frac{\Delta^2 p}{p^{\frac{1}{2}}},\ p^{\frac{1}{2}}\Delta^2 q \in \ell^1, \qquad p(t) = o(t^2). \tag{4.6}$$

These conditions are more strict than their differential equation counterparts.

In order to diagonalise the propagator matrix $S(t,z)$, approximations of its eigenvalues are required, which are simply the roots of the Fourier polynomial (4.3). By absorbing z into q, and using (4.4), one has for the two $\zeta = \lambda + \lambda^{-1}$ values

$$\zeta_{1/2} = \begin{cases} 2 + \frac{q}{p} + \frac{q^2}{p^3} & \operatorname{mod}(p^{-5}) \\ 2 + p - \frac{q}{p} - \frac{q^2}{p^3} & \operatorname{mod}(p^{-5}) \end{cases}. \tag{4.7}$$

From the ζ-values (4.7), the S-eigenvalues can be determined from $\zeta = \lambda + \lambda^{-1}$. Because $p(t)$ is unbounded, the roots λ are calculated modulo p^{-3}. Thus for the small ζ-values, one has

$$\begin{aligned} \lambda^1_{\pm} = 1 \pm \left(\frac{q}{p}\right)^{\frac{1}{2}} + \frac{q}{2p} \pm \frac{1}{8}\left(\frac{q}{p}\right)^{\frac{3}{2}} \\ \pm \frac{q^{\frac{3}{2}}}{2p^{\frac{5}{2}}} \mp \frac{1}{128}\left(\frac{q}{p}\right)^{\frac{5}{2}} \pm \cdots, \quad \operatorname{mod}(p^{-3}) \end{aligned} \tag{4.8}$$

while from the large ζ-values, one has

$$\lambda^2_{\pm} \approx \begin{cases} \zeta - \zeta^{-1} - \zeta^{-3} + \cdots & \operatorname{mod}(\zeta^{-5}) \\ \zeta^{-1} + \zeta^{-3} + \cdots & \operatorname{mod}(\zeta^{-5}) \end{cases}. \tag{4.9}$$

This leads to the following eigenvalues

$$\lambda^2_+ \approx 2 + p - p^{-1}(1+q) + 2p^{-2} + O(p^{-3}), \quad \lambda^2_- \approx p^{-1} - 2p^{-2} + O(p^{-3}).$$

For the eigenvalue λ, the corresponding λ-eigenvector, $v(\lambda)$, is given by

$$\begin{aligned} v(\lambda) = \big(\lambda^{-1},\ (\lambda-1)\lambda^{-2}, \\ p(\lambda-1)\lambda^{-1} - (\lambda-1)^3\lambda^{-2},\ \lambda^{-2}(\lambda-1)^2\big)^{tr}. \end{aligned} \tag{4.10}$$

Thus, the eigenvalues and eigenvectors admit an expansion in terms of $q^{\frac{1}{2}}$ and $p^{-\frac{1}{2}}$. It is well known that the matrix $T(t)$ formed with the eigenvectors as columns will diagonalise S. In order to do this, the eigenvectors (4.10) will be normalised. For the small ζ-values, the third component is normalised to 1 while the large ζ-eigenvectors have their fourth component normalised to 1. Assume $\lambda_{1/2}$ to be the small ζ- eigenvalues while $\lambda_3 \approx p+2$ and its inverse λ_4, with an accuracy of $O(p^{-3})$, one finds with some rather tedious computations

$$T(t) = \begin{bmatrix} \alpha - \frac{1}{2p} & -\alpha - \frac{1}{2p} & \frac{1}{p} & \frac{1}{p} \\ \frac{1}{p} + \frac{q}{2p^2} - b & \frac{1}{p} + \frac{q}{2p^2} + b & \frac{1}{p} - \frac{1}{p^2} & -1 - \frac{1}{p} + \frac{1}{p^2} \\ 1 & 1 & \frac{q}{p} + \frac{q}{p^2} & -\frac{q}{p^2} \\ b - \frac{q}{2p^2} & -b - \frac{q}{2p^2} & 1 & 1 \end{bmatrix},$$

