A category of pseudo-tangles with classifying space $\Omega^\infty S^\infty$ and applications

Dissertation

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von

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Contents

Introduction

For this paper a monoidal category means a strict monoidal category in the usual sense, i.e. it consists of a category $\mathcal{C}$, a functor $\oplus : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (called tensor product) and an object $0 \in \mathcal{C}$ (called unit) such that $\oplus$ is strictly associative and unital on objects and morphisms.

A monoidal category is called permutative if there exists a family of natural isomorphisms

$$c_{V,W} : V \oplus W \longrightarrow W \oplus V$$

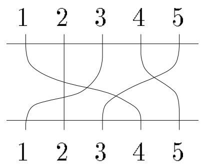
(indexed by all pairs of objects (V, W) in $\mathcal{C}$) which again satisfy certain relations and in particular $c_{W,V} \circ c_{V,W} = id$.

The forgetful functor from the category of permutative categories to the category of all categories has a left adjoint. The value of this left adjoint on the trivial category (i.e. one object, one morphism) is $\mathcal{F}_P$, the "free permutative category generated by one object". It has the following description:

$$\begin{aligned} ob\, \mathcal{F}_P &= \mathbb{N} \\ mor_{\mathcal{F}_P}(m,n) &= \begin{cases} \emptyset & \text{for} \quad m \neq n \\ \Sigma_n & \text{for} \quad m = n \end{cases} \end{aligned}$$

where Σ_n denotes the permutation group of the set $\{1, \ldots, n\}$. Every concrete permutation σ can be pictured by tracing the n elements:

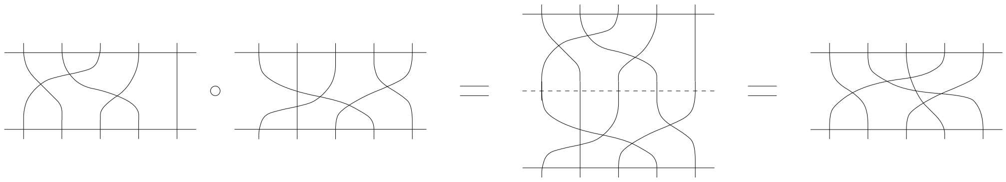
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} \quad \hat{=}$$



(By convention the direction of the tracks is from the bottom to the top.) This looks like a braid, but it does not matter if one string crosses over or underneath another one. Hence it is called a "pseudo-braid".

In this pictorial interpretation, the composition of two permutations can be computed by stacking the pictures onto each other.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 3 & 2 \end{pmatrix}$$



Further, the tensor product of $\mathcal{F}_P$ is given by juxtaposition.

The classifying space functor B assigns to each small category $\mathcal{C}$ (i.e. the objects form a set) a topological space $B\mathcal{C}$, which mirrors the algebraic properties of $\mathcal{C}$.
For instance, it is obvious that $B\mathcal{C}$ of a monoidal category is a topological monoid, and it is commutative up to homotopy if $\mathcal{C}$ is permutative. It has been known for some time that the group completion of $B\mathcal{C}$ of any permutative category is an infinite loop space [9]. The algebraic structure on $\mathcal{C}$ which leads to an exactly n-fold loop space structure on the group completion of $B\mathcal{C}$ has recently been discovered [1].

If a monoidal category $\mathcal{S}$ satisfies certain conditions, one obtains a new category $\mathcal{S}^{-1}\mathcal{S}$ and a canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ which yields a group completion $B\mathcal{S} \longrightarrow B(\mathcal{S}^{-1}\mathcal{S})$ on the level of classifying spaces by Quillen's "$S^{-1}S$ construction" [3].
In this paper it is shown that in the special case $\mathcal{S} = \mathcal{F}_P$ the $S^{-1}S$ construction has a pictorial interpretation. This means there is a pictorially described category $\mathcal{G}$ which is equivalent to $\mathcal{F}_P^{-1}\mathcal{F}_P$. This is of interest, since it is well known that the group completion of the classifying space $B\mathcal{F}_P$ is $\Omega^\infty S^\infty$, i.e. stable homotopy of the 0-sphere [10]. Hence the category $\mathcal{G}$ is a combinatorial model for this space, and it is possible to give combinatorial proofs of $\pi_0(\Omega^\infty S^\infty) = \mathbb{Z}$ and $\pi_1(\Omega^\infty S^\infty) = \mathbb{Z}/2$. In addition, it can be shown that $\pi_2(\Omega^\infty S^\infty)$ has at least a non-trivial element of order 2.

Little is known about the effect of a duality structure to classifying spaces. A duality structure in a monoidal category $\mathcal{C}$ assigns to each object V another object V^* and two morphisms

$$b_V : 0 \longrightarrow V \oplus V^*, \qquad d_V : V^* \oplus V \longrightarrow 0$$

with $\quad (id \oplus d)(b \oplus id) = id_V \quad$ and $\quad (d \oplus id)(id \oplus b) = id_{V^*} \quad$ for all objects V.

The classifying space of such a category is a grouplike monoid (i.e. $\pi_0(B\mathcal{C})$ is a group under the induced multiplication) since V^* defines the inverse component of the component of V, but there seems to be no further information in the literature so far.
In this paper the "free permutative category with duality generated by one object" is investigated. Again there is a pair of adjoint functors (forgetful and free, respectively) between the category of permutative categories with duality and the category of all categories. The value of the free functor on the trivial category is a category denoted by $\mathcal{F}_{P,D}$; it has again a pictorial description.
$\mathcal{F}_P$ is a subcategory of $\mathcal{F}_{P,D}$, and $\pi_0 B\mathcal{F}_{P,D}$ is the algebraic group completion of the monoid $\pi_0 B\mathcal{F}_P$. Hence it is a natural question (which was suggested to me by Zig Fiedorowicz) whether the process of adding a free duality structure to $\mathcal{F}_P$ gives a topological group completion on classifying spaces.

Unfortunately, this question has a negative answer as will be shown in this paper in the following way:
$\mathcal{G}$ can be interpreted as a subcategory of $\mathcal{F}_{P,D}$ and the inclusion $\mathcal{F}_P \hookrightarrow \mathcal{F}_{P,D}$ factors as $\mathcal{F}_P \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{F}_{P,D}$. If $B\mathcal{F}_{P,D}$ was the group completion of $B\mathcal{F}_P$, then the map $B\mathcal{G} \longrightarrow B\mathcal{F}_{P,D}$ would have to be a homotopy equivalence. There is a well-known theorem to investigate the homotopy fibre of such a map, Quillen's Theorem B. This theorem can be applied to the inclusion functor $\mathcal{G} \hookrightarrow \mathcal{F}_{P,D}$ and yields a categorical model for the homotopy fibre, which turns out to be exactly simply-connected. An obstruction for higher connectivity is caused by the non-trivial element of $\pi_2(B\mathcal{G})$ which can be lifted into the homotopy fibre. In any case we obtain:

Theorem: The canonical map $\quad \Omega^\infty S^\infty \longrightarrow B\mathcal{F}_{P,D} \quad$ is 1-connected but not 2-connected .—

This leads to the homotopy groups $\pi_i(B\mathcal{F}_{P,D})$ for $i = 0, 1, 2$, and it is possible to give generators for these groups.

1 Basic definitions and short reviews

1.1 Definition: A (strict) monoidal category is a category $\mathcal{C}$ equipped with a functor $\oplus : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (called tensor product) and an object $0 \in ob\,\mathcal{C}$ (called unit) such that

$$(U \oplus V) \oplus W = U \oplus (V \oplus W),$$

$$0 \oplus V \;=\; V \;=\; V \oplus 0$$

for all objects (U, V, W) in $\mathcal{C}$.—

1.2 Remark: The usual definition for a monoidal category does not require the tensor product to be strictly associative but only associative up to a family of natural isomorphisms $a_{U,V,W} : (U \oplus V) \oplus W \longrightarrow U \oplus (V \oplus W)$ which are to satisfy a coherence condition (the pentagon rule). Also the unit is often defined to be neutral only up to natural isomorphisms with appropriate coherence conditions. Since this paper only deals with strict structures, we have omitted these more general definitions and the term "strict". Interested readers are referred to [8] .—

1.3 Definition: (a) Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, φ_0 is an isomorphism from 0 to $F(0)$ and

$$\varphi_2(V, W) : F(V) \oplus F(W) \longrightarrow F(V \oplus W)$$

is a family of natural isomorphisms indexed by all pairs (V, W) of objects of $\mathcal{C}$ such that the diagrams below commute for all objects (U, V, W) in $\mathcal{C}$:

$$\begin{tikzcd}
F(U) \oplus F(V) \oplus F(W) \arrow[r, "id_{F(U)} \oplus \varphi_2(V,W)"] \arrow[d, "\varphi_2(U,V) \oplus id_{F(W)}"'] & F(U) \oplus F(V \oplus W) \arrow[d, "\varphi_2(U,V \oplus W)"] \\
F(U \oplus V) \oplus F(W) \arrow[r, "\varphi_2(U \oplus V,W)"] & F(U \oplus V \oplus W)
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(V) \arrow[dl, "\varphi_0 \oplus id_{F(V)}"'] \arrow[dr, "id_{F(V)} \oplus \varphi_0"] \arrow[dd, equal] & \\
F(0) \oplus F(V) \arrow[dr, "\varphi_2(0,V)"'] & & F(V) \oplus F(0) \arrow[dl, "\varphi_2(V,0)"] \\
 & F(V) &
\end{tikzcd}$$

A monoidal functor is called strict if φ_0 and all φ_2 are identities.

(b) A natural tensor transformation $\eta : (F, \varphi_0, \varphi_2) \longrightarrow (F', \varphi_0', \varphi_2')$ between monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \longrightarrow F'$ such that the following diagrams commute for each pair (V, W) of objects in $\mathcal{C}$:

$$\begin{tikzcd} & F(0) \arrow[dd, "\eta(0)"] \\ 0 \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & \\ & F'(0) \end{tikzcd} \qquad \begin{tikzcd} F(V) \oplus F(W) \arrow[r, "\varphi_2(V,W)"] \arrow[d, "\eta(V)\oplus\eta(W)"] & F(V \oplus W) \arrow[d, "\eta(V\oplus W)"] \\ F'(V) \oplus F'(W) \arrow[r, "\varphi_2'(V,W)"] & F'(V \oplus W) \end{tikzcd}$$

A natural tensor isomorphism is a natural tensor transformation that is also a natural isomorphism.

(c) A tensor equivalence between monoidal categories is a monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that there exists a monoidal functor $F' : \mathcal{D} \longrightarrow \mathcal{C}$ and natural tensor isomorphisms $\eta : id_{\mathcal{D}} \longrightarrow F \circ F'$ and $\eta' : F' \circ F \longrightarrow id_{\mathcal{C}}$.—

1.4 Definition: A permutative category is a monoidal category $\mathcal{C}$ equipped with a family of natural isomorphisms $c_{V,W} : V \oplus W \longrightarrow W \oplus V$ indexed by all pairs (V, W) of objects of $\mathcal{C}$ with $c_{W,V} \circ c_{V,W} = id_{V \oplus W}$ for all (V, W) and such that the following diagrams commute:

$$\begin{tikzcd} U \oplus V \oplus W \arrow[rr, "c_{U \oplus V, W}"] \arrow[dr, "id_U \oplus c_{V,W}"'] & & W \oplus U \oplus V \\ & U \oplus W \oplus V \arrow[ur, "c_{U,W} \oplus id_V"'] & \end{tikzcd}$$

$$\begin{tikzcd} U \oplus V \oplus W \arrow[rr, "c_{U, V \oplus W}"] \arrow[dr, "c_{U,V} \oplus id_W"'] & & V \oplus W \oplus U \\ & V \oplus U \oplus W \arrow[ur, "id_V \oplus c_{U,W}"'] & \end{tikzcd}$$

.—

1.5 Definition: A monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between permutative categories $\mathcal{C}$ and $\mathcal{D}$ is said to be permutative when the following diagram commutes for all pairs (V, W) of $\mathcal{C}$:

$$\begin{tikzcd} F(V) \oplus F(W) \arrow[r, "\varphi_2(V,W)"] \arrow[d, "c_{F(V),F(W)}"'] & F(V \oplus W) \arrow[d, "F(c_{V,W})"] \\ F(W) \oplus F(V) \arrow[r, "\varphi_2(W,V)"] & F(W \oplus V) \end{tikzcd}$$

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1.6 Review of classifying spaces: Let $\mathcal{C}$ be a small category (all occurring categories in this paper are "small", i.e. the class of objects forms a set, hence this term is always omitted in the sequel). Its nerve is the simplicial set $N\mathcal{C}$ whose n–simplices are diagrams in $\mathcal{C}$ of the form $C_0 \overset{\alpha_1}{\to} C_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_n}{\to} C_n$. The i-th face d_i (resp. degeneracy s_i) of this simplex is obtained by deleting the object C_i (resp. replacing C_i by $C_i \xrightarrow{id} C_i$) in the evident way.

The classifying space $B\mathcal{C}$ of $\mathcal{C}$ is the topological realization of its nerve $N\mathcal{C}$. It is a CW complex whose n-cells are in one-one correspondence with the n-simplices of the nerve which are not degenerate, i.e. such that none of the arrows is an identity map. By the homotopy type of a category $\mathcal{C}$, we mean the homotopy type of its classifying space.

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, it induces a cellular map $BF : B\mathcal{C} \longrightarrow B\mathcal{D}$. In this way we obtain a functor B from the category of small categories to the category of CW complexes and cellular maps.

By the compatibility of the geometric realization with products [7], one knows that the canonical map $B(\mathcal{C} \times \mathcal{D}) \longrightarrow B\mathcal{C} \times B\mathcal{D}$ is a homeomorphism if either $B\mathcal{C}$ or $B\mathcal{D}$ is a finite complex. A natural transformation $\eta : F \longrightarrow G$ of functors from $\mathcal{C}$ to $\mathcal{D}$ can be viewed as a functor $\mathcal{C} \times \mathcal{L}_1 \longrightarrow \mathcal{D}$, where $\mathcal{L}_1$ is the category which consists of two objects and only one non-trivial morphism between them, and $B\mathcal{L}_1$ is the unit interval I. Hence a natural transformation induces a homotopy $B\mathcal{C} \times I \longrightarrow B\mathcal{D}$ between BF and BG. In particular, if a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has either a left or a right adjoint, then $BF : B\mathcal{C} \longrightarrow B\mathcal{D}$ is a homotopy equivalence. So for instance any category with either an initial or a final object is contractible.

The homology groups $H_n(B\mathcal{C})$ are obtained as the homology groups of the chain complex

$$\cdots \xrightarrow{\partial} K'_{n+1} \xrightarrow{\partial} K'_n \xrightarrow{\partial} K'_{n-1} \xrightarrow{\partial} \cdots$$

where K'_n is the free abelian group on n-simplices in $N\mathcal{C}$ and $\partial : K'_n \longrightarrow K'_{n-1}$ is determined by $\partial(\alpha_n, \ldots, \alpha_1) := \sum_{i=0}^{n} d_i(\alpha_n, \ldots, \alpha_1)$.

For computations it is easier to work with the normalized chain complex

$$\cdots \xrightarrow{\partial} K_{n+1} \xrightarrow{\partial} K_n \xrightarrow{\partial} K_{n-1} \xrightarrow{\partial} \cdots$$

where K_n is the quotient of K'_n by the subgroup which is generated by the degenerated simplices. It has the same homology groups.

The homotopy sets $\pi_0(B\mathcal{C})$ and $\pi_1(B\mathcal{C})$ can also be computed directly from the category as follows:

$B\mathcal{C}$ is a CW complex whose vertices correspond to the objects and whose 1-simplexes correspond to the non-identity morphisms of $\mathcal{C}$. Every path in a CW complex which starts and ends at vertices is homotopic to a cellular map and can be represented within its homotopy class as a combination of 1-cells. Hence, up to homotopy, any path in $B\mathcal{C}$ which starts and ends at vertices C_0 and C_n, respectively, may be represented by a finite combination of morphisms in $\mathcal{C}$

$$\omega: \quad C_0 \overset{\alpha_1}{\text{——}} C_1 \overset{\alpha_2}{\text{——}} C_2 \overset{\alpha_3}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} C_{n-1} \overset{\alpha_n}{\text{——}} C_n$$

where α_i is any morphism from C_{i-1} to C_i or vice versa. This determines $\pi_0(B\mathcal{C})$.
The fundamental group $\pi_1(B\mathcal{C}, C)$ is given by paths which start and end at some fixed basepoint object C of $\mathcal{C}$. The 2-cells in a classifying space are given by composable pairs of morphisms $A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} A_2$. Such a pair defines a triangle with edges β_1, β_2 and $\beta_2 \circ \beta_1$. Hence there is a homotopy between ω and another path ω' if and only if the diagram

$$\begin{tikzcd}
C \arrow[r, dashed, dash, "\omega"] \arrow[d, equal] & C \arrow[d, equal] \\
C \arrow[r, dashed, dash, "\omega'"] & C
\end{tikzcd}$$

can be filled by additional objects and morphisms to a diagram which consists only of commutative triangles .— (See [13] and [7]).

1.7 Remark: The classifying space of a monoidal category is a topological monoid since the tensor functor induces a multiplication which is associative and unital.
If the category is even permutative, the isomorphisms $c_{V,W}$ describe a natural transformation from the functor $\oplus$ to the functor $\oplus \circ \tau$, where $\tau : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ is the switch functor $(V, W) \longmapsto (W, V)$. Hence the classifying space of a permutative category is a homotopy commutative monoid.—

1.8 Definition: An H-space X is called admissible if it is homotopy associative and if left translation by any given element is homotopic to right translation by the same element. X is called grouplike if $\pi_0 X$ is a group under the induced multiplication .—

Obviously the classifying space of a permutative category is admissible.

1.9 Definition: Let $f : X \longrightarrow Y$ be an H-map between admissible H-spaces, where Y is grouplike. The map f is called a group completion if $f_* : H_*(X; R) \longrightarrow H_*(Y; R)$ is a localization of the ring $H_*(X; R)$ at its multiplicative submonoid $\pi_0 X$ for every commutative coefficient ring R .—

1.10 Theorem: Let $\mathcal{C}$ and $\mathcal{D}$ be permutative categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a permutative functor. Then the map $BF : B\mathcal{C} \longrightarrow B\mathcal{D}$ induces an infinite loop map between the group completions of $B\mathcal{C}$ and $B\mathcal{D}$. In particular, if both $B\mathcal{C}$ and $B\mathcal{D}$ are grouplike, then BF itself is an infinite loop map .— (See [9]).