where $\alpha = \frac{1}{(pq)^{\frac{1}{2}}}\left[1 + \frac{q}{8p} + \frac{q}{2p^2} - \frac{q^2}{128p^2}\right]$ and $b = \frac{q^{\frac{1}{2}}}{p^{\frac{3}{2}}} + \frac{q^{\frac{3}{2}}}{8p^{\frac{5}{2}}}$. While

$$T^{-1}(t) = \frac{1}{\Omega}\begin{bmatrix} -\mu_1 & \mu_2 + \frac{q}{2p^2} & \mu_3 - \gamma_1 & \mu_2 + \mu_4 \\ \mu_1 & \mu_2 - \frac{q}{2p^2} & \mu_3 + \gamma_1 & \mu_2 - \mu_4 \\ \frac{2b}{p} & -\gamma_2 & \frac{2\alpha}{p} & -\gamma_2 + \gamma_3 \\ 2b + \frac{2b}{p} & \gamma_2 & \gamma_4 & \gamma_3 + \frac{2\alpha q}{p^2} - \frac{2b}{p} \end{bmatrix},$$

where $\det T = \Omega$ and

$$\begin{aligned} \Omega &= -2\alpha - \frac{4\alpha}{p} + \frac{4\alpha}{p^2} + \frac{2\alpha q}{p^2} + \frac{2b}{p} \\ &\approx -\frac{2}{(pq)^{\frac{1}{2}}} - \frac{q^{\frac{1}{2}}}{4p^{\frac{3}{2}}} - \frac{4}{q^{\frac{1}{2}}p^{\frac{3}{2}}} + \frac{5q^{\frac{1}{2}}}{2p^{\frac{5}{2}}} + \frac{4}{q^{\frac{1}{2}}p^{\frac{5}{2}}} + \frac{q^{\frac{3}{2}}}{64p^{\frac{5}{2}}} \end{aligned}$$

with

$$\mu_1 = 1 + \frac{2}{p} - \frac{(2+q)}{p^2}, \qquad \mu_2 = \frac{\alpha q}{p} + \frac{2\alpha q}{p^2}$$

$$\mu_3 = -\alpha - \frac{2\alpha}{p} + \frac{b}{p} + \frac{2\alpha}{p^2}, \qquad \mu_4 = \frac{q}{2p^2} + \frac{1}{p} + \frac{2}{p^2}$$

$$\gamma_1 = \frac{1}{2p} + \frac{1}{p^2}, \qquad \gamma_2 = 2\alpha - \frac{2b}{p}$$

$$\gamma_3 = -\frac{2\alpha}{p} + \frac{2\alpha}{p^2}, \qquad \gamma_4 = -\frac{2\alpha}{p} - \frac{\alpha q}{p^2} + \frac{b}{p}.$$

Now one can transform (4.2) by

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = T(t)v(t). \tag{4.11}$$

This transformation leads to

$$\begin{aligned} v(t+1) &= T^{-1}(t+1)S(t)T(t)v(t) \\ &= T^{-1}(t+1)(T(t)-T(t+1))\Lambda(t)v(t) + \Lambda(t)v(t) \\ &= (\Lambda + \mathcal{R})(t)v(t). \end{aligned} \tag{4.12}$$

Here, $\Lambda(t) = T^{-1}(t)S(t)T(t) = \mathrm{diag}(\lambda_i(t))$ is the diagonal matrix formed with the eigenvalues of S. Even though T^{-1} is unbounded, the perturbing term $\mathcal{R}$ is $O(p^{-\frac{1}{2}})$.

Lemma 4.2.1. *Assume (4.5) as $t \to \infty$. Moreover, assume the λ-roots of (4.3) to be distinct and given by (4.4). Then, the λ-roots of $\mathcal{P}(t, \lambda, z)$ satisfy the uniform dichotomy condition.*

Proof. Because of Theorem 3.2.2, it suffices to prove uniform dichotomy condition only for the eigenvalues that are obtained from the small ζ-value. The large ζ-value resulting into $\lambda_3 \approx 2 + p$ and $\lambda_4 \approx p^{-1} - 2p^{-2}$ leads to two solutions which are non-square summable and square summable respectively, irrespective of the dichotomy condition. Now assume that $\lambda^1_+ = \lambda_1$ and $\lambda^1_- = \lambda_2$, $q(t)$ bounded and $p(t) \nearrow \infty$ as $t \to \infty$, then the uniform dichotomy condition between these two roots are proved off the real axis. Let $z = z_0 + i\eta$ with $z \in \mathcal{K}_\epsilon(z_0)$ such that $\mathrm{Im} z = \eta$. One can then absorb $\mathrm{Re} z = z_0$ into q and write $q - z = q - i\eta$. Then there are two cases to consider for $q(t)$ namely: $q(t) \geq \epsilon > 0$ and $q(t) \leq -\epsilon < 0$ for $\epsilon > 0$. The proof is now done for one case only since the other case can be shown in a similar fashion. Now expand λ_1 and λ_2 (4.8) in terms of p, q and η, and without loss of generality, this can be done with accuracy of $O(p^{-1})$. If $\eta > 0$, then $\mid \lambda_1 \mid = 1 - \frac{\eta}{(|qp|)^{\frac{1}{2}}}$ and $\mid \lambda_2 \mid = 1 + \frac{\eta}{(|qp|)^{\frac{1}{2}}}$. In this case, one obtains $\frac{|\lambda_1(t,z)|}{|\lambda_2(t,z)|} < 1 - \delta$ and if $\eta < 0$, then $\frac{|\lambda_1(t,z)|}{|\lambda_2(t,z)|} > 1 + \delta$ where $\delta \approx \frac{2|\eta|}{(|qp|)^{\frac{1}{2}}} + O(p^{-1}) > 0$. This is, however, the uniform dichotomy condition required for λ_1 and λ_2 as $t \to \infty$. $\square$

For differential operators, it was shown in [20] that the large eigenvalues contribute $(1, 1)$ to the deficiency index and discrete eigenvalues to the spectrum only, while the small eigenvalues contribute $(1, 1)$ to the deficiency index and $[\overline{q}, \infty)$ to $\sigma_{ac}(H, 1)$, while $(-\infty, \underline{q}) \cap \sigma_{ess}(H) = \emptyset$. Here,

$$\underline{q} = \liminf \ q(t), \qquad \overline{q} = \limsup \ q(t). \tag{4.13}$$

In this case, the situation is not different and one has:

Theorem 4.2.2. *Assume p and q have second order difference with $\Delta^2 p(t), \Delta^2 q(t) \to 0$ as $t \to \infty$. Moreover, assume (4.5) and (4.6), then the difference operator L generated by (4.1) is limit point, $def L = (2, 2)$ and $[\overline{q}, \infty) \subset \sigma_{ac}(H, 1)$ while $\sigma_{ess}(H) \subset [\underline{q}, \infty)$ and $(-\infty, \underline{q}] \cap \sigma_{ess}(H) = \emptyset$.*

Proof. The proof follows the by now well established routine: system formulation, diagonalisations, dichotomy condition and the M-matrix. One thus writes the difference equation (4.1) into its propagator form (4.2). The eigenvalues of the matrix $S(t, z)$ are then approximated from the Fourier polynomial (4.3) via (4.4). The uniform dichotomy condition will then follow from Lemma 4.2.1.