1.11 Review of the $S^{-1}S$ construction: Suppose S is an abelian monoid which acts on a set X. The monoid S is said to act invertibly on X if for each $s \in S$ the translation $X \longrightarrow X, \quad x \longmapsto s \cdot x$ is a bijection. Define $S^{-1}X$ to be the quotient $S \times X/S$, where S acts on each factor of the product. Let S act on $S^{-1}X$ by $t \cdot (s, x) := (s, t \cdot x)$, and define $X \longrightarrow S^{-1}X$ to be $x \longmapsto (0, x)$. Then the translation $(s, x) \longmapsto (s, t \cdot x)$ has an inverse given by $(s, x) \longmapsto (t + s, x)$, so S acts invertibly on $S^{-1}X$. The arrow $X \longrightarrow S^{-1}X$ respects the S-action and is a universal arrow from X to a set upon which S acts invertibly.

Now let $\mathcal{S}$ be a monoidal category. Define a category $\mathcal{S}^{-1}\mathcal{S}$ in the following way: $\mathcal{S}^{-1}\mathcal{S}$ has the same objects as $\mathcal{S} \times \mathcal{S}$, and a morphism from (V_-, V_+) to (W_-, W_+) is an isomorphism class of triples

$$(K, \quad \alpha_- : K \oplus V_- \longrightarrow W_-, \quad \alpha_+ : K \oplus V_+ \longrightarrow W_+)$$

where K is an object and α_-, α_+ are morphisms of $\mathcal{S}$. An isomorphism between triples (K, α_-, α_+) and $(K', \alpha'_-, \alpha'_+)$ is an isomorphism $K \xrightarrow{\cong} K'$ in $\mathcal{S}$ which makes the following diagrams commute:

$$\begin{tikzcd} K \oplus V_- \arrow[rr, "\cong"] \arrow[dr, "\alpha_-"'] & & K' \oplus V_- \arrow[dl, "\alpha'_-"] \\ & W_- & \end{tikzcd} \qquad \begin{tikzcd} K \oplus V_+ \arrow[rr, "\cong"] \arrow[dr, "\alpha_+"'] & & K' \oplus V_+ \arrow[dl, "\alpha'_+"] \\ & W_+ & \end{tikzcd}$$

The isomorphism class of a triple (K, α_-, α_+) is denoted by $[K, \alpha_-, \alpha_+]$.

If F and G are morphisms, represented by triples (K, α_-, α_+) and (L, β_-, β_+), respectively, then the composition $G \circ F$ is represented by

$$(L \oplus K, \beta_- \circ (id_L \oplus \alpha_-), \beta_+ \circ (id_L \oplus \alpha_+)).$$

$\mathcal{S}^{-1}\mathcal{S}$ has a canonical monoidal structure: On objects the tensor product is given by $(V_-, V_+) \oplus (W_-, W_+) := (V_- \oplus W_-, V_+ \oplus W_+)$, and for morphisms $F = [K, \alpha_-, \alpha_+],\ G = [L, \beta_-, \beta_+]$ we have

$$F \oplus G = [K \oplus L,\ \ (\alpha_- \oplus \beta_-) \circ (id \oplus \tau \oplus id),\ \ (\alpha_+ \oplus \beta_+) \circ (id \oplus \tau \oplus id)],$$

where τ denotes the switch map. The canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$, given by $V \longmapsto (0, V)$, strictly preserves the monoidal structure.

If $\mathcal{S}$ is a permutative category, then $\mathcal{S}^{-1}\mathcal{S}$ is also permutative by

$$c_{(V_-,V_+),(W_-,W_+)} = [0, c_{V_-,W_-}, c_{V_+,W_+}].$$

In this situation the canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ is permutative, and there is the following theorem:

Theorem: Suppose $(\mathcal{S}, \oplus)$ is a permutative category which satisfies the following conditions:

1. All morphisms of $\mathcal{S}$ are isomorphisms.
2. All translations $\mathcal{S} \longrightarrow \mathcal{S},\ V \longmapsto W \oplus V$ are faithful.

Then the canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ induces a group completion on the level of classifying spaces.

A more precise formulation of the last statement is as follows:
The monoid $S = \pi_0(B\mathcal{S})$ acts on $H_p(B\mathcal{S})$, hence there is a universal arrow $H_p(B\mathcal{S}) \longrightarrow (\pi_0(B\mathcal{S}))^{-1}H_p(B\mathcal{S})$. Further, S acts invertibly on $H_p(B\mathcal{S}^{-1}\mathcal{S})$, and the canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ gives a map $H_p(B\mathcal{S}) \longrightarrow H_p(B\mathcal{S}^{-1}\mathcal{S})$ which respects the S-action. So we get an induced map

$$(\pi_0(B\mathcal{S}))^{-1}H_p(B\mathcal{S}) \longrightarrow H_p(B\mathcal{S}^{-1}\mathcal{S}).$$

This map is an isomorphism for all p .— (See [3]).

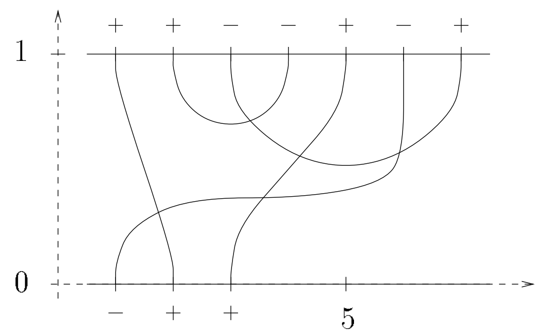
2 The category $\mathcal{G}$

2.1 The construction of $\mathcal{G}$

2.1 Construction: Let $\mathcal{G}$ be the following category: The set of objects is the free monoid on the alphabet $\{+,-\}$. We think of an object w with k letters as the subset $\{1,\ldots,k\} \subset \mathbb{R}$ with an assigned $+$ or $-$ to each i.
A morphism of a word w_1 with k_1 letters to another word w_2 with k_2 letters is represented by an unordered set $\{f_\lambda\}_{\lambda\in\Lambda}$ of $\frac{1}{2}(k_1+k_2)$ paths $f_\lambda : [0,1] \longrightarrow \mathbb{R}\times[0,1]$ subject to the following conditions:

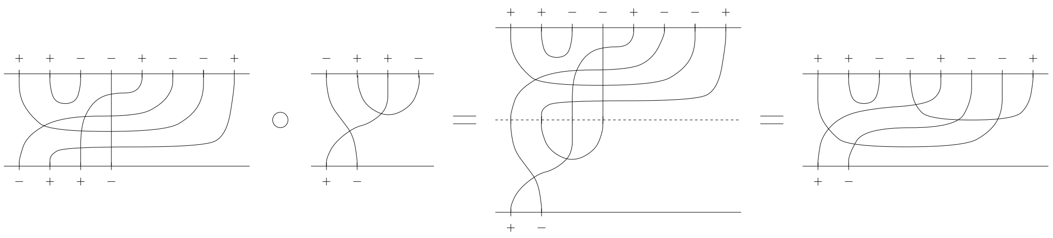
1. Each f_λ connects two different points of the subset $\{1,\ldots,k_1\}\times\{0\} \cup \{1,\ldots,k_2\}\times\{1\}$.
2. Each point of $\{1,\ldots,k_1\}\times\{0\}\cup\{1,\ldots,k_2\}\times\{1\}$ is in the image of exactly one f_λ.
3. Each f_λ connects either an element from $\{1,\ldots,k_1\}\times\{0\}$ with an element from $\{1,\ldots,k_2\}\times\{1\}$ having the same sign (called "vertical strings") or two elements from $\{1,\ldots,k_2\}\times\{1\}$ having different signs (called "arcs").

The term "string" is used for both kinds of paths.
Two such systems $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{f'_\lambda\}_{\lambda\in\Lambda}$ represent the same morphism if there is a family of homotopies $\{H_\lambda : f_\lambda \simeq f'_\lambda \;\text{ or }\; -f_\lambda \simeq f'_\lambda\}_{\lambda\in\Lambda}$ which keep boundary points fixed. ($-f_\lambda$ denotes the reversed parametrization of f_λ.)
Morphisms can be illustrated by pictures like this:

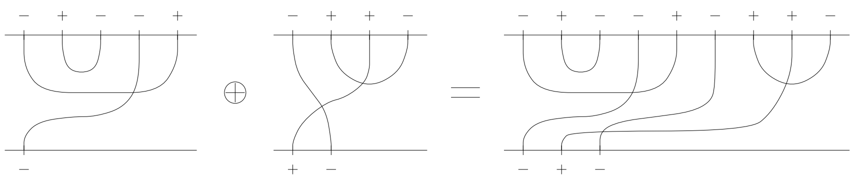


We will call such a picture a "pseudo-tangle" because it looks like a tangle with unconsidered crossings.

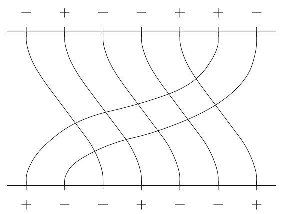
To find the picture for the composition of two pseudo-tangles $w_1 \longrightarrow w_2$ and $w_2 \longrightarrow w_3$, one stacks the latter picture onto the first one, compresses the second coordinate into the unit interval and forgets the intermediate state:



$\mathcal{G}$ has a natural monoidal structure: On objects the tensor product is given by juxtaposition of words (with the empty word 0 as a unit) and on morphisms by putting the appropriate pseudo-tangles side by side and making the necessary adjustments to the endpoints:



Further, $\mathcal{G}$ has a natural permutative structure, hence its classifying space is an infinite loop space and admissible. The symmetry isomorphisms $c_{v,w} : v \oplus w \longrightarrow w \oplus v$ are given by pictures of the following pattern:

$c_{+-,-+--+} \quad =$  .—

The object $+ + \ldots + +$ of $\mathcal{G}$ which consists of k letters "+" is sometimes simply denoted by k, and $-k = - - \ldots - -$ denotes the word of k letters "−". Using this notation we have, for example, $2 \oplus -1 \oplus 1 \oplus -3 = + + - + - - -$.

2.2 Remark: There are two canonical embeddings $i_+, i_- : \mathcal{F}_P \longrightarrow \mathcal{G}$ which are induced by $+ \longmapsto +$ and $+ \longmapsto -$, respectively, and which preserve the monoidal structure strictly.
$\mathcal{F}_P = \coprod_{n \geq 0} \Sigma_n$ will be identified with its image $i_+(\mathcal{F}_P)$, i.e. the full subcategory of $\mathcal{G}$ which consists of those words that are given by the free monoid on the letter "+". Its morphisms will be called "pseudo-braids" .—

2.2 $\mathcal{G}$ has the homotopy type of $\Omega^\infty S^\infty$

The following theorem is the main result of this section:

2.3 Theorem: There is a permutative functor $\mathcal{F}_P^{-1}\mathcal{F}_P \longrightarrow \mathcal{G}$ which induces a homotopy equivalence on classifying spaces. Hence $B\mathcal{G}$ has the homotopy type of $\Omega^\infty S^\infty$ as an infinite loop space.—

The multiplication on $\Omega^\infty S^\infty$ is given by the wedge product of maps $S^k \to S^k$. The first statement of 2.3 implies the latter:
$B\mathcal{F}_P^{-1}\mathcal{F}_P$ and $B\mathcal{G}$ are grouplike because of the properties of the $S^{-1}S$ construction and 2.8 below; so 1.10 can be applied. Further, the canonical functor $\mathcal{F}_P \longrightarrow \mathcal{F}_P^{-1}\mathcal{F}_P$ is a group completion on classifying spaces ($\mathcal{F}_P$ satisfies the sufficient conditions given in the $S^{-1}S$ construction). The rest follows from $B\mathcal{F}_P = \coprod_{n\geq 0} B\Sigma_n$ and the following well-known theorem:

2.4 Theorem: The canonical map $\coprod_{n\geq 0} B\Sigma_n \longrightarrow \Omega^\infty S^\infty$ is a group completion .— (See [10] or [14]).

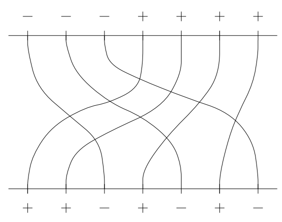
Proof of 2.3: The required functor $\mathcal{F}_P^{-1}\mathcal{F}_P \longrightarrow \mathcal{G}$ is given by a composition

$$\mathcal{F}_P^{-1}\mathcal{F}_P \xrightarrow{\Phi} \mathcal{RG} \xrightarrow{J} \mathcal{G}$$

where $\mathcal{RG}$ is a subcategory of $\mathcal{G}$, the inclusion J an equivalence of permutative categories and Φ an isomorphism.

An object of $\mathcal{G}$ is called ordered if it is of the pattern $w = - - \cdots - + + \cdots +$. Let $\mathcal{RG}$ be the full subcategory of $\mathcal{G}$ which consists of all ordered words, and let $J : \mathcal{RG} \longrightarrow \mathcal{G}$ be the inclusion.

Consider the following functor $R : \mathcal{G} \longrightarrow \mathcal{RG}$:
For a given word w let $\hat{w} := R(w)$ be the word which is obtained by ordering its letters, i.e. $R(+ + - + -) = - - + + +$ etc. Let $s_w : w \longrightarrow \hat{w}$ be the isomorphism which orders the letters in such a way that the order among the letters "+" and "−" is preserved separately.

$s_{++-+-+-} \quad =$ 

Define R on morphisms $\alpha : w_1 \longrightarrow w_2$ by $R(\alpha) := s_{w_2} \circ \alpha \circ s_{w_1}^{-1}$.
R is obviously well-defined, and the functorial properties can easily be checked:

$$R(id_w) = s_w \circ s_w^{-1} = id_{\hat{w}} \qquad \text{for all } \; w,$$

and for any given composition $w_1 \xrightarrow{\alpha_1} w_2 \xrightarrow{\alpha_2} w_3$ we have

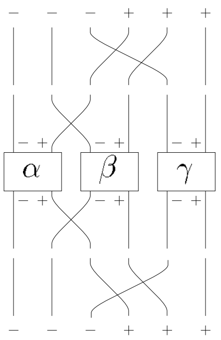
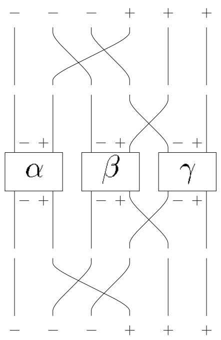
$$R(\alpha_2 \circ \alpha_1) = s_{w_3} \circ \alpha_2 \circ \alpha_1 \circ s_{w_1}^{-1} = s_{w_3} \circ \alpha_2 \circ s_{w_2}^{-1} \circ s_{w_2} \circ \alpha_1 \circ s_{w_1}^{-1} = R(\alpha_2) \circ R(\alpha_1).$$

2.5 Lemma: $\mathcal{RG}$ is a permutative category with tensor product $\hat{\oplus} := R \circ \oplus$ and symmetry isomorphisms $\hat{c}_{v,w} := R(c_{v,w})$. It is equivalent as a permutative category to $\mathcal{G}$ by the functors J and R. In particular, the classifying spaces of $\mathcal{RG}$ and $\mathcal{G}$ are homotopy equivalent as infinite loop spaces .—

Proof: The following identities are immediate for any objects u, v, w:

$$\begin{array}{lrcl} (1) & s_{(u\hat{\oplus}v)\oplus w} \circ (s_{u\oplus v} \oplus id_w) & = & s_{u\oplus(v\hat{\oplus}w)} \circ (id_u \oplus s_{v\oplus w}), \\ (2) & s_{v\oplus w} & = & s_{\hat{v}\oplus\hat{w}} \circ (s_v \oplus s_w). \end{array}$$

$\hat{\oplus} : \mathcal{RG} \times \mathcal{RG} \longrightarrow \mathcal{RG}$ is given by a composition of functors and hence a functor. Obviously the empty word yields a strict unit for $\hat{\oplus}$, and the associativity on objects is also clear. The associativity on morphisms can be shown algebraically by using (1) or is to be seen from the following self-explanatory picture:

$(\alpha\hat{\oplus}\beta)\hat{\oplus}\gamma \;=$  $=$  $= \;\alpha\hat{\oplus}(\beta\hat{\oplus}\gamma)$

R strictly preserves the monoidal structure:
On objects the equation $R(v \oplus w) = R(v)\hat{\oplus}R(w)$ is evident.
Given any two morphisms $v_1 \xrightarrow{\alpha} v_2$ and $w_1 \xrightarrow{\beta} w_2$ one uses (2):

$$\begin{aligned} R(\alpha \oplus \beta) &= s_{v_2\oplus w_2}(\alpha \oplus \beta)s_{v_1\oplus w_1}^{-1} \\ &= s_{\hat{v_2}\oplus\hat{w_2}}(s_{v_2} \oplus s_{w_2})(\alpha \oplus \beta)(s_{v_1} \oplus s_{w_1})^{-1}s_{\hat{v_1}\oplus\hat{w_1}}^{-1} \\ &= s_{\hat{v_2}\oplus\hat{w_2}}(R(\alpha) \oplus R(\beta))s_{\hat{v_1}\oplus\hat{w_1}}^{-1} \\ &= R(\alpha)\hat{\oplus}R(\beta). \end{aligned}$$

The inclusion J is monoidal but not strictly. The necessary isomorphisms $\varphi_2(v,w) : J(v) \oplus J(w) \longrightarrow J(v \hat{\oplus} w)$ are in fact given by $s_{v\oplus w} : v \oplus w \longrightarrow v \hat{\oplus} w$, and the coherence condition turns out to be formula (1) (cf. 1.3 (a)).

The identity $R \circ J = Id_{\mathcal{RG}}$ is obvious, and a natural tensor isomorphism $Id_{\mathcal{G}} \longrightarrow J \circ R$ is given by the isomorphisms s_w:

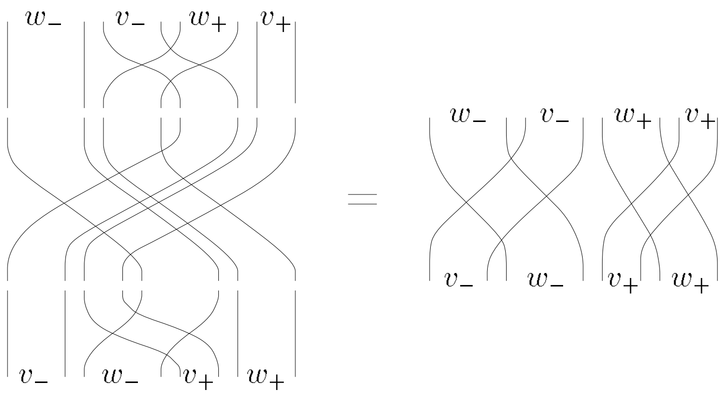
$$\begin{tikzcd} Id(w_1) \arrow[r, "s_{w_1}"] \arrow[d, "Id(\alpha)"'] & J \circ R(w_1) \arrow[d, "J\circ R(\alpha)"] \\ Id(w_2) \arrow[r, "s_{w_2}"] & J \circ R(w_2) \end{tikzcd} \quad = \quad \begin{tikzcd} w_1 \arrow[r, "s_{w_1}"] \arrow[d, "\alpha"'] & \hat{w}_1 \arrow[d, "s_{w_2}\circ\alpha\circ s_{w_1}^{-1}"] \\ w_2 \arrow[r, "s_{w_2}"] & \hat{w}_2 \end{tikzcd}$$

In this case the appropriate coherence condition (cf. 1.3 (b)) is nothing else but formula (2).