The matrix $T(t)$ formed from the eigenvectors (4.10), is then used to diagonalise $S(t, z)$. With the transformation (4.11), one obtains the system (4.12) where $\mathcal{R} = -T^{-1}(t+1)\Delta T(t)\Lambda(t)$. Even though $T^{-1}(t+1)$ is unbounded, the perturbing term $\mathcal{R}(t)$ is $O(p^{-\frac{1}{2}})$. In fact, all matrix elements outside the $(1, 2)$ block and different from $(1, 3)$ and $(2, 3)$ elements are $O(\frac{\Delta p}{p})$ or $O(\frac{\Delta q}{p})$. Thus, the system (4.12) can be diagonalised again with a matrix of the form $(I + B(t))$, see [14], whose components can be calculated explicitly using (1.67). The resulting system is then in the Levinson-Benzaid-Lutz form

$$w(t + 1) = (\Lambda(t) + \widetilde{\mathcal{R}}(t))w(t).$$

For this, the conditions (4.6) are needed. Then the matrix elements outside the $(1, 2)$-block can be estimated by $\frac{\Delta p}{p^{\frac{3}{2}}}, \frac{\Delta q}{p^{\frac{1}{2}}}$.

Responsible for the rather strict conditions (4.6) are the small ζ eigenvalues, which are too close together. The corresponding eigenvectors are likewise rather close, making T^{-1} unbounded. With higher order smoothness of p and q, these conditions could be weakened. It should be noted that this second diagonalisation gives only summable, that is, neglegible contribution to the diagonal outside the $(1, 2)$ block. Thus $(I + B)$ is essentially the matrix for diagonalising the $(1, 2)$-block. one may base the analysis on the eigenvalues (4.8) and (4.9). The eigenvalue $\lambda_4 \approx p^{-1}$ leads to a z-uniformly square summable solution. The large ζ-eigenvalues contribute $(1,1)$ to the deficiency index and to the discrete spectrum at most, as expected.

In order to analyse the small ζ-eigenvalues, absorb Rez into q and write $q-z$ as $q-i\eta$, $\eta \geq 0$. As is well known from the differential operators, reasonable spectral results can be obtained only outside $[\underline{q}, \overline{q}]$. Here, this means that one has to consider the cases $q(t) \geq \epsilon > 0$ and $q(t) \leq -\epsilon < 0$. The case $q(t) \geq \epsilon > 0$ or Re$z \in (-\infty, \underline{q})$ will in general only lead to uniformly square summable solutions. In order to see this, note that $p(t) = o(t^2)$ implies that $\prod^k \left(1 \pm \left(\frac{q}{p}\right)^{\frac{1}{2}}\right)$ can be estimated against $\prod^k (1 \pm (\frac{c}{t})) \approx k^{\pm c}$ for large c. Thus, def$L = (1,1)$ and $\sigma_{ess}(H) \cap (-\infty, \underline{q}) = \emptyset$.

It remains, therefore, to analyse the case $q \leq -\epsilon < 0$. Expanding $\lambda_\pm = 1 \pm \left(\frac{q-i\eta}{p}\right)^{\frac{1}{2}} + \frac{q-i\eta}{2p} + \cdots$ with respect to η shows that λ_- leads for $\eta > 0$ to a square summable solution which loses its square summability as $\eta \to 0^+$. Alternatively, note that in this case $\mid \zeta \mid < 2$ for $\eta = 0$. In order to determine the M-matrix, one needs the approximate forms of the eigenfunctions. By the LBL- theorem, Theorem 1.2.2, they have the form

$$
\begin{aligned}
u_k(t) &= T(t)(I + B(t))(e_k + r_k(t)) \prod_{s=t_0}^{t-1} \lambda_k(s) \\
&= (v_k(t) + r_k(t)) \prod_{s=t_0}^{t-1} \lambda_k(s). \qquad (4.14)
\end{aligned}
$$

Here, λ_k is the kth eigenvalue of S and v_k is the normalised kth eigenvector. For the small ζ-eigenfunctions, this means for $t \geq t_0$ suitably large

$$
(v_{1/2})_1(t) = p^{-1}(t)(\lambda_{1/2} - 1)^{-1}(1 + r_{1/2}(t)) \prod_{s=t_0}^{t-1} \lambda_{1/2}(s). \qquad (4.15)
$$

Following the approach in [14, Sect. 2], one has to evaluate the square summable solutions v_1 in dependence on $\eta = \text{Im}z > 0$. For this, one needs the expansion of $\lambda_{1/2} = \lambda_\pm$ in terms of p, q and η. One finds

$$\begin{aligned}(4.16)\qquad \lambda_{\pm} &= 1 \pm \left(\frac{q-i\eta}{p}\right)^{\frac{1}{2}} + \frac{(q-i\eta)}{2p} \\ &\pm \frac{1}{8}\left(\frac{q-i\eta}{p}\right)^{\frac{3}{2}} \pm \cdots \qquad \mathrm{mod}(p^{-\frac{5}{2}}) \\ &\approx 1 \mp \frac{i\eta}{(qp)^{\frac{1}{2}}} + \frac{q}{2p} \pm \left(\frac{q}{p}\right)^{\frac{1}{2}} - \frac{i\eta}{2p} + \cdots\end{aligned}$$

Thus $\lambda_+ = \lambda_1$ leads to square summable solutions with $\mid \lambda_+ \mid \approx 1 - \frac{\eta}{(|qp|)^{\frac{1}{2}}}$. In order to evaluate $\prod_{t_0}^{t-1} \lambda_+(s)$, take logarithms and use the Euler summation formula

$$\ln \prod_{t_0}^{t-1} \mid \lambda_+(s) \mid^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{\mid qp(s) \mid^{\frac{1}{2}}} \approx -2\eta \int_{t_0}^{t} \mid qp(s) \mid^{-\frac{1}{2}} ds$$

since

$$\mid p^{-1}(t)(\lambda_1 - 1)^{-1}(t) \mid \approx (pq)^{-\frac{1}{2}} \left(1 + \left(\frac{q}{p}\right)^{\frac{1}{2}}\right)^{-1}(t).$$

Thus for $\eta > 0$, one has modulo a constant factor, which is $1 + o\left(p^{-\frac{1}{2}}(t_0)\right)$,

$$\|v_1(.,\eta)\|^2 \approx \int_{t_0}^{\infty} (pq)^{-\frac{1}{2}}(t)(\exp(-2\eta \int_{t_0}^{t} \mid qp(s) \mid^{-\frac{1}{2}} ds))dt$$

and thus $\lim_{\eta\to 0^+} \|v_1(.,\eta)\|^2$ exists nontrivially and consequently $\mathrm{Im}M(z_+)$ is nontrivial.

□

Remark 4.2.3. Apart from the approximation devices, logarithms and Euler summations, the derivation of the nontriviality of $\mathrm{Im}M(z_+)$ is rather similar to that for differential equations.

Theorem 4.2.2 extends to the case $q(t) \to +\infty$ or $q(t) \to -\infty$ as well provided q does not grow too fast. In this case, one needs

$$(4.17)\qquad \frac{\Delta p}{p^{\frac{1}{2}}}.q^{\frac{1}{2}},\ p^{\frac{1}{2}}\frac{\Delta q}{q^{\frac{1}{2}}} \in \ell^2, \qquad \frac{\Delta^2 p}{p^{\frac{1}{2}}}.q^{\frac{1}{2}},\ p^{\frac{1}{2}}\frac{\Delta^2 q}{q^{\frac{1}{2}}} \in \ell^1.$$

Theorem 4.2.4. *Assume p and q have second order difference with $\Delta^2 p(t), \Delta^2 q(t) \to 0$ as $t \to \infty$. Moreover, assume (4.17) with $q(t) \to \infty$, then it implies that $\underline{q} = \infty$ so that $\sigma(H)$ is discrete as expected. For $q(t) \to -\infty$ we have correspondingly $\overline{q} = -\infty$ and $\sigma_{ac}(H) = \mathbb{R}$ provided q does not grow too fast.*

4.2.2 $p(t) \to -\infty$, $q(t)$ bounded

The analysis of this case proceeds exactly as the case $p(t) \to \infty$. Therefore, only the main results are stated.