Finally, we have to check the assertion about the permutative structure.
For any object w in $\mathcal{G}$ let w_- and w_+ denote the negative and positive part of $\hat{w}$, i.e. for $w = ++-+-$ we have $w_- = --$, $w_+ = +++$ etc.
With this notation we obtain $\hat{w} = w_- \oplus w_+$ and for ordered v, w:

$$\hat{c}_{v,w} = R(c_{v,w}) = s_{w\oplus v} \circ c_{v,w} \circ s_{v\oplus w}^{-1} = c_{v_-,w_-} \oplus c_{v_+,w_+}.$$

The latter equation follows from the picture below.



The naturality of the isomorphisms $\hat{c}_{v,w}$ is derived from the naturality of $c_{v,w}$.
For morphisms $\alpha : v_1 \longrightarrow v_2$, $\beta : w_1 \longrightarrow w_2$ we have

$$\begin{aligned} \hat{c}_{v_2,w_2} \circ (\alpha \hat{\oplus} \beta) &= R(c_{v_2,w_2}) \circ R(\alpha \oplus \beta) \\ &= R(c_{v_2,w_2} \circ (\alpha \oplus \beta)) \\ &= R((\beta \oplus \alpha) \circ c_{v_1,w_1}) \quad = \quad (\beta \hat{\oplus} \alpha) \circ \hat{c}_{v_1,w_1}. \end{aligned}$$

Now consider the conditions of 1.4.

$$\begin{aligned}\hat{c}_{u\hat{\oplus}v,w} &= (\hat{c}_{u,w}\hat{\oplus}id_v)\circ(id_u\hat{\oplus}\hat{c}_{v,w}),\\ \hat{c}_{u,v\hat{\oplus}w} &= (id_v\hat{\oplus}\hat{c}_{u,w})\circ(\hat{c}_{u,v}\hat{\oplus}id_w),\\ \hat{c}_{w,v}\circ\hat{c}_{v,w} &= id_{v\hat{\oplus}w}.\end{aligned}$$

Using the obvious equations $(u\hat{\oplus}v)_- = u_-\oplus v_-$ and $(u\hat{\oplus}v)_+ = u_+\oplus v_+$, we obtain

$$\begin{array}{rclclcl}\hat{c}_{u\hat{\oplus}v,w} &=& R(c_{u\hat{\oplus}v,w}) &=& c_{u_-\oplus v_-,w_-} &\oplus& c_{u_+\oplus v_+,w_+},\\ \hat{c}_{u,w}\hat{\oplus}id_v &=& R(\hat{c}_{u,w}\oplus id_v) &=& c_{u_-,w_-}\oplus id_{v_-} &\oplus& c_{u_+,w_+}\oplus id_{v_+},\\ id_u\hat{\oplus}\hat{c}_{v,w} &=& R(id_u\oplus\hat{c}_{v,w}) &=& id_{u_-}\oplus c_{v_-,w_-} &\oplus& id_{u_+}\oplus c_{v_+,w_+}.\end{array}$$

In this way the first two of the above equations follow from the corresponding equations for $c_{v,w}$, and the third equation is obvious.

For R to be a permutative functor, we have to show $R(c_{v,w}) = \hat{c}_{\hat{v},\hat{w}}$. This is clear for ordered words v, w by definition, but it has to be true for all words in $\mathcal{G}$. The reader may convince himself that the equation

$$R(c_{v,w}) = c_{v_-,w_-}\oplus c_{v_+,w_+}$$

is true for arbitrary words v, w, which immediately leads to the assertion.

Finally, we check the analogous condition for the inclusion J. This is not a strict monoidal functor; it has coherence isomorphisms $\varphi_2(v,w) = s_{v\oplus w}$. Hence 1.5 leads to the following diagram which commutes because of $\hat{c}_{v,w} = R(c_{v,w}) = s_{w\oplus v}\circ c_{v,w}\circ s^{-1}_{v\oplus w}$:

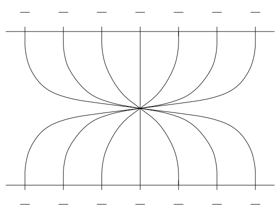
$$\begin{tikzcd} v\oplus w \arrow[r, "s_{v\oplus w}"] \arrow[d, "c_{v,w}"'] & v\hat{\oplus}w \arrow[d, "\hat{c}_{v,w}"] \\ w\oplus v \arrow[r, "s_{w\oplus v}"] & w\hat{\oplus}v \end{tikzcd}$$

.—

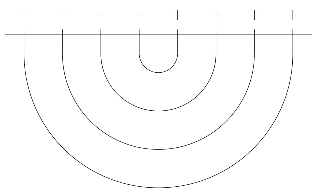
2.6 Remark: The proof of 2.5 in fact shows that $B\mathcal{RG}$ is a strong deformation retract of $B\mathcal{G}$. Hence all questions about the homotopy type of $\mathcal{G}$ can be treated equivalently in $\mathcal{RG}$, which will be undertaken later .—

2.7 Lemma:
There is an isomorphism of permutative categories $\Phi : \mathcal{F}_P^{-1}\mathcal{F}_P \longrightarrow \mathcal{RG}$.
Hence $B\mathcal{F}_P^{-1}\mathcal{F}_P$ and $B\mathcal{RG}$ are homeomorphic as infinite loop spaces .—

Proof: To be able to give an explicit formula for Φ, some special morphisms of $\mathcal{G}$ have to be introduced.
Let $t_{-k} : -k \longrightarrow -k$ be the morphism of $\mathcal{G}$, the picture of which is given by the permutation $i \longmapsto k - i + 1$ of Σ_k. We will depict it as follows:

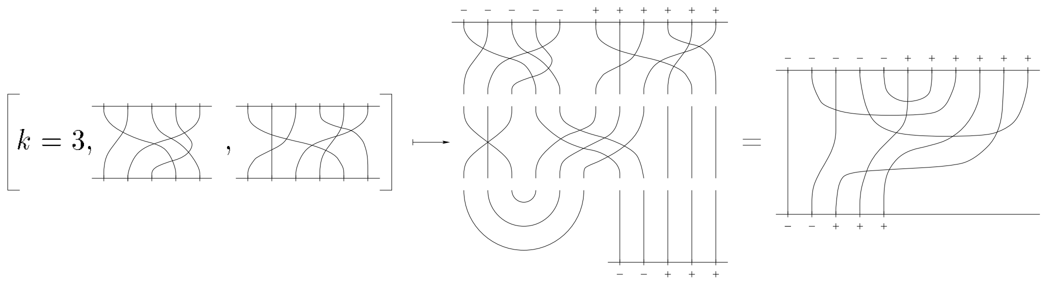
$t_{-7} \quad =$ 

Let $b_{-k} : 0 \longrightarrow -k \oplus k$ denote the morphism which is represented by k arcs without any intersection among them.

$b_{-4} \quad =$ 

Finally, recall the symmetry isomorphism $c_{k,-m} : k \oplus -m \longrightarrow -m \oplus k$ of $\mathcal{G}$ and from 2.2 the embeddings $i_+, i_- : \mathcal{F}_P \longrightarrow \mathcal{G}$.
Now define $\Phi(m,n) := -m \oplus n$ on objects, and if we are given a morphism $\alpha : (m,n) \longrightarrow (k+m, k+n)$, represented by some triple (k, α_-, α_+), let

$$\Phi(\alpha) := (i_-(\alpha_-) \oplus i_+(\alpha_+)) \circ (t_{-k} \oplus c_{k,-m} \oplus id_n) \circ (b_{-k} \oplus id_{-m\oplus n}).$$



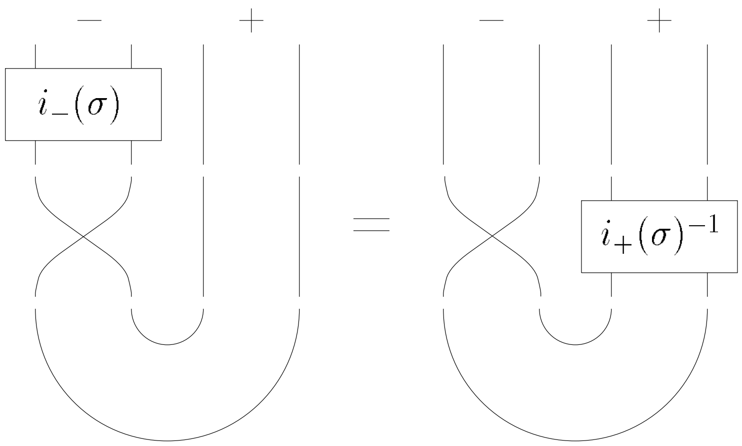
Note: $\Phi(\alpha)$ is indeed a morphism in $\mathcal{RG}$ although its constituents are morphisms in $\mathcal{G}$, and the number of arcs in $\Phi(\alpha)$ is equal to k.

The definition of Φ does not depend on the representing triple (k, α_-, α_+): Let $(k', \alpha'_-, \alpha'_+)$ be another triple which represents the same morphism. Then we have $k = k'$, and there exists an isomorphism σ in $\mathcal{F}_P$ (i.e. $\sigma \in \Sigma_k$) with

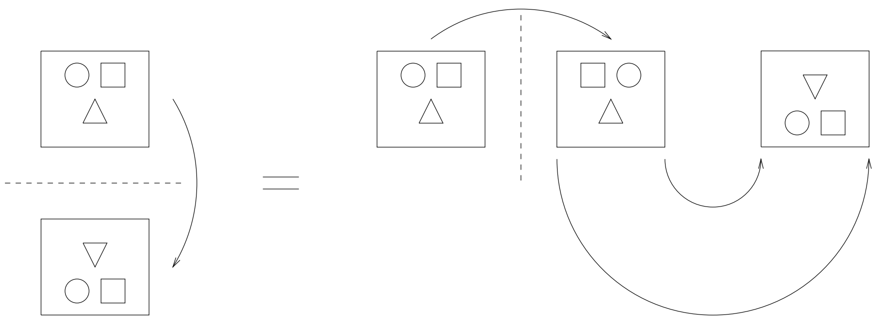
$$\alpha_- = \alpha'_- \circ (\sigma \oplus id_m) \quad \text{and} \quad \alpha_+ = \alpha'_+ \circ (\sigma \oplus id_n).$$

The crucial step in this proof is the observation of the following equation:

$$((i_-(\sigma) \circ t_{-k}) \oplus id_k) \circ b_{-k} = (t_{-k} \oplus i_+(\sigma)^{-1}) \circ b_{-k} \qquad \text{for all } \sigma \in \Sigma_k.$$

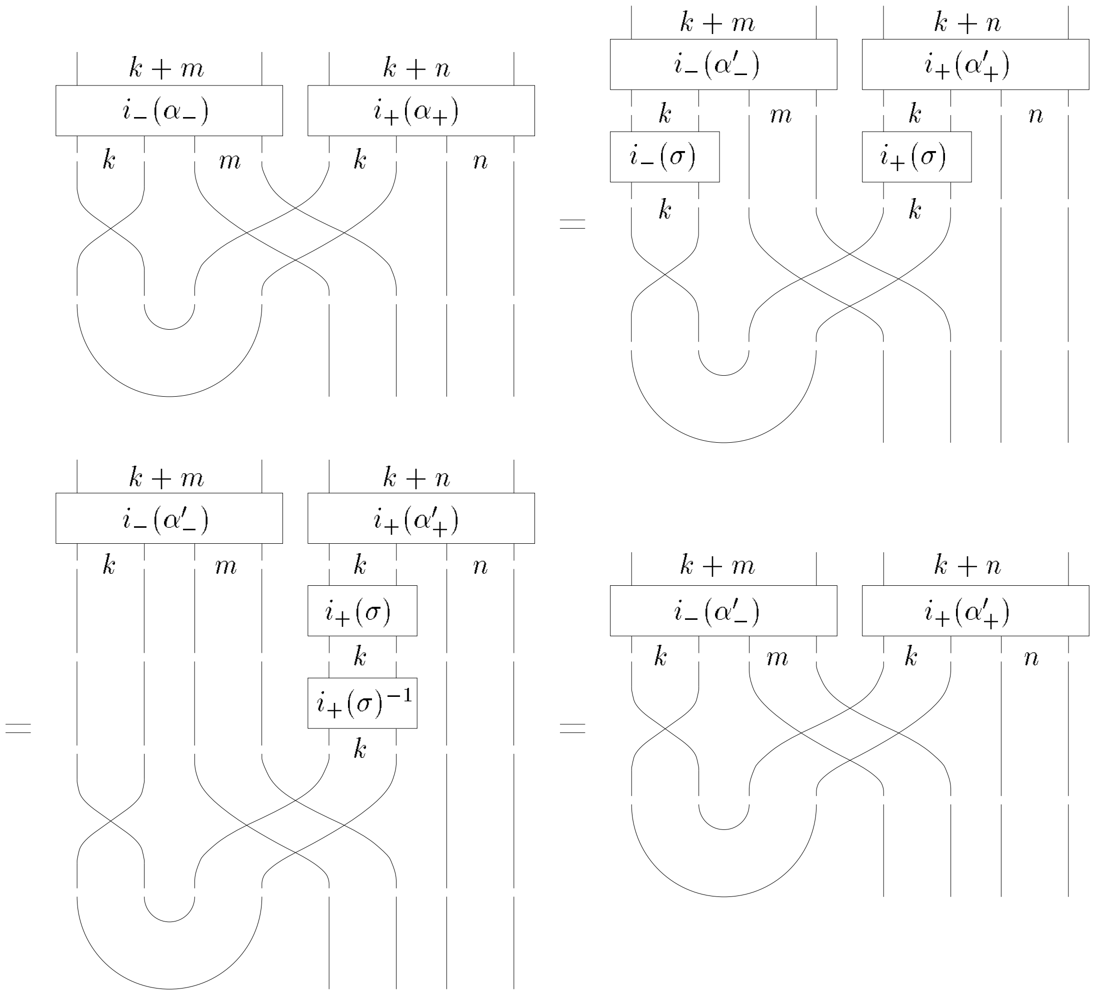


It is clear that the inverse of a given pseudo-braid is obtained by reflecting it at its bottom base. But it also can be built by reflecting it at a vertical axis and then rotating it by 180°:



This is exactly what is done by pushing $i_-(\sigma)$ through t_{-k} and b_{-k}; hence the desired relation is true.

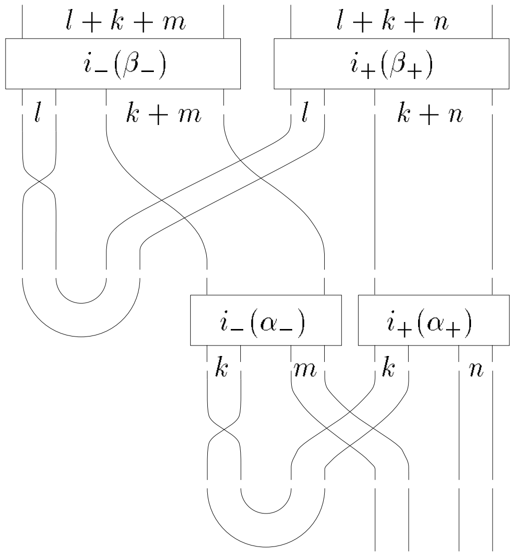
Using this relation we have:



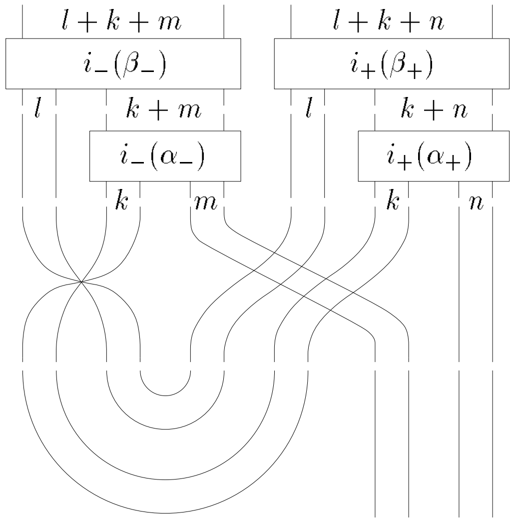
This shows the well-definition of Φ.

The functoriality of Φ is checked as follows: The identity of some object (m,n) in $\mathcal{F}_P^{-1}\mathcal{F}_P$ only has the representative $(0, id_m, id_n)$. In this case the morphisms t, c and b are identities, and the equation $\Phi(id_{m,n}) = id_{-m\oplus n}$ becomes obvious.
For any given composition $(m,n) \xrightarrow{\alpha} (k+m, k+n) \xrightarrow{\beta} (l+k+m, l+k+n)$ the relation $\Phi(\beta \circ \alpha) = \Phi(\beta) \circ \Phi(\alpha)$ is deduced from the following pictures:

$\Phi(\beta) \circ \Phi(\alpha) \quad =$



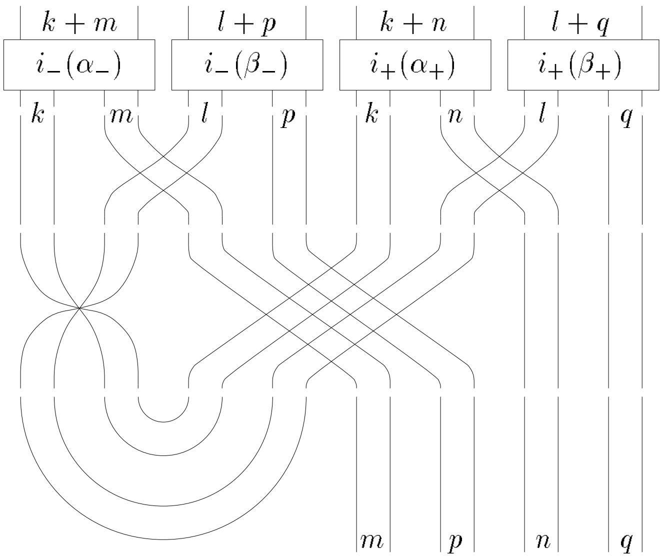
$=$



$= \quad \Phi(\beta \circ \alpha)$

Next we show the preservation of the monoidal structure by Φ, i.e. the equation $\Phi(\alpha \oplus \beta) = \Phi(\alpha)\hat{\oplus}\Phi(\beta)$ which is directly clear for all two objects. So let (k, α_-, α_+), (l, β_-, β_+) be representatives for α and β, respectively. Then the desired equation is proven by the following pictures:

$\Phi(\alpha \oplus \beta) \quad =$



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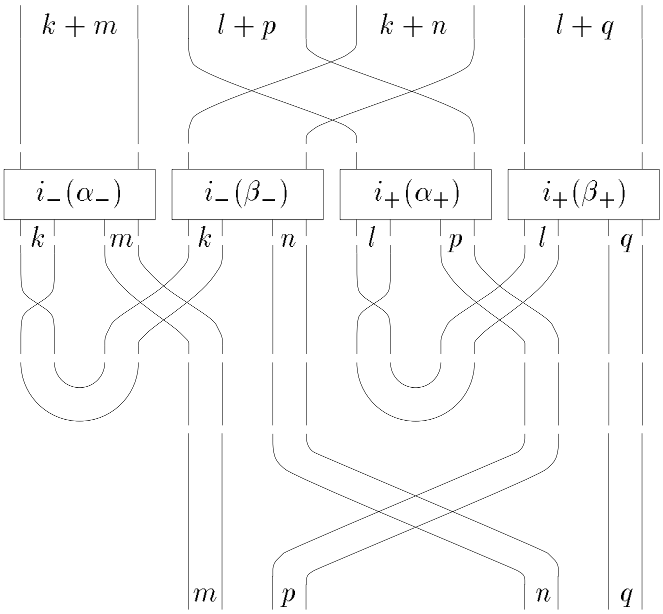


$= \quad \Phi(\alpha)\hat{\oplus}\Phi(\beta)$

Following 1.11, the permutative structure of $\mathcal{F}_P^{-1}\mathcal{F}_P$ is given by

$$c_{(m_1,n_1),(m_2,n_2)} = [0, c_{m_1,m_2}, c_{n_1,n_2}].$$

If Φ is applied to this triple, the morphisms t, c and b are again identities, which leads to

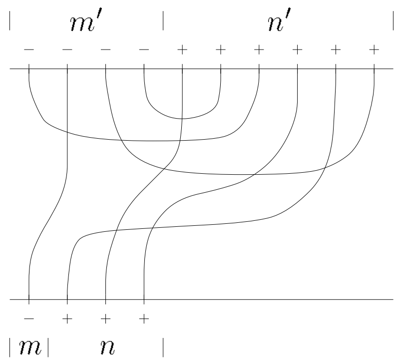
$$\begin{aligned}
\Phi(c_{(m_1,n_1),(m_2,n_2)}) &= \Phi([0, c_{m_1,m_2}, c_{n_1,n_2}]) \\
&= i_-(c_{m_1,m_2}) \oplus i_+(c_{n_1,n_2}) \\
&= R(c_{-m_1\oplus n_1, -m_2\oplus n_2}) \\
&= \hat{c}_{-m_1\oplus n_1, -m_2\oplus n_2} \\
&= \hat{c}_{\Phi(m_1,n_1),\Phi(m_2,n_2)}.
\end{aligned}$$

(For the third equality compare the proof of 2.5).
This shows that Φ strictly preserves the permutative structure.

Finally, we have to show that Φ is an isomorphism of categories.
The functor Φ is an isomorphism if it is fully faithful, since it is obviously bijective on objects. Suppose we are given a pseudo-tangle

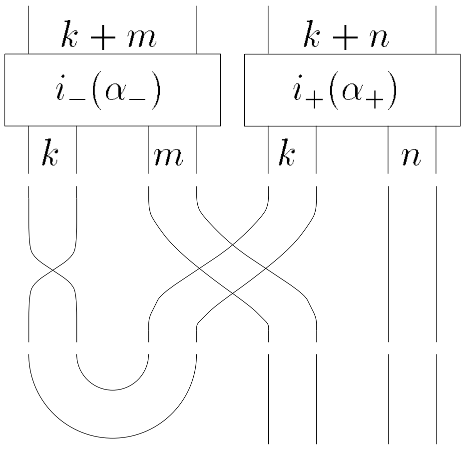
$$\varphi : -m \oplus n \longrightarrow -m' \oplus n'$$

and we are asking for triples (k, α_-, α_+) representing morphisms in $\mathcal{F}_P^{-1}\mathcal{F}_P$ which map to φ.

$\varphi \quad =$ 

Firstly, we observe that $m' = k+m, \;\; n' = k+n$ where $k \in \mathbb{N}$ is the number of arcs in φ, because every arc adds two letters with different signs to the source object. This number k is inevitably the object needed in the representing triple, and α_- and α_+ are morphisms $k+m \longrightarrow k+m$ and $k+n \longrightarrow k+n$, respectively, in $\mathcal{F}_P$.

As we have already seen, Φ applies the following machinery to α_- and α_+:



So, those strings of α_- (resp. α_+) which have their lower end among the last m (resp. n) strings have already been completely determined. The other strings contribute to arcs and are not unique. But every concrete choice for the remaining k strings of α_- determines the last k strings of α_+ and vice versa. There are $k!$ such choices which correspond by the simultaneous action of Σ_k on the first k strings of the lower bound of α_- and α_+, respectively. But this leads precisely to the notion of isomorphic triples in the construction of $\mathcal{F}_P^{-1}\mathcal{F}_P$. Hence all representing triples are isomorphic, and there is exactly one morphism in $\mathcal{F}_P^{-1}\mathcal{F}_P$ which maps to φ, i.e. Φ is in fact an isomorphism of categories.

This completes the proof of this lemma and of Theorem 2.3 .—

2.3 Generators for the lower homotopy groups of $\mathcal{G}$

For any choice of basepoint, the first three homotopy groups of $\Omega^\infty S^\infty$ are well known as

$$\pi_0(\Omega^\infty S^\infty) = \mathbb{Z}, \qquad \pi_1(\Omega^\infty S^\infty) = \mathbb{Z}/2, \qquad \pi_2(\Omega^\infty S^\infty) = \mathbb{Z}/2.$$

The first two of these can be computed directly from the model $\mathcal{G}$, and further we can show that $\pi_2(B\mathcal{G})$ has at least a non-trivial element of order 2 by giving an explicit representative. This is done as follows (recall 1.6):

2.8 Theorem: $\pi_0(B\mathcal{G}) = \mathbb{Z}$ as a group. A representative for the component which corresponds to $k \in \mathbb{Z}$ is given by the object k of $\mathcal{G}$.—

Proof: Let $n_+, n_- : ob\ \mathcal{G} \longrightarrow \mathbb{N}$ be the functions which associate to a word w the number of its letters "+" and "−", respectively, and define

$$K(w) := n_+(w) - n_-(w) \in \mathbb{Z}.$$

Then any two objects v and w belong to the same path component if and only if $K(v) = K(w)$:
Indeed, every morphism of $\mathcal{G}$ has the same invariant K on its source and target, since the letters of the bottom base are preserved by the vertical strings, and every arc adds only two letters with different signs to the source object. Hence the condition is necessary.
Conversely, given two objects v, w with the same invariant K one has

$$n_+(w) - n_+(v) = n_-(w) - n_-(v) =: d.$$

Because of symmetry we can assume $d \geq 0$. A morphism $v \longrightarrow w$ can be found by tensoring id_v with d arcs (which generates the additional letters) and composing this with a suitable pseudo-braid to get the right order.

$\mathbb{Z}$ can be interpreted as a monoidal category with $\mathbb{Z}$ being the set of objects and only identity morphisms. In this sense we obtain $B\mathbb{Z} = \mathbb{Z}$, which is now a discrete topological monoid, and $\pi_0(B\mathbb{Z}) = \mathbb{Z}$. Now $K : \mathcal{G} \longrightarrow \mathbb{Z}$ can be interpreted as a strict monoidal functor because of the obvious equation

$$K(v \oplus w) = K(v) + K(w)$$

and the application of $\pi_0 B$ yields the stated isomorphism of groups.
The second statement is clear because of $K(k) = k$.—

The component of $\mathcal{G}$, which corresponds to the object $k \in \mathbb{Z}$, is denoted by $\mathcal{G}^{(k)}$ in the sequel.

The next goal is the computation of $\pi_1(B\mathcal{G}, 0)$ from the model $\mathcal{G}$.
For simplification, we work in the subcategory $\mathcal{RG}$ of ordered words (cf. 2.6). $\mathcal{RG}^{(0)}$ is the full category of $\mathcal{G}$, which consists of the objects

$$0,\ -+,\ --++,\ ---+++,\ \dots$$

and these words are denoted by $0,\ (1),\ (2),\ (3),\ \dots$ according to the number of letters "$-$". Any path in $B\mathcal{RG}$ is homotopic to a path represented by a finite combination of morphisms in $\mathcal{RG}$. Further, after composing all morphisms which can be composed, the arrows are pointing in turn in different directions, and the only morphism which has the empty word as the target is the identity. Therefore any closed path ω can be represented as

$$\omega :\quad 0 \xrightarrow{\alpha_1} (n_1) \xleftarrow{\alpha_2} (n_2) \xrightarrow{\alpha_3} \cdots \xleftarrow{\alpha_{k-2}} (n_{k-2}) \xrightarrow{\alpha_{k-1}} (n_{k-1}) \xleftarrow{\alpha_k} 0.$$

Every pseudo-tangle α can be represented in a generic way (i.e. no triple points, no self-intersections, no overlaps, etc. of the strings). Define the intersection number $\#\alpha$ to be the number of crossing points modulo 2.
Note that this leads to the equation $\#(\beta \circ \alpha) = \#\beta + \#\alpha$ for any given composable pair of morphisms.
In this way we get a notion of odd and even morphisms. A path is called odd (resp. even) if the sum of the intersection numbers of its defining morphisms is odd (resp. even).

Before formulating the theorem, we fix three essential properties of $\mathcal{RG}$ in the following lemma:

2.9 Lemma:

(a) To any two morphisms $\alpha, \alpha' : (k) \longrightarrow (n)$ there exits a morphism $\sigma : (n) \longrightarrow (n)$ with $\sigma \circ \alpha' = \alpha$.

(b) To any two morphisms $(n') \xleftarrow{\alpha'} (k) \xrightarrow{\alpha} (n)$ with $n' \leq n$ there exists a morphism $(n') \xrightarrow{\beta} (n)$ with $\beta \circ \alpha' = \alpha$.

(c) The path defined by an even endomorphism $\sigma : (n) \longrightarrow (n)$ is based null homotopic.

Proof: (a) The group $End(n) = \Sigma_n \times \Sigma_n$ operates transitively on $mor((k),(n))$ for all $k \leq n$, since every two pseudo-tangles $(k) \longrightarrow (n)$ only differ by a suitable permutation of the strings at their upper bound.

(b) follows immediately from (a).

(c) For $n < 2$ there is nothing to show, $End(0)$ and $End(1)$ are trivial.
The symmetric group Σ_n is generated by simple transpositions, i.e. by pseudo-braids with exactly one crossing point. Pairs of such simple pseudo-braids form even morphisms of $End(n) = \Sigma_n \times \Sigma_n$ and these generate all even endomorphisms. This is proven algebraically as follows:
Let $(\sigma, \tau) \in \Sigma_n \times \Sigma_n$ be an even endomorphism. Decompose σ and τ into simple transpositions

$$\sigma = \sigma_p \circ \cdots \circ \sigma_1, \qquad \tau = \tau_q \circ \cdots \circ \tau_1.$$

Then $p+q$ is even. Because of symmetry we may assume $p \leq q$ and define

$$\gamma_i := (\sigma_i, \tau_i) \text{ for } 1 \leq i \leq p.$$

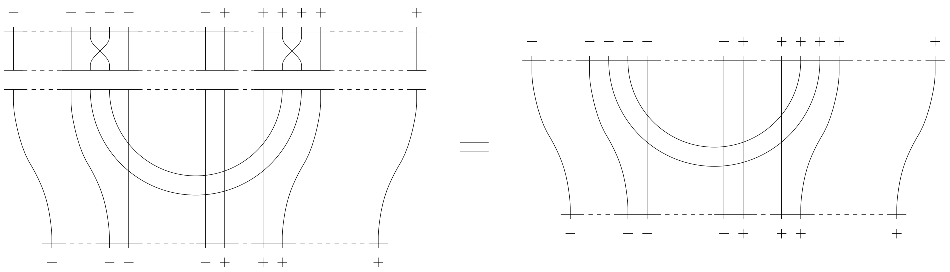
If $p = q$, we have $(\sigma, \tau) = \gamma_p \circ \cdots \circ \gamma_1$, and we have finished.
Otherwise, choose any simple transposition ε of Σ_n and define

$$\gamma_i := (\varepsilon, \tau_i) \text{ for } p+1 \leq i \leq q.$$

$q - p$ is even and $\varepsilon \circ \varepsilon = id$. It follows that

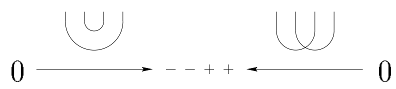
$$(\sigma, \tau) = \gamma_q \circ \cdots \circ \gamma_{p+1} \circ \gamma_p \circ \cdots \circ \gamma_1.$$

Now it is sufficient to prove the assertion of (c) for an endomorphism σ, which has exactly two crossing points, one of which is among the negative strings and the other among the positive strings. Define a morphism $\alpha : (n-2) \longrightarrow (n)$ by connecting the positions which belong to the crossing strings of σ with two arcs (there are two possibilities to do so, but it does not matter which one is chosen), and keep all $n-2$ vertical strings without any crossing among them. Then we obtain a morphism α with $\sigma \circ \alpha = \alpha$:



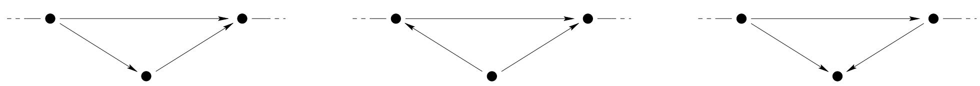
Therefore the path defined by $(n) \xrightarrow{\sigma} (n)$ is based homotopic to the path $(n) \xleftarrow{\alpha} (n-2) \xrightarrow{\alpha} (n)$, which is clearly based null homotopic .—

2.10 Theorem: $\pi_1(B\mathcal{G}, 0) = \mathbb{Z}/2$, a generator is given by the path

$\kappa :$  .—

Proof:
Denote the left of these morphisms $0 \longrightarrow (2)$ by u and the other by t.

To investigate edge paths up to homotopy it suffices to consider those homotopies which are determined by 2-cells. These are given by composable pairs $(n) \xrightarrow{\alpha} (n') \xrightarrow{\beta} (n'')$ which form a triangle with edges α, β and $\beta \circ \alpha$. Hence there are three elementary manipulations to replace a morphism in the chain of ω up to homotopy by two others and vice versa:



From the equation $\#(\beta \circ \alpha) = \#\beta + \#\alpha$ it follows that ω does not change its intersection number under the above manipulations. Therefore an odd path cannot be homotopic to an even path, and the theorem reduces to the following two assertions:

(1) All even paths are null homotopic.

(2) All odd paths are homotopic to κ.

By 2.9 (b), every sequence $(n) \longleftarrow (k) \longrightarrow (n')$ of ω can be replaced by a single morphism $(n) \longrightarrow (n')$ or $(n) \longleftarrow (n')$ depending on whether $n \leq n'$ or $n \geq n'$. This single morphism can be composed with one of its neighbours and we obtain a representative of ω, which is shortened by two morphisms. Finite repetition leads to a representative of the form

$$\omega : \quad 0 \xrightarrow{\alpha} (n) \xleftarrow{\alpha'} 0.$$

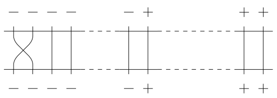
Now applying 2.9 (a) to $(n) \xleftarrow{\alpha'} 0 \xrightarrow{\alpha} (n)$ we find an endomorphism $\sigma : (n) \longrightarrow (n)$ with $\sigma \circ \alpha' = \alpha$, i.e. $\alpha' = \sigma^{-1} \circ \alpha$, and get the sequence

$$\omega : \quad 0 \xrightarrow{\alpha} (n) \xrightarrow{\sigma} (n) \xleftarrow{\alpha} 0.$$

σ has the same intersection number as ω. Hence if ω is even, σ is null homotopic because of 2.9 (c), and this leads to statement (1).

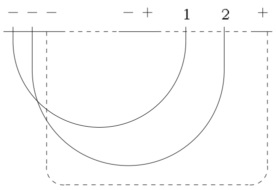
Now assume that ω is odd, i.e. σ is also odd.

ε :=

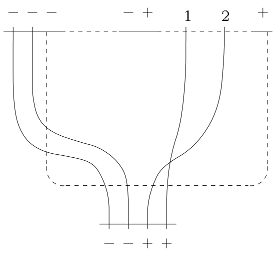


Define $\varepsilon : (n) \longrightarrow (n)$ to be the endomorphism, which has exactly one crossing point between the first two strings from the left side and no others.

α =

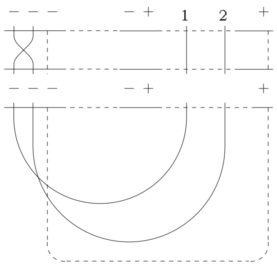


$\overline{\alpha}$:=

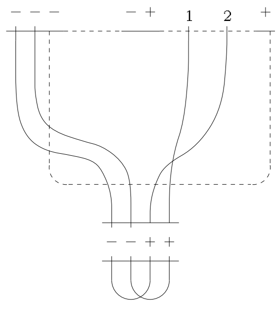


Take α and define a new morphism $\overline{\alpha} : (2) \longrightarrow (n)$ by "opening" those two arcs which belong to the first two letters of (n) in the following way: Connect the first letter "$-$" of (n) with the first letter of (2) and the corresponding "$+$" of (n) with the last letter of (2). Similarly, connect the second "$-$" of (n) and the corresponding "$+$" with the second and third letters of (2). All other arcs of α remain the same.