Theorem 4.2.5. *Assume that p and q have second order difference with $\Delta^2 p(t), \Delta^2 q(t) \to 0$ as $t \to \infty$. Moreover, assume (4.6) with $p(t) < 0$ and $p(t) \to -\infty$ as $t \to \infty$, then the difference operator L generated by (4.1) is limit point, $def L = (2,2)$ and $(-\infty, \underline{q}] \subset \sigma_{ac}(H,1)$ while $\sigma_{ess}(H) \subset (-\infty, \overline{q}]$ and $[\overline{q}, \infty) \cap \sigma_{ess}(H) = \emptyset$.*

An analogue of Theorem 4.2.4 as $\mid q(t) \mid \to \infty$ slowly can be obtained easily.

4.2.3 $\mid q(t) \mid \to \infty$

If q is dominant, the spectrum tends to be discrete as the next result shows.

Theorem 4.2.6. *Let $a_n, d_n \in \mathbb{R}$ and $b_n, c_n \in \mathbb{C}$ satisfying $a_n > d_n > 0$, $d_n \nearrow \infty$*

$$\begin{aligned} &\mid b_n \mid (a_n - d_n)^{-\frac{1}{2}} (a_{n+1} - d_{n+1})^{-\frac{1}{2}} \\ &+ \mid c_n \mid (a_n - d_n)^{-\frac{1}{2}} (a_{n+2} - d_{n+2})^{-\frac{1}{2}} \leq \frac{1}{2} \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\mid b_n \mid (a_n - d_n)^{-\frac{1}{2}} (a_{n+1} - d_{n+1})^{-\frac{1}{2}} \\ &+ \mid c_{n-1} \mid (a_{n-1} - d_{n-1})^{-\frac{1}{2}} (a_{n+1} - d_{n+1})^{-\frac{1}{2}} \leq \frac{1}{2}. \end{aligned} \tag{4.19}$$

Then the matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & \cdots & \cdots \\ \overline{b_1} & a_2 & b_2 & c_2 & \cdots \\ \overline{c_1} & \overline{b_2} & a_3 & b_3 & \ddots \\ \vdots & \overline{c_2} & \overline{b_3} & a_4 & \ddots \\ \vdots & \vdots & \ddots & \cdots & \ddots \end{pmatrix}$$

defines an operator with a compact resolvent, that is, an operator with a discrete spectrum.

Proof. We will show $A \geq D = \mathrm{diag}(d_i)$. Since $D > 0$ with $d_i \nearrow \infty$, D^{-1} is compact. However, $D^{-1} \geq A^{-1}$ shows that A has compact resolvent likewise. Now $A-D \geq 0$ is equivalent to $Q^{-\frac{1}{2}}(A-D)Q^{-\frac{1}{2}} = I+C+C^* \geq 0$ where $Q = \text{diag}(a_i - d_i)$ and where the non-zero matrix elements of C are given by $C_{i,i+1} = b_i.(a_i - d_i)^{-\frac{1}{2}}(a_{i+1} - d_{i+1})^{-\frac{1}{2}}$ and $C_{i,i+2} = c_i.(a_i - d_i)^{-\frac{1}{2}}(a_{i+2} - d_{i+2})^{-\frac{1}{2}}$. Applying the Schur test [[51], Theorem. 4.1.2] with $p = r = 2$, $\phi_i = 1$, $\mu_i = 1$, $i = 1, 2, \ldots$ shows that $\mid C_{i,i+1} \mid + \mid C_{i,i+2} \mid \leq M_1^2$, $\mid C_{j-1,j} \mid + \mid C_{j-2,j} \mid \leq M_2^2$ and which implies that $\|C\| \leq M_1 M_2$, $i, j = 1, 2, \ldots$. □

Remark 4.2.7. The theorem will also be valid if the conditions (4.18) and (4.19) hold only for $n \geq n_0$ for some n_0. The result can easily be extended to finite band matrices. For three band matrices (4.18) suffices and gives a condition, which is rather similar to that in [30].