This obviously gives the equation $\overline{\alpha} \circ u = \alpha$. But we also obtain $\varepsilon \circ \alpha = \overline{\alpha} \circ t$:



=



Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r, "\alpha"] \arrow[d, equal] & (n) \arrow[rr, "\sigma"] \arrow[dr, "\varepsilon\circ\sigma"] & & (n) \arrow[dl, "\varepsilon"] & 0 \arrow[l, "\alpha"'] \arrow[dd, equal] \\
0 \arrow[rr, "\alpha"] \arrow[d, equal] & & (n) & & \\
0 \arrow[rr, "u"] & & (2) \arrow[u, "\overline{\alpha}"] & & 0 \arrow[ll, "t"']
\end{tikzcd}$$

All parts of this diagram are commutative, except the upper left corner. But since σ and ε are odd, their composition $\varepsilon \circ \sigma$ is even and therefore null homotopic by 2.9 (c). Hence the upper left corner can be filled somehow by a commutative diagram, and the whole diagram yields a homotopy between ω and the path κ .—

$B\mathcal{G}$ is an infinite loop space, hence all path components are homotopy equivalent and we have $\pi_1(B\mathcal{G}, w) = \mathbb{Z}/2$ for any chosen basepoint w. The generators of the other path components can be found as the images of the above generator under explicit functors, which induce homotopy equivalences between the various components:

2.11 Proposition: The functors $l_+,\ l_- : \mathcal{G} \longrightarrow \mathcal{G}$ which are defined by

$$l_+(w) := + \oplus w,\ l_+(\alpha) := id_+ \oplus \alpha, \quad l_-(w) := - \oplus w,\ l_-(\alpha) := id_- \oplus \alpha,$$

map $\mathcal{G}^{(k)}$ into $\mathcal{G}^{(k+1)}$ and vice versa and induce homotopy inverse maps on classifying spaces. The same is true for $r_+, r_- : \mathcal{G} \longrightarrow \mathcal{G}$ given by

$$r_+(w) := w \oplus +,\ r_+(\alpha) := \alpha \oplus id_+,\ r_-(w) := w \oplus -,\ r_-(\alpha) := \alpha \oplus id_- \ .\text{—}$$

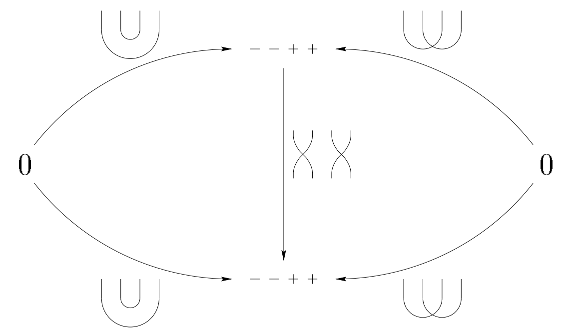
Proof: The first statement is trivial.
A natural transformation $\eta : Id_{\mathcal{G}} \longrightarrow l_- \circ l_+$ is given by $\eta(w) := b_- \oplus id_w$, where b_- is the only morphism $0 \longrightarrow -+$. For any morphism $w_1 \xrightarrow{\alpha} w_2$ we have

$$\begin{aligned} (l_- \circ l_+)(\alpha) \circ \eta(w_1) &= (id_{-+} \oplus \alpha) \circ (b_- \oplus id_{w_1}) \\ &= (b_- \oplus id_{w_2}) \circ (id_0 \oplus \alpha) \\ &= \eta(w_2) \circ Id(\alpha). \end{aligned}$$

The other direction is given by exchanging "+" and "−", and the proof for r_+, r_- is analogous.—

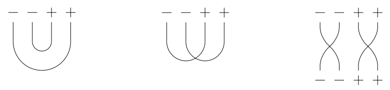
2.12 Theorem: $\pi_2(B\mathcal{G}, 0)$ has an element of order 2, given by the realization of the following diagram:

$\Theta \ :=$



.—

Proof:
The given picture defines both a 2-sphere and a 2-cycle in $B\mathcal{G}$, and the Hurewicz homomorphism $\pi_2(B\mathcal{G}) \longrightarrow H_2(B\mathcal{G})$ leads from the first interpretation to the second. As a cycle Θ is defined by $(b,u)-(b,t)$, where u,t and b denote the morphisms



u $\qquad$ t $\qquad$ b .

It will be shown that Θ represents a non-trivial $\mathbb{Z}/2$–element in $H_2(B\mathcal{G})$. Then by the Hurewicz map it has to be non-trivial in $\pi_2(B\mathcal{G})$ as well.

At first, we see that the order of Θ is at most 2, since 2Θ is the boundary of $(b,b,t)-(b,b,u)$ and therefore zero in $H_2(B\mathcal{G})$:

$$\begin{aligned}
\partial((b,b,t)-(b,b,u)) &= \partial(b,b,t)-\partial(b,b,u)\\
&= (b,b)-(b,b\circ t)+(b\circ b,t)-(b,t)\\
&\quad -(b,b)+(b,b\circ u)-(b\circ b,u)+(b,u)\\
&= (b,b)-(b,t)+0-(b,t)\\
&\quad -(b,b)+(b,u)-0+(b,u)\\
&= 2((b,u)-(b,t))\\
&= 2\Theta.
\end{aligned}$$

Note that the realization of $(b,b,t)-(b,b,u)$ actually gives a contraction of the sphere 2Θ in $B\mathcal{G}$. Therefore, as an element of $\pi_2(B\mathcal{G})$, Θ also has at most order 2.

The proof that Θ itself is not trivial is done in several steps:

Recall that $\mathcal{F}_P^{(k)} = End(k) = \Sigma_k$, and there is the following presentation:
Let $t_1,\ldots,t_{k-1}$ be transpositions of Σ_k defined by

$$t_1=(12), \qquad t_2=(23), \qquad \ldots \qquad t_{k-1}=(k-1,k).$$

Then we have

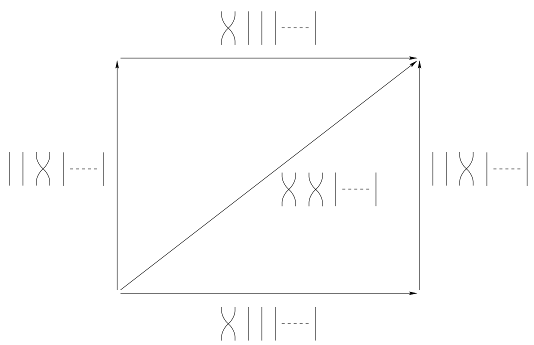
$$\Sigma_k = \langle t_1,\ldots,t_{k-1} \;\mid\; t_i^2=(t_jt_{j+1})^3=(t_mt_l)^2=1\rangle$$

with $\quad i,m,l\in\{1,\ldots,k-1\}, \quad j\in\{1,\ldots,k-2\}, \quad |m-l|\geq 2.$
(See [4], Prop. 2.12.1).

Step 1: $$H_2(B\Sigma_k) = \begin{cases} 0 & \text{for} \quad k \leq 3 \\ \mathbb{Z}/2 & \text{for} \quad k \geq 4 \end{cases},$$

and a generator for $k \geq 4$ is represented by

$$\Gamma_k := (t_1, t_3) - (t_3, t_1)$$



.

The above picture gives the graphical interpretation of Γ_k. The upper and lower sides correspond to t_1, the left and right sides correspond to t_3.

An abstract computation of these homology groups is the combination of the theorems 2.7.3 and 2.12.3 of [4], and the assertion about the generator is contained in the proofs of those theorems. Here is a brief summary:

If a group G is given by any free presentation $G = F/R$ (i.e. F is free), then the second homology group can be computed by the well-known Hopf formula

$$H_2(G) \cong (F' \cap R)/[F, R].$$

Here F' denotes the commutator subgroup of F, and $[F, R]$ is the subgroup which is generated by all elements of the form $frf^{-1}r^{-1}$ with $f \in F, r \in R$. Up to isomorphism, the group $(F' \cap R)/[F, R]$ is independent of the choice of the free presentation ([4, Thm. 2.6.6]), hence we suppose that G is given by the standard free presentation, i.e. $F = F_0 = \langle X_g \mid g \in G, g \neq 1 \rangle$, and R_0 is the kernel of the canonical homomorphism $F_0 \longrightarrow G$. Further, recall from 1.6 the sequence of the chain complex which defines $H_2(G)$:

$$\cdots \longrightarrow K_3 \xrightarrow{\partial_3} K_2 \xrightarrow{\partial_2} K_1 \longrightarrow \cdots$$

Then the map

$$\sigma : K_2 \longrightarrow R_0/R_0', \qquad (g, h) \longmapsto (X_g X_h X_{gh}^{-1})R_0'$$

is an isomorphism of free abelian groups with the properties

$$\sigma(Ker\ \partial_2) = (F_0' \cap R_0)/R_0', \qquad \sigma(Im\ \partial_3) = [F_0, R_0]/R_0'$$

which leads immediately to Hopf's formula [4, p. 79].

In our case we have $G = \Sigma_k$ and invoking $t_1t_3 = t_3t_1$ (in G) we find

$$\begin{aligned}
\sigma(\Gamma_k) &= \sigma(t_1,t_3)(\sigma(t_3,t_1))^{-1}\\
&= X_{t_1}X_{t_3}X_{t_1t_3}^{-1}(X_{t_3}X_{t_1}X_{t_3t_1}^{-1})^{-1}R_0'\\
&= X_{t_1}X_{t_3}X_{t_1}^{-1}X_{t_3}^{-1}R_0'.
\end{aligned}$$

Since Hopf's formula is independent of the choice of the free presentation, we can pass to the above presentation of Σ_k, i.e.

$$\begin{aligned}
F_1 &= \langle t_1,\ldots,t_{k-1}\rangle,\\
R_1 &= \text{normal closure of } \langle t_i^2, (t_jt_{j+1})^3, (t_mt_l)^2\rangle,\ |m-l|\geq 2.
\end{aligned}$$

This change is described by the canonical map of presentations $t_i \longmapsto X_{t_i}$ which maps $t_1t_3t_1^{-1}t_3^{-1}$ to $X_{t_1}X_{t_3}X_{t_1}^{-1}X_{t_3}^{-1}$. Hence it suffices to show that $w := t_1t_3t_1^{-1}t_3^{-1}[F_1,R_1]$ is a non-trivial element in $(F_1'\cap R_1)/[F_1,R_1]$. This is done explicitly in the proof of Theorem 2.12.3 of [4].

<u>*Step 2:*</u> The inclusion $\Sigma_{k_0} = \mathcal{F}_P^{(k_0)} \xrightarrow{i_+} \mathcal{G}^{(k_0)}$ (cf. 2.2) induces an isomorphism

$$(i_+)_* : H_2(B\Sigma_{k_0}) \xrightarrow{\cong} H_2(B\mathcal{G}^{(k_0)}) \text{ for } k_0 \geq 4.$$

Recall the statements of the $S^{-1}S$ construction 1.11 and of Theorem 2.3. The monoid $S = \pi_0(B\mathcal{F}_P) = \mathbb{N}$ acts on

$$H_2(B\mathcal{F}_P) = \bigoplus_{k\in\mathbb{N}} H_2(B\Sigma_k) \cong \bigoplus_{k\geq 4} \mathbb{Z}/2,$$

the k-th summand of which is generated by $\overline{\Gamma}_k$, i.e. the class of Γ_k. The action is translation, i.e. adding positive trivial strings. So for $n \in \mathbb{N} = \pi_0(B\mathcal{F}_P)$ we have

$$n\cdot\overline{\Gamma}_k = \overline{\Gamma}_{k+n}.$$

(By definition Γ_{k+n} is created from Γ_k by adding n trivial strings on the right side, while the translation from $\pi_0(B\mathcal{F}_P)$ is given from the left side. But obviously this does not matter, since an evident automorphism of Σ_{k+n} exchanges these different 2-cycles, which shows that both represent $\overline{\Gamma}_{k+n}$).

Invoking the theorem of 1.11, we find an isomorphism

$$
\begin{aligned}
H_2(B\mathcal{G}) &\cong (\pi_0(B\mathcal{F}_P))^{-1}H_2(B\mathcal{F}_P) \\
&= (\pi_0(B\mathcal{F}_P)\times H_2(B\mathcal{F}_P))/\pi_0(B\mathcal{F}_P) \\
&\cong (\mathbb{N}\times\bigoplus_{k\geq 4}\mathbb{Z}/2)/\mathbb{N} \\
&\cong \bigoplus_{k\in\mathbb{Z}}\mathbb{Z}/2.
\end{aligned}
$$

The latter isomorphism follows from the definition of the action of $\mathbb{N}$ on $\mathbb{N}\times\bigoplus_{k\geq 4}\mathbb{Z}/2$, which, by using $n\cdot\overline{\Gamma}_k=\overline{\Gamma}_{k+n}$, leads to

$$
\begin{aligned}
(m,(x_4,x_5,x_6,\ldots)) &\sim n\cdot(m,(x_4,x_5,x_6,\ldots)) \\
&= (n+m,n\cdot(x_4,x_5,x_6,\ldots)) \\
&= (n+m,(\underbrace{0,\ldots,0}_{n},x_4,x_5,\ldots)).
\end{aligned}
$$

Now look at the following commutative diagram:

$$
\begin{tikzcd}
H_2(B\Sigma_{k_0}) \arrow[r, "(i_+)_*"] \arrow[d, hook] & H_2(B\mathcal{G}^{(k_0)}) \arrow[d, hook] \\
\bigoplus_{k\geq 4} H_2(B\mathcal{F}_P^{(k)}) \arrow[r, "(i_+)_*"] \arrow[d, equal] & \bigoplus_{k\in\mathbb{Z}} H_2(B\mathcal{G}^{(k)}) \arrow[d, equal] \\
H_2(B\mathcal{F}_P) \arrow[r, "(i_+)_*"] \arrow[d, "\xi"'] & H_2(B\mathcal{G}) \\
(\pi_0(B\mathcal{F}_P))^{-1}H_2(B\mathcal{F}_P) \arrow[ur, dashed, "\cong", "\Psi"']
\end{tikzcd}
$$

The universal arrow is $\xi:\gamma\longmapsto(0,\gamma)$, and the induced map can be expressed as $\Psi:(n,\gamma)\longmapsto(l_-)_*^n(i_+)_*(\gamma)$ with l_- being the functor of 2.11.

This shows that the map $(i_+)_*: H_2(B\mathcal{F}_P)\longrightarrow H_2(B\mathcal{G})$ corresponds to the inclusion $\bigoplus_{k\geq 4}\mathbb{Z}/2\hookrightarrow\bigoplus_{k\in\mathbb{Z}}\mathbb{Z}/2$. Now the assertion follows immediately.

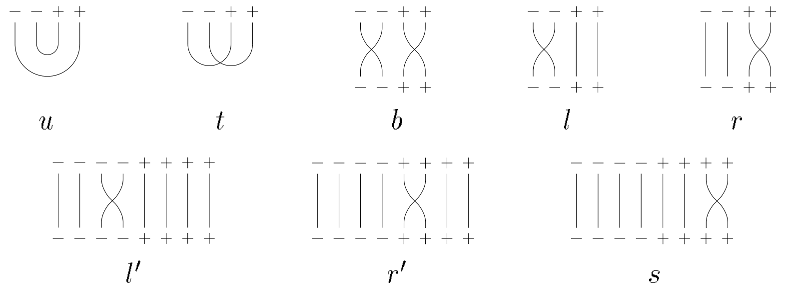
<u>*Step 3:*</u> Using the functor l_- of 2.11, a generator for $H_2(B\mathcal{G}^{(0)})$ is given by

$$\Gamma := (l_-)_*^4(i_+)_*(\Gamma_4).$$

This is obvious, since $(i_+)_*(\Gamma_4)$ is a generator of $H_2(B\mathcal{G}^{(4)})$ according to step 2, and l_- is a homotopy equivalence.

Step 4: Θ and Γ are homologous.

Consider the following morphisms of $\mathcal{G}^{(0)}$:



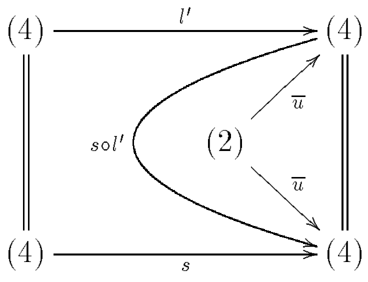
We have $\Theta = (b,u) - (b,t)$ and obviously $\Gamma = (r',s) - (s,r')$.
Compute the following boundary:

$$
\begin{aligned}
\partial((l,r,u) - (l,l,u) + (b,l,u) - (b,l,l)) &= \quad (l,r) - (l,t) + (b,u) - (r,u) \\
&\qquad -(l,l) + (l,t) - 0 + (l,u) \\
&\qquad +(b,l) - (b,t) + (r,u) - (l,u) \\
&\qquad -(b,l) + 0 - (r,l) + (l,l) \\
&= ((b,u) - (b,t)) - ((r,l) - (l,r)).
\end{aligned}
$$

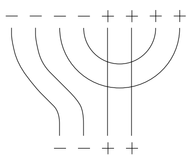
This shows that Θ is homologous to $(r,l) - (l,r)$.

The latter cycle is homotopic (and hence homologous) to $(r',l')-(l',r')$, since it only has two additional positive and negative trivial strings (cf. 2.11).

The path defined by l' is based homotopic to the path defined by s, as the following commutative diagram shows:



with $\quad \overline{u} \;=$



This homotopy leads to a homotopy between the cycle $(r',l') - (l',r')$ and $(r',s) - (s,r')$, which is Γ.

This completes the proof of 2.12 .—

2.4 An easy counterexample for a known mistake

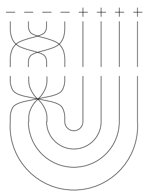
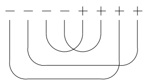
In 1980, R.W. Thomason noticed an irreparable mistake in Z. Fiedorowicz's definition of a pairing $K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B)$ for the algebraic K-groups of rings, which used Quillen's $S^{-1}S$ construction. (See [2] and [12]).

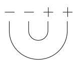
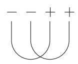
For every permutative category $\mathcal{S}$ there is a canonical functor $\imath : \mathcal{S}^{-1}\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ which is given by $(V, W) \longmapsto (W, V)$ on objects and by $[K, \alpha, \beta] \longmapsto [K, \beta, \alpha]$ on morphisms. It seems plausible that the induced map $B\imath$ is the homotopy inverse for the H-space structure on $B\mathcal{S}^{-1}\mathcal{S}$, as it induces the inverse on the group $\pi_0(B\mathcal{S}^{-1}\mathcal{S})$. Indeed this is true, but the argument used in [2] is wrong. It is claimed in [2] that, if all morphisms of $\mathcal{S}$ are isomorphisms, there would be a natural transformation $\eta : 0 \longrightarrow \imath \oplus id$ given by

$$\eta(V, W) := [V \oplus W, c_{V,W}, id_{V \oplus W}]$$

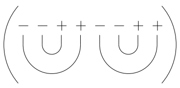
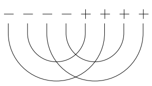
which is the canonical candidate from an algebraic point of view. Thomason refutes this, but the proof is purely algebraic and somewhat technical. The mistake becomes obvious in the case $\mathcal{S} = \mathcal{F}_P$ with the help of our geometric description of the category $\mathcal{F}_P^{-1}\mathcal{F}_P \cong \mathcal{RG}$ (cf. 2.7).

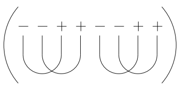
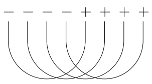
If η is translated by the isomorphism Φ of Lemma 2.7 to $\mathcal{RG}$, we have, for example,

$\eta(--++) = \Phi(\eta(2,2)) = \Phi([4,\ c_{2,2},\ id_4])$ $=$  $=$  .

Consider the two morphisms $0 \longrightarrow (2)$, namely $u =$  and $t =$ .

We obtain $\imath(u) = u$, since Φ maps both of the triples $[2,\ id_2,\ c_{1,1}]$ and $[2,\ c_{1,1},\ id_2]$ to u. Analogously, we find $\imath(t) = (t)$. Recall the tensor product $\hat{\oplus} = R \circ \oplus$ of $\mathcal{RG}$ (2.5). This gives

$(\imath \hat{\oplus} id)(u) = R\Big($  $\Big)$ $=$ 

$(\imath \hat{\oplus} id)(t) = R\Big($  $\Big)$ $=$  .

If η were a natural transformation, the following diagram would commute for all morphisms $\alpha : 0 \longrightarrow (2)$:

$$\begin{tikzcd}
0 \arrow[r, "\eta(0)=id_0"] \arrow[d, "id_0"'] & 0 \arrow[d, "(\imath \hat{\oplus} id)(\alpha)"] \\
0 \arrow[r, "\eta(--++)"] & (4)
\end{tikzcd}$$

This cannot be satisfied simultaneously for $\alpha = u$ and $\alpha = t$, neither by the given definition of η, nor by any other.

3 The relation between $\mathcal{G}$ and the free permutative category with duality

3.1 The construction of $\mathcal{F}_{P,D}$

3.1 Definition: A duality in a monoidal category associates to every object V an object V^* and two morphisms

$$b_V : 0 \longrightarrow V \oplus V^*, \qquad d_V : V^* \oplus V \longrightarrow 0$$

subject to the following relations:

$$(id_V \oplus d_V)(b_V \oplus id_V) = id_V, \qquad (d_V \oplus id_{V^*})(id_{V^*} \oplus b_V) = id_{V^*} \qquad .—$$

3.2 Construction: Let $\mathcal{F}_{P,D}$ be the following category: The set of objects is the free monoid on the alphabet $\{+,-\}$. We think of an object with k letters as the subset $\{1,\ldots,k\} \subset \mathbb{R}$ with an assigned $+$ or $-$ to each i.
A morphism of a word w_1 with k_1 letters to another word w_2 with k_2 letters is represented by an unordered finite set $\{f_\lambda\}_{\lambda\in\Lambda}$ of paths $f_\lambda : [0,1] \longrightarrow \mathbb{R}\times[0,1]$ subject to the following conditions:

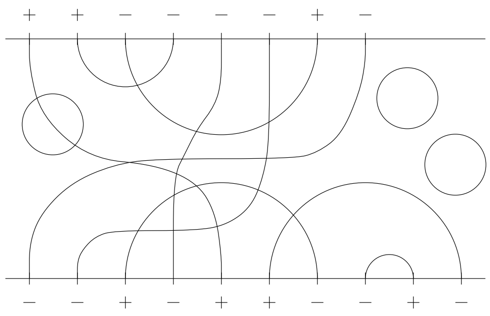
1. Each f_λ connects two different points of the subset $\{1,\ldots,k_1\} \times \{0\} \cup \{1,\ldots,k_2\} \times \{1\}$, or it is a closed path, i.e. $f_\lambda(0) = f_\lambda(1)$.

2. Each point of $\{1,\ldots,k_1\} \times \{0\} \cup \{1,\ldots,k_2\} \times \{1\}$ is in the image of exactly one f_λ, which is not closed.

3. Each f_λ which is not closed connects either an element from $\{1,\ldots,k_1\}\times\{0\}$ with an element from $\{1,\ldots,k_2\}\times\{1\}$ having the same sign (called "vertical strings")
   or two elements from $\{1,\ldots,k_2\} \times \{1\}$ having different signs (called "upper arcs")
   or two elements from $\{1,\ldots,k_1\} \times \{0\}$ having different signs (called "lower arcs").

The closed paths are called "cycles". The term "string" is used for all kinds of paths. Two such systems $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{f'_\lambda\}_{\lambda\in\Lambda}$ represent the same morphism if there is a family of homotopies $\{H_\lambda : f_\lambda \simeq f'_\lambda \;\text{ or }\; -f_\lambda \simeq f'_\lambda\}_{\lambda\in\Lambda}$ which keep the boundary points of the non-closed paths fixed.

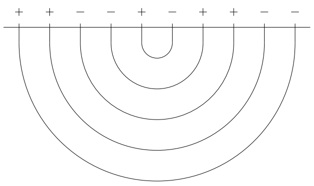
Morphisms can be illustrated by pictures like this:



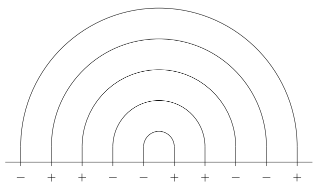
Again, such a picture is called a "pseudo-tangle".
Composition in $\mathcal{F}_{P,D}$ is defined in the same way as for $\mathcal{G}$, and it has an obvious monoidal structure which is again given by juxtaposition of words and morphisms, respectively. Further, the permutative structure of $\mathcal{G}$ also defines a permutative structure for $\mathcal{F}_{P,D}$.

$\mathcal{F}_{P,D}$ has a duality structure. One obtains the dual word w^* to a given word w by reading it from right to left and replacing "+" by "−" and vice versa. Example: $(++--+)^* = -++--$.
The morphisms b_w and d_w are given by the unique pseudo-tangles, which have no cycles and no crossing point among the strings:

b_{++--+} =



d_{++--+} =



.—

3.3 Theorem:
(a) $\mathcal{F}_{P,D}$ is isomorphic to the free permutative category with duality which is generated by one object.

(b) Under tensor product and composition, $\mathcal{F}_{P,D}$ is generated by the duality morphisms b_w, d_w and the symmetry isomorphisms $c_{v,w}$, where v, w run through all objects of $\mathcal{F}_{P,D}$.—

Proof: This is derived from Proposition 4.1, Theorem 8.2 and §9 of [6] by using the methods of [11]. $\mathcal{F}_{P,D}$ is isomorphic to the category $F''\mathcal{A}$ of [6, §9], where $\mathcal{A}$ is the trivial category. So in [6, Prop. 4.1] the morphisms a, a^{-1}, r, r^{-1} and f are identities .—

Define $\mathcal{RF}_{\mathcal{P},\mathcal{D}}$ to be the full subcategory of $\mathcal{F}_{P,D}$ which consists of all ordered words, and define functors $R : \mathcal{F}_{P,D} \longrightarrow \mathcal{RF}_{\mathcal{P},\mathcal{D}}$ and $J : \mathcal{RF}_{P,D} \longrightarrow \mathcal{F}_{P,D}$ as on page 13. The following lemma has the same proof as 2.5.

3.4 Lemma: $\mathcal{RF}_{\mathcal{P},\mathcal{D}}$ is a permutative category with tensor product $\hat{\oplus} := R \circ \oplus$ and symmetry isomorphisms $\hat{c}_{v,w} := R(c_{v,w})$. It is equivalent as a permutative category to $\mathcal{F}_{P,D}$ by the functors J and R. In particular, the classifying spaces of $\mathcal{RF}_{\mathcal{P},\mathcal{D}}$ and $\mathcal{F}_{P,D}$ are homotopy equivalent as infinite loop spaces .—

3.5 Remarks:
(a) $B\mathcal{RF}_{\mathcal{P},\mathcal{D}}$ is a strong deformation retract of $B\mathcal{F}_{P,D}$.

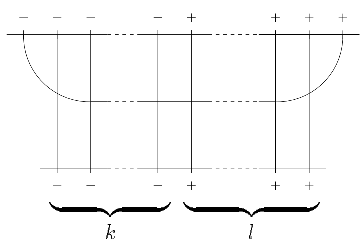
(b) $\mathcal{G}$ is a subcategory of $\mathcal{F}_{P,D}$ and the retraction R commutes with the inclusion, i.e. the inclusion $(B\mathcal{RG}, B\mathcal{RF}_{P,D}) \hookrightarrow (B\mathcal{G}, B\mathcal{F}_{P,D})$ is a homotopy equivalence of pairs .—

Proof: For (a) compare 2.6, and (b) is a trivial consequence .—

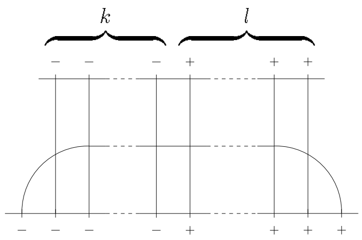
Directly from 3.3 we obtain

3.6 Proposition: Under composition, $\mathcal{F}_{P,D}$ is generated by the symmetry isomorphisms $c_{v,w}$ and the following morphisms $\beta_- = \beta_-^{(k,l)}$, $\beta_+ = \beta_+^{(k,l)}$

$\beta_+ :=$



$\beta_- :=$



where the numbers k and l vary in $\mathbb{N}$.—

3.2 The homotopy fibre of the inclusion $\mathcal{G} \hookrightarrow \mathcal{F}_{P,D}$

The homotopy type of $\mathcal{F}_{P,D}$ is to be investigated. $\mathcal{G}$ is a subcategory of $\mathcal{F}_{P,D}$, the homotopy type of which is already known to be $\Omega^\infty S^\infty$. Let $I : \mathcal{G} \hookrightarrow \mathcal{F}_{P,D}$ denote the inclusion functor. We start with the following observation:

3.7 Observation:
$\pi_0(B\mathcal{F}_{P,D}) = \mathbb{Z}$ as a group. In particular, $B\mathcal{F}_{P,D}$ is an infinite loop space. $\pi_0 BI : \pi_0(B\mathcal{G}) \longrightarrow \pi_0(B\mathcal{F}_{P,D})$ is an isomorphism of groups .—

Proof: The proof is essentially the same as for 2.8.
Again, define $n_+, n_- : ob\,\mathcal{F}_{P,D} \longrightarrow \mathbb{N}$ to be the functions which associate to a word w the number of its letters "+" and "−" respectively, and

$$K(w) := n_+(w) - n_-(w) \in \mathbb{Z}.$$

According to 3.3 (b), $\mathcal{F}_{P,D}$ is generated by the structure morphisms b_w, d_w and $c_{v,w}$. For these morphisms the invariant K yields the same value on the source and target. The same holds for tensor products of these generators because of the equation $K(v \oplus w) = K(v) + K(w)$. Every morphism can be expressed as a composition of tensor products of the generating morphisms. We see that again $K(v) = K(w)$ is necessary for v and w being of the same component. The rest of the proof of the first statement is exactly the same as in 2.8.

The inclusion I is surjective on objects, hence $\pi_0 BI$ is also a surjection. Since $\pi_0(B\mathcal{G}) = \mathbb{Z}$, we have a surjection $\mathbb{Z} \longrightarrow \mathbb{Z}$ on path components which is a monoid homomorphism, because the inclusion functor is strictly monoidal. Hence this has to be an isomorphism .—

Recall Quillen's famous Theorem B [13]:
Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor and denote objects of $\mathcal{C}$ by X, X' and objects of $\mathcal{C}'$ by Y, Y'. If Y is a fixed object of $\mathcal{C}'$, let F/Y denote the category consisting of pairs (X, u) with $u : F(X) \longrightarrow Y$, in which a morphism from (X, u) to (X', u') is a map $v : X \longrightarrow X'$ such that $u = u' \circ F(v)$. In particular, when F is the identity functor of $\mathcal{C}'$, we obtain the category $\mathcal{C}'/Y$ of objects over Y which is contractible, because (Y, id_Y) is a final object.
Every arrow $f : Y \longrightarrow Y'$ in $\mathcal{C}'$ induces a functor $f_* : F/Y \longrightarrow F/Y'$ by

$$f_*(X, u) = (X, f \circ u), \qquad f_*(v) = v.$$

3.8 Theorem: (Quillen's Theorem B (dual version), [13]) Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor such that for every arrow $f : Y \longrightarrow Y'$ in $\mathcal{C}'$ the induced functor $f_* : F/Y \longrightarrow F/Y'$ gives a homotopy equivalence on classifying spaces. Then for any object Y of $\mathcal{C}'$ the cartesian square of categories

$$\begin{tikzcd} F/Y \arrow[r, "j"] \arrow[d, "F'"'] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C}'/Y \arrow[r, "j'"] & \mathcal{C}' \end{tikzcd} \qquad \begin{aligned} j(X,v) &= X \\ F'(X,v) &= (F(X),v) \\ j'(Y',v) &= Y' \end{aligned}$$

is homotopy-cartesian (i.e. homotopy-cartesian after applying the functor B). Consequently, for any X in $F^{-1}(Y)$ we have an exact sequence

$$\cdots \to \pi_{i+1}(B\mathcal{C}', Y) \to \pi_i(B(F/Y), \overline{X}) \xrightarrow{j_*} \pi_i(B\mathcal{C}, X) \xrightarrow{F_*} \pi_i(B\mathcal{C}', Y) \to \cdots$$

where $\overline{X} = (X, id_Y)$.—

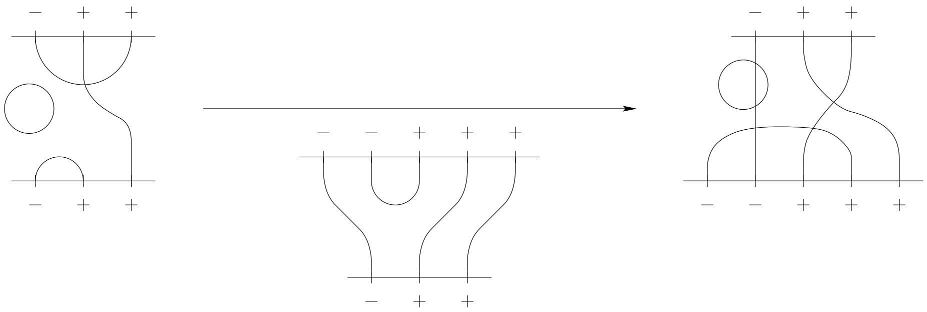
The spaces $B\mathcal{G}$ and $B\mathcal{F}_{P,D}$ are infinite loop spaces. So, all their path components are homotopy equivalent, and for homotopy questions it is sufficient to concentrate on a single chosen component. Fortunately, Theorem B can be applied after restricting $I : \mathcal{G} \hookrightarrow \mathcal{F}_{P,D}$ to the component of the object $+$. Again, the appropriate subcategories are denoted $\mathcal{G}^{(1)}$ and $\mathcal{F}^{(1)}_{P,D}$, respectively.

3.9 Theorem: The inclusion functor $I : \mathcal{G}^{(1)} \hookrightarrow \mathcal{F}^{(1)}_{P,D}$ satisfies the condition of Theorem B. Hence $I/+$ is a category whose classifying space describes the homotopy fibre of BI .—

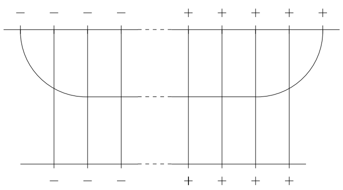
Proof: According to 3.4, the category $\mathcal{F}_{P,D}$ may be retracted into its subcategory of ordered words $\mathcal{RF}_{P,D}$ (cf. 2.5), and the inclusion $(B\mathcal{RG}, B\mathcal{RF}_{P,D}) \hookrightarrow (B\mathcal{G}, B\mathcal{F}_{P,D})$ is a homotopy equivalence of pairs (3.5). This remains the same after restriction to the component of the object $+$. Hence, for technical simplification we work in these subcategories and consider the inclusion $I : \mathcal{RG}^{(1)} \hookrightarrow \mathcal{RF}^{(1)}_{P,D}$. Note that $\mathcal{RG}^{(1)}$ and $\mathcal{RF}^{(1)}_{P,D}$ are still full subcategories of $\mathcal{G}$ and $\mathcal{F}_{P,D}$, respectively.

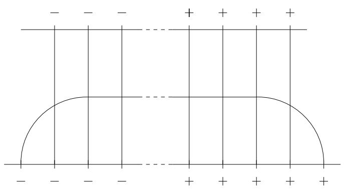
The set of objects for both $\mathcal{RG}^{(1)}$ and $\mathcal{RF}^{(1)}_{P,D}$ is $\{+, -++, --+++, \ldots\}$ and these words will be denoted $w_0, w_1, w_2, \ldots$ according to the number of letters "$-$".
An object of I/w_k is a pair (w, α) with $w \in ob\, \mathcal{RG}^{(1)}$ and $\alpha : I(w) = w \longrightarrow w_k$ being a morphism in $\mathcal{F}_{P,D}$. Obviously, w is determined by α and can be

dropped from the notation. But for a better distinction, objects are sometimes denoted by $(w \uparrow_\alpha w_k)$. A morphism from $(w \uparrow_\alpha w_k)$ to $(w' \uparrow_{\alpha'} w_k)$ is a morphism $w \xrightarrow{\varphi} w'$ in $\mathcal{G}$ with $\alpha = \alpha' \circ I(\varphi)$, i.e. $\alpha = \alpha' \circ \varphi$.
For example, a typical morphism in I/w_1 can be pictured as follows:



Every arrow $\beta : w_k \longrightarrow w_l$ in $\mathcal{RF}^{(1)}_{P,D}$ induces a functor $\beta_* : I/w_k \longrightarrow I/w_l$ by $\beta_*(w \uparrow_\alpha w_k) = (w \uparrow_{\beta\circ\alpha} w_l)$ on objects and $\beta_*(\varphi) = \varphi$ on morphisms, which has to be a homotopy equivalence on classifying spaces.
Clearly isomorphisms induce homotopy equivalences. Because of 3.6 it suffices to prove the assertion for the morphisms $\beta_+ : w_k \longrightarrow w_{k+1}$ and $\beta_- : w_{k+1} \longrightarrow w_k$:

$\beta_+ \;:=$ 

$\beta_- \;:=$ 

We will prove the following claim:

β_+ and β_- induce homotopy inverse maps between $B(I/w_k)$ and $B(I/w_{k+1})$.