Remark 4.2.8. The conditions of the proof need only hold for $i, j \geq n_0$. For a_n unbounded and b_n, c_n bounded, one may also use a Simon-Spencer type of argument, see [57].

Remark 4.2.9. As in the case of differential operators, one can weaken the decay conditions if higher order smoothness is required for the coefficients.

4.3 Examples

The following example shows the location of absolutely continuous spectrum when the term $p(t) = 0$ and $q(t)$ is bounded or unbounded.

Example 4.3.1. Consider a fourth order difference operator L generated by

$$\Delta^4 y(t-2) + t^{-\alpha} y(t) = zw(t),$$

where z is a spectral parameter and defined on $\ell^2_w(\mathbb{N})$. Moreover, assume $w(t) = 1$ for all $t \in \mathbb{N}$. Thus the Fourier polynomial $\mathcal{F}(t, \zeta, z)$, where $\zeta = \lambda + \lambda^{-1}$ is given by

$$(2-\zeta)^2 + t^{-\alpha} - z = 0.$$

If $\zeta_- = 2 - (z - t^{-\alpha})^{\frac{1}{2}}$ and $\mid \zeta_- \mid \leq 2$, then $t^{-\alpha} \leq z \leq 16 + t^{-\alpha}$. Now assume that $\alpha > 0$, then $t^{-\alpha} \to 0$ as $t \to \infty$. Thus $\sigma_{ac}(H, 1) \subset [0, 16]$. On the other hand, if $\alpha < 0$, then $t^{-\alpha} \to \infty$ as $t \to \infty$ and the spectrum is discrete.

The next example illutrates the results of Theorems 4.2.2 and 4.2.5.

Example 4.3.2. Let

$$\Delta^4 y(t-2) - \Delta[bt^\alpha \Delta y(t-1)] + ay(t) = zw(t)$$

be a fourth order difference equation with a, b and α as finite constants, and assume that L is the fourth order difference operator generated by this equation on $\ell^2_w(\mathbb{N})$. Moreover, assume that $w(t) = 1$. Thus the corresponding Fourier polynomial $\mathcal{F}(t, \zeta, z)$ is given by

$$(2-\zeta)^2 + bt^\alpha(2-\zeta) + a - z = 0.$$

Assume $\alpha < 0$, then all the coefficients are uniformly bounded, and bt^α tends to zero as $t \to \infty$. As $t \to \infty$, it follows that $\sigma_{ac}(H, 1) = [a, a+16]$.

Now assume that $b > 0$ and $\alpha > 0$, then as $t \to \infty$, $bt^\alpha \to \infty$ and the λ-roots of the polynomial can only be approximated using (4.7). The large ζ-value contributes $(1, 1)$ to the deficiency index and at most discrete spectrum only while the small ζ-value can be analysed as follows. Assume $\text{Re}z < a$, then by (4.8), the small ζ-value leads to deficiency index $(1, 1)$ and only discrete spectrum can be obtained. On the other hand, if $\text{Re}z \geq a$, then the small ζ-value leads to two roots whose solutions are square summable though one of the solutions loses its square summability as $\text{Im}z \searrow 0$. It is this solution which loses its square summability that contributes to absolutely continuous spectrum. One has $[a, \infty) \subset \sigma_{ac}(H, 1)$ while $\sigma_{ess}(H) \subset [a, \infty)$ and $\sigma_{ess}(H) \cap (-\infty, a) = \emptyset$.

Similarly, if one assumes that $b < 0$ and $\alpha > 0$, then absolutely continuous spectrum can only be obtained for $\text{Re}z \leq a$ while $\text{Re}z > a$ will only lead to discrete spectrum with no accumulation of eigenvalues. Thus $(-\infty, a] \subset \sigma_{ac}(H, 1)$ and $\sigma_{ess}(H) \cap (a, \infty) = \emptyset$.

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