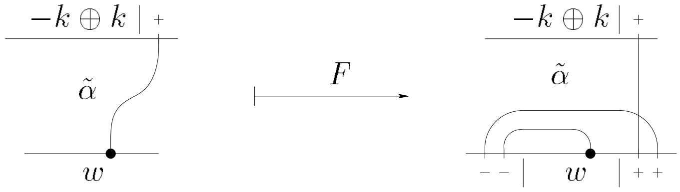
The value of the functor $(\beta_- \circ \beta_+)_* : I/w_k \longrightarrow I/w_k$ is given on an object by placing $\beta_- \circ \beta_+$ on top of it, which is nothing but the identity plus one cycle. So $(\beta_- \circ \beta_+)_*$ simply adds one cycle to every object and there is no natural transformation between Id_{I/w_k} and this functor, since the morphisms of $\mathcal{G}$ contain no cycles. But there is a certain functor $F : I/w_k \longrightarrow I/w_k$ and there are natural transformations

$$Id_{I/w_k} \xrightarrow{\eta_t} F \xleftarrow{\eta_u} (\beta_- \circ \beta_+)_*$$

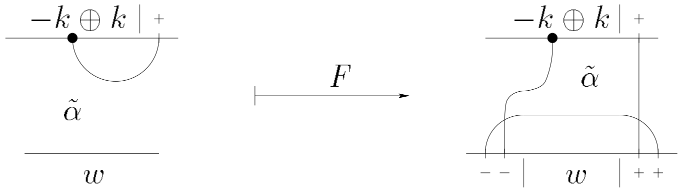
which together yield the desired homotopy.

The functor F associates to an object $(w \uparrow_\alpha w_k)$ a slightly modified object $((-2 \oplus w \oplus 2) \uparrow w_k)$ and the concrete definition depends on whether the final letter "+" of w_k belongs to a vertical string or to an upper arc (then the object α is said to be of type (S) or type (A), respectively).

(S)



(A)

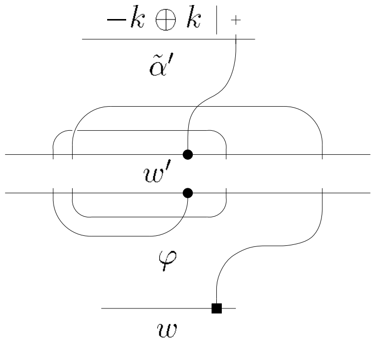


These pictures are to be understood as follows: The only important string of α is that which belongs to the final letter "+" of $w_k = -k \oplus k \oplus +$, called the "final string". The black dot marks the position where this string ends. For the definition of $F(w \uparrow_\alpha w_k)$ the final string is changed with the help of the additional letters at the bottom base and two further strings. All other strings of $F(w \uparrow_\alpha w_k)$ are given by the remaining strings of α (indicated by $\tilde{\alpha}$).

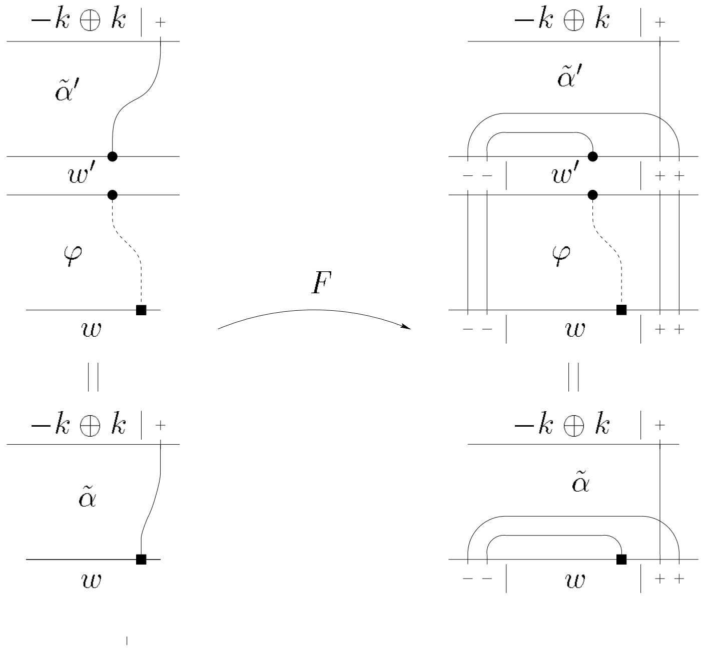
On morphisms $\alpha \xrightarrow{\varphi} \alpha'$ the functor F is given by $F(\varphi) := id_{-2} \oplus \varphi \oplus id_2$. It is clear that F satisfies the identities $F(id) = id$, $F(\varphi \circ \varphi') = F(\varphi) \circ F(\varphi')$. It has only to be shown that F is well-defined, i.e. that $F(\varphi)$ is a morphism from $F(w \uparrow_\alpha w_k)$ to $F(w' \uparrow_{\alpha'} w_k)$. The following cases, depending on the type of α and α', have to be distinguished:

Case 1: α and α' are both of type (S).

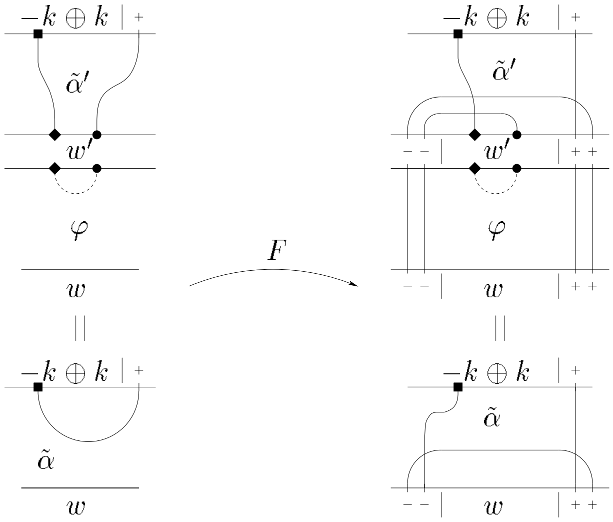
After composition with φ, the black dot of α' is somehow connected with the distinguished letter at the bottom base which belongs to the final string of α (indicated by the black square in the pictures below). This may happen by a simple vertical string, but it is also possible that several upper arcs of φ and lower arcs of α' are involved.



In any case, this does not matter, because F preserves all these possibly involved arcs, and the assertion can be seen from the following picture, in which the connection between the distinguished letters is drawn by a dashed line.



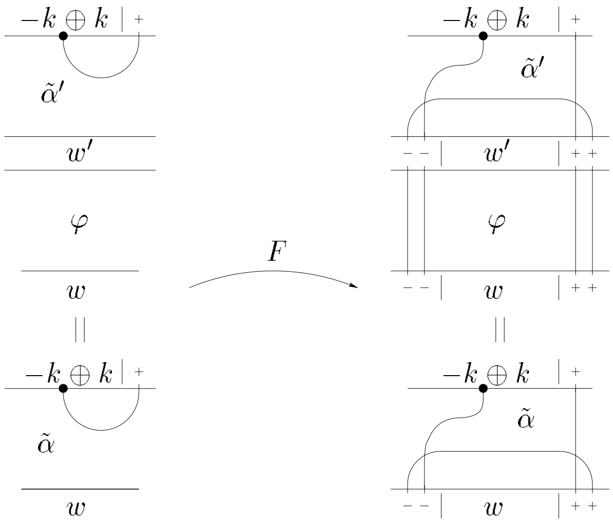
Case 2: α is of type (A) and α' is of type (S).



Again, there is some connection between the black dot and the black square, but the square is now located at the top of α' and is connected with another distinguished letter (black rhombus) at the bottom of α'. Hence there is a link between the dot and the rhombus which is given by a simple upper arc of φ or by a similar detour as in the preceding case.

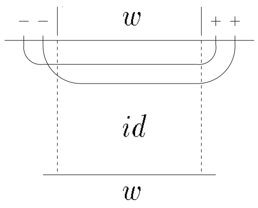
Case 3: α and α' are both of type (A).

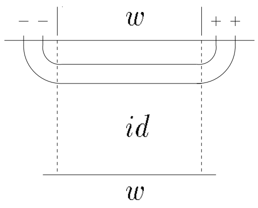
Here φ has no effect on the relevant arc of α and the situation is very easy:



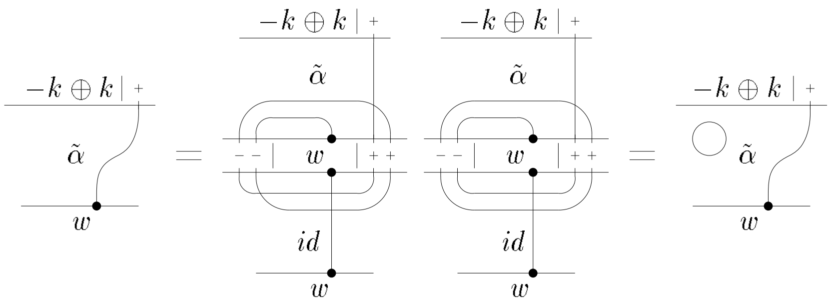
Obviously, the fourth case (α being of type (S), α' of type (A)) does not exist. Hence the well-definition of F is proved.

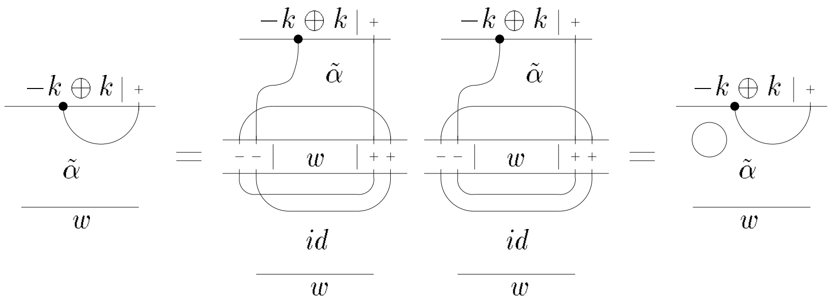
Now we give the stated natural transformations $Id_{I/w_k} \xrightarrow{\eta_t} F \xleftarrow{\eta_u} (\beta_- \circ \beta_+)_*$. For a given object $(w \uparrow_\alpha w_k)$ of I/w_k the transformations are defined by

$\eta_t(w \uparrow_\alpha w_k) :=$ 

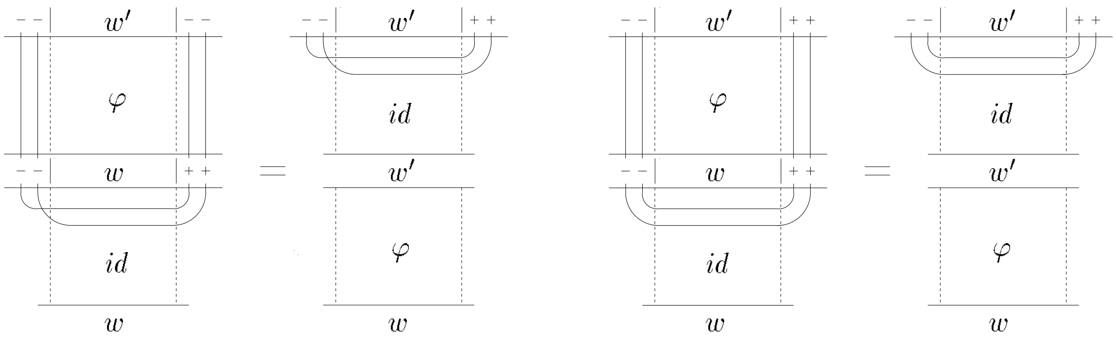
$\eta_u(w \uparrow_\alpha w_k) :=$ 

The following pictures show that $\eta_t(\alpha)$ and $\eta_u(\alpha)$ are actually morphisms from $Id(w \uparrow_\alpha w_k)$ and $(\beta_- \circ \beta_+)_*(w \uparrow_\alpha w_k)$, respectively, to $F(w \uparrow_\alpha w_k)$.

(S)


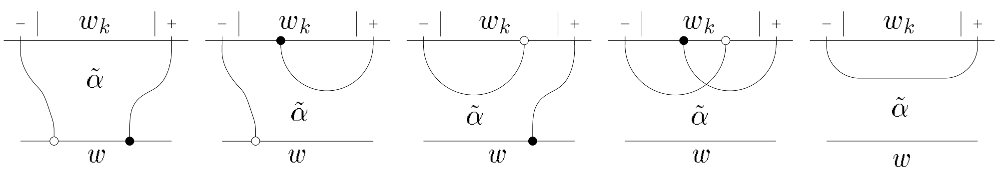
(A)


Finally, the equation $F(\varphi) \circ \eta(\alpha) = \eta(\alpha') \circ \varphi$ has to be checked for $\eta = \eta_t, \eta_u$. (Note that $Id(\varphi) = (\beta_- \circ \beta_+)_*(\varphi) = \varphi$). This is done by the next picture:



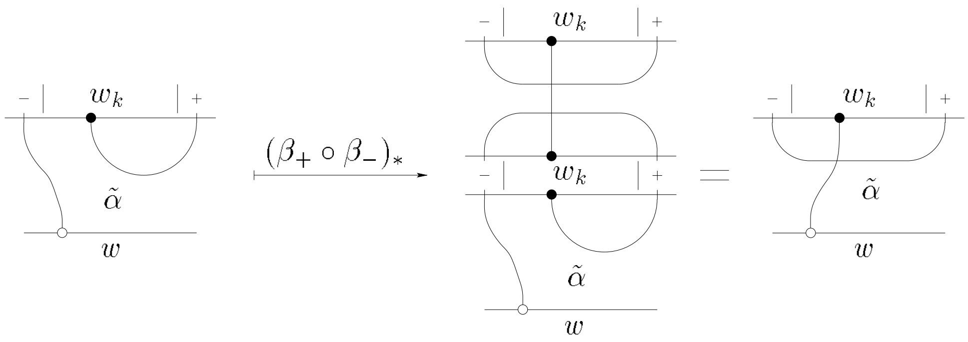
This completes the proof that there is a homotopy $B(\beta_- \circ \beta_+)_* \simeq id_{B(I/w_k)}$. Note that this also covers the case $k = 0$, when we have $w_k = +$. Then the object type (A) cannot occur, but this only reduces the number of cases which have to be checked in the following.

The way to prove $B(\beta_+ \circ \beta_-)_* \simeq id_{B(I/w_{k+1})}$ is very similar: one gives a functor $G : I/w_{k+1} \longrightarrow I/w_{k+1}$ and natural transformations $Id \xrightarrow{\eta_t} G \xleftarrow{\eta_u} (\beta_+ \circ \beta_-)_*$, but there are more cases which have to be distinguished. We have five types of objects $(w \uparrow_\alpha w_{k+1})$ in I/w_{k+1}, depending on the shape of the strings belonging to the initial and to the final letter of $w_{k+1} = - \oplus w_k \oplus +$:

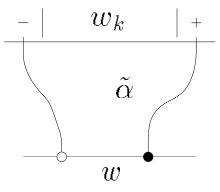
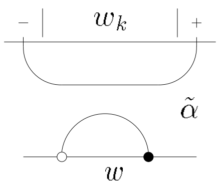
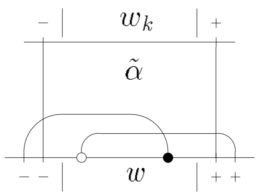
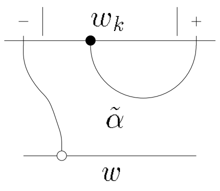
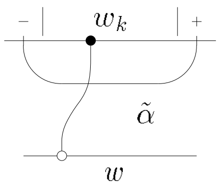
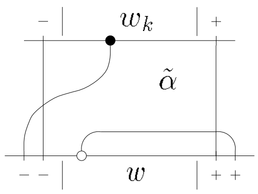
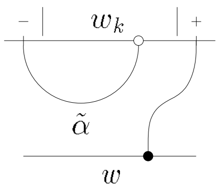
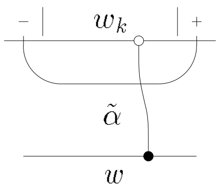
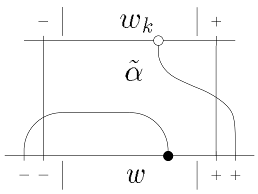
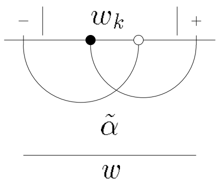
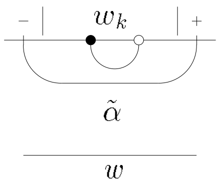
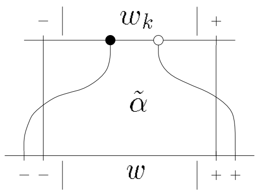
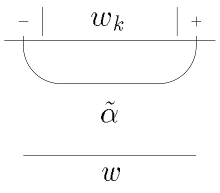
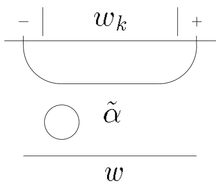
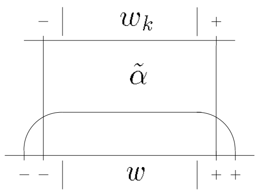


(SS) (SA) (AS) (AA) (C)

The values $(\beta_+ \circ \beta_-)_*(w \uparrow_\alpha w_{k+1})$ are computed by placing $\beta_+ \circ \beta_-$ on top of these objects. As an example, the following picture shows the computation for the type (SA):

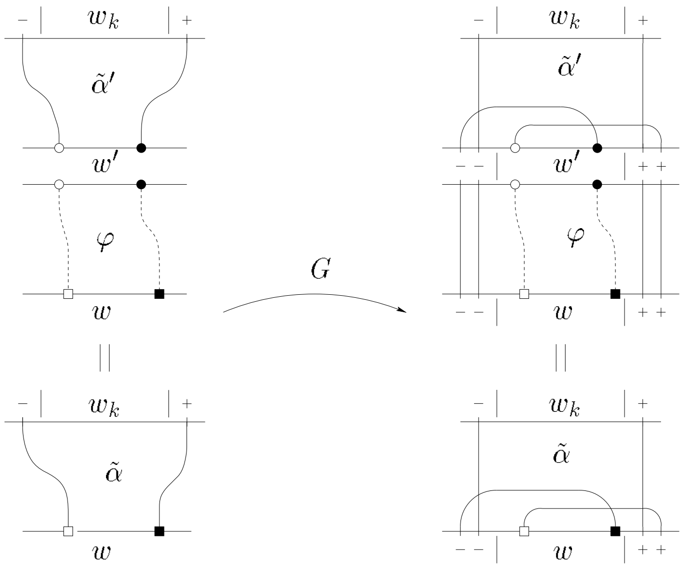


The following table shows the different types in the first column, the appropriate values under the functor $(\beta_+ \circ \beta_-)_*$ in the second column and the definition of G on these different types in the final column:

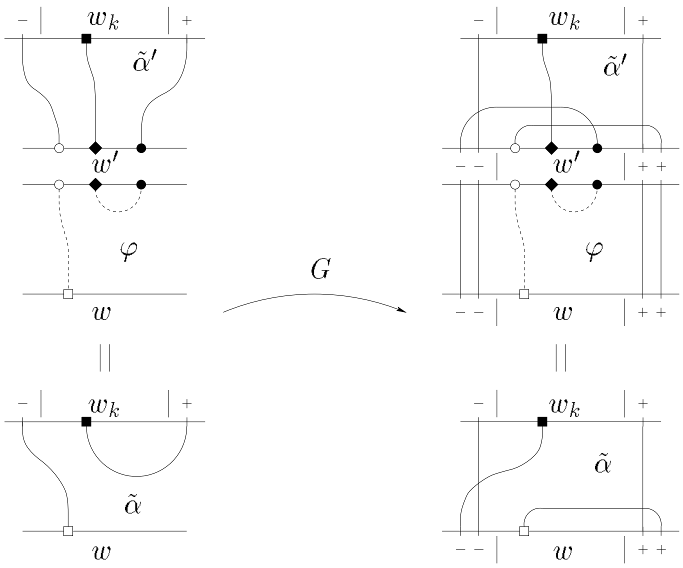
	$(w \uparrow_\alpha w_{k+1})$	$(\beta_+ \circ \beta_-)_*(w \uparrow_\alpha w_{k+1})$	$G(w \uparrow_\alpha w_{k+1})$
(SS)			
(SA)			
(AS)			
(AA)			
(C)			

Again, G is defined by $G(\varphi) = id_{-2} \oplus \varphi \oplus id_2$ on morphisms. The following pictures show that G is well-defined:

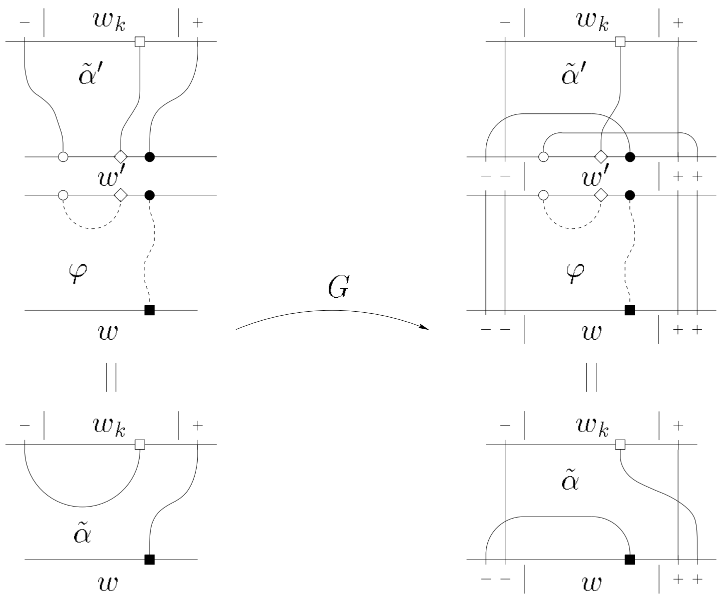
Case 1: α and α' are both of type (SS).



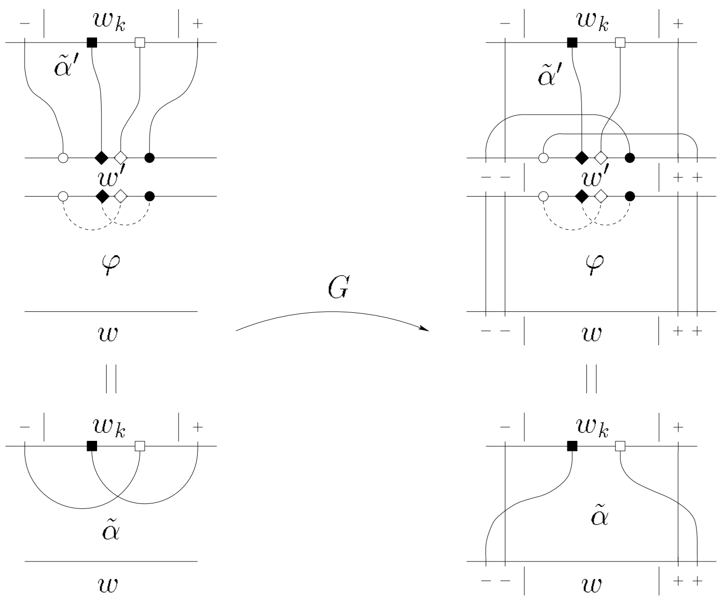
Case 2: α is of type (SA) and α' is of type (SS).



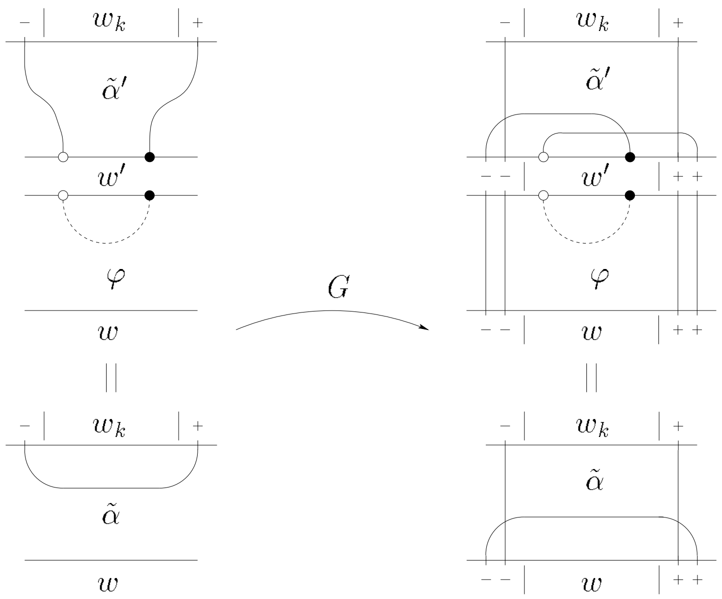
Case 3: α is of type (AS) and α' is of type (SS).



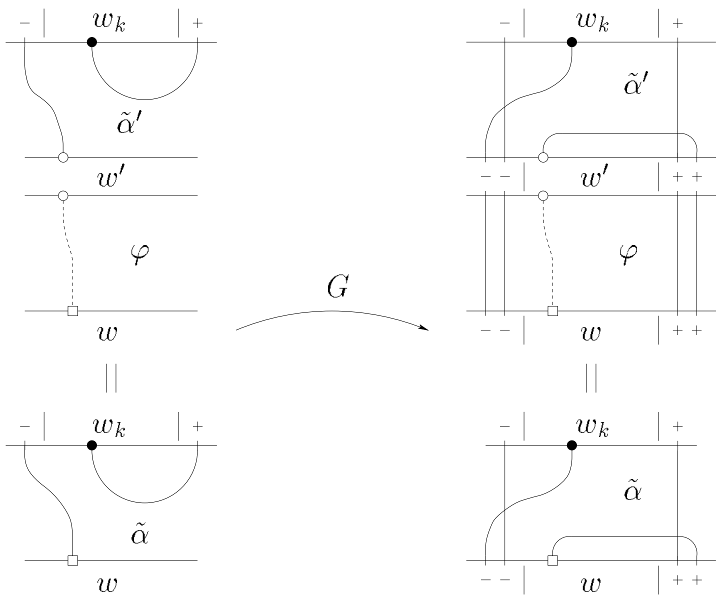
Case 4: α is of type (AA) and α' is of type (SS).



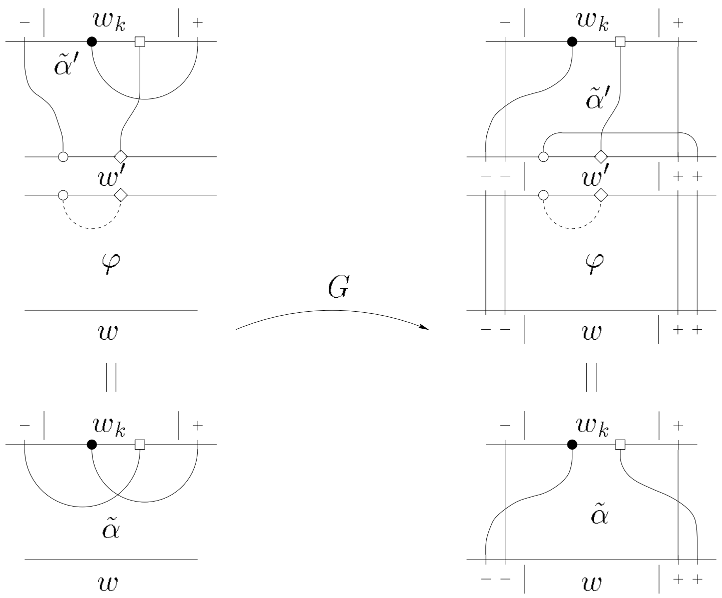
Case 5: α is of type (C) and α' is of type (SS).



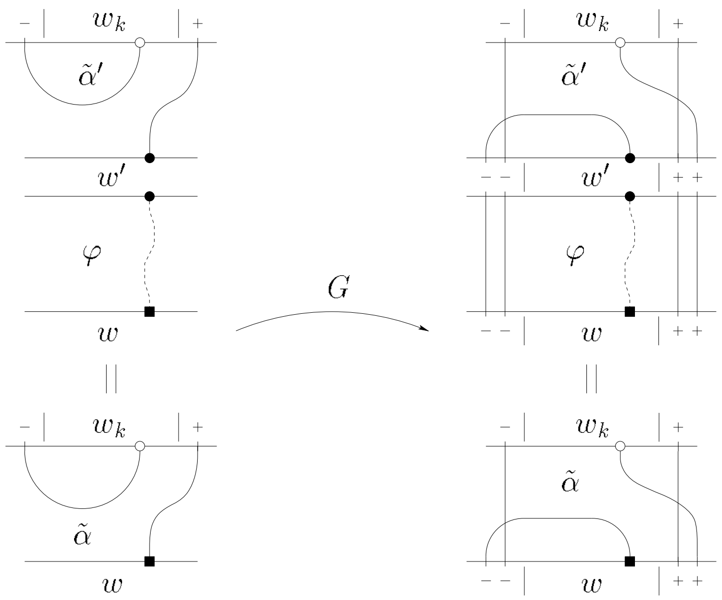
Case 6: α and α' are both of type (SA).



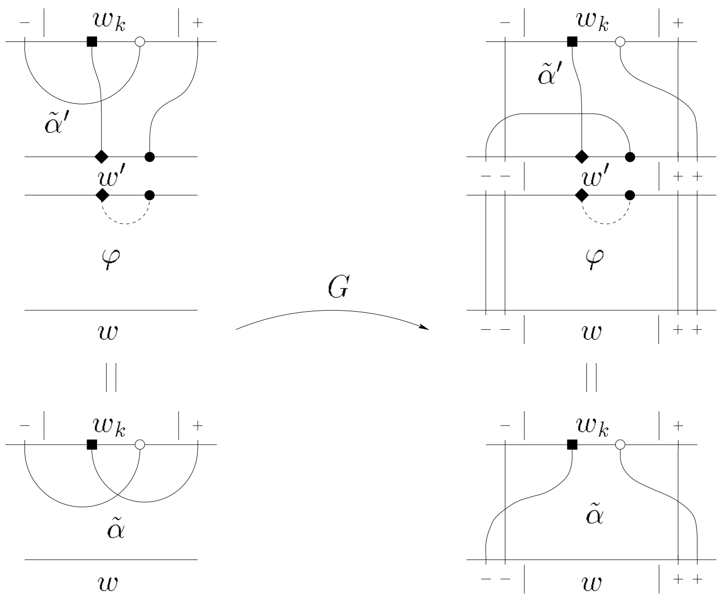
Case 7: α is of type (AA) and α' is of type (SA).



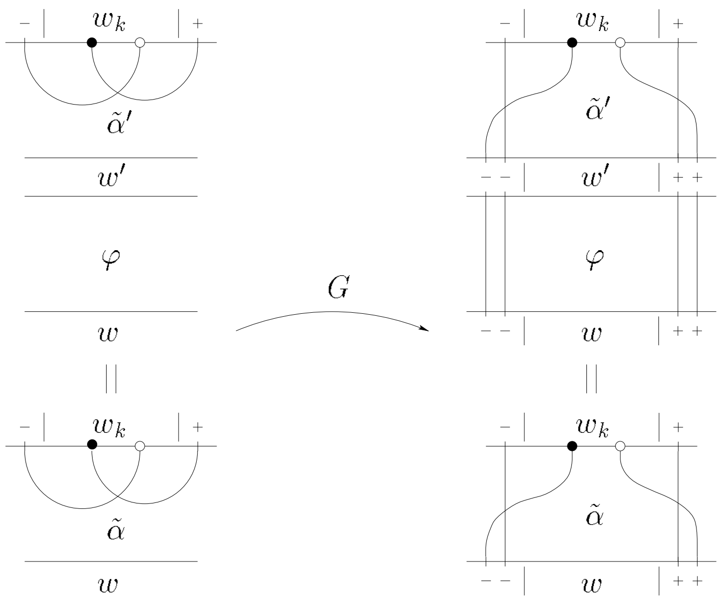
Case 8: α and α' are both of type (AS).



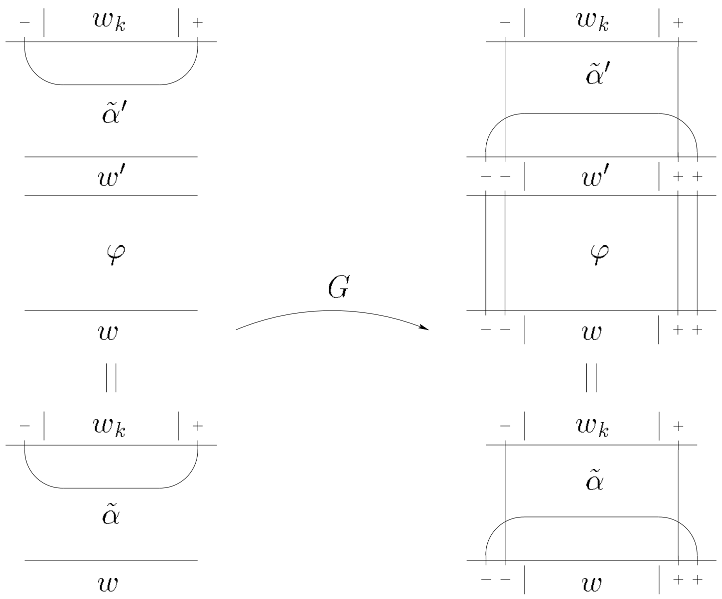
Case 9: α is of type (AA) and α' is of type (AS).



Case 10: α and α' are both of type (AA).

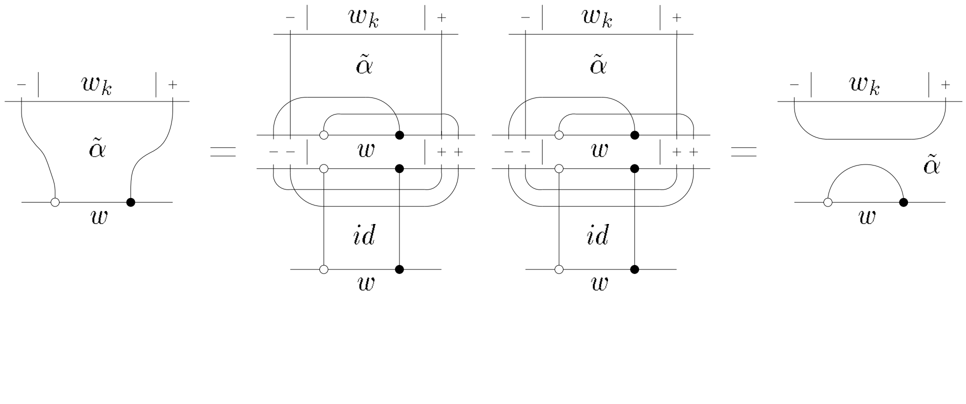


Case 11: α and α' are both of type (C).

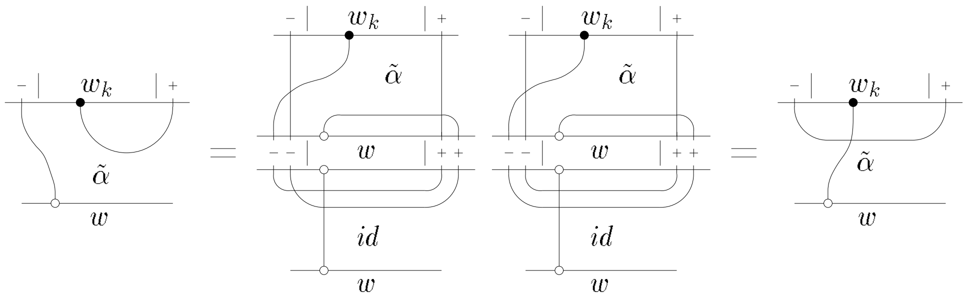


These are all possible transitions between the different types, hence G is well-defined.

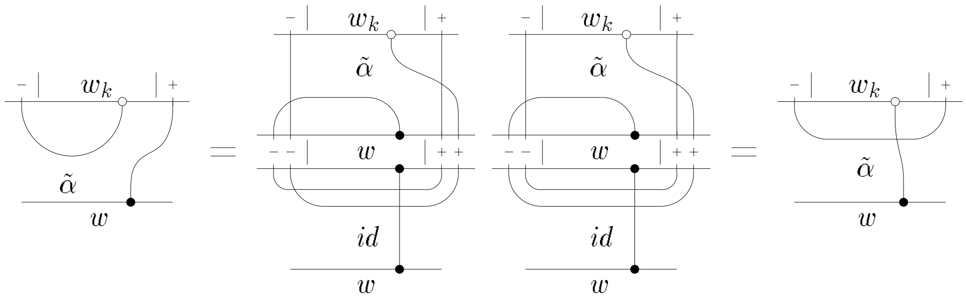
The natural transformations $Id \xrightarrow{\eta_t} G \xleftarrow{\eta_u} (\beta_+ \circ \beta_-)_*$ are defined exactly in the same way as before. Hence the only thing which remains to be shown is that $\eta_t(\alpha)$ and $\eta_u(\alpha)$ are actually morphisms from $Id(\alpha)$ and $(\beta_+ \circ \beta_-)_*(\alpha)$, respectively, to $G(\alpha)$. This is done by the following pictures:

(SS) 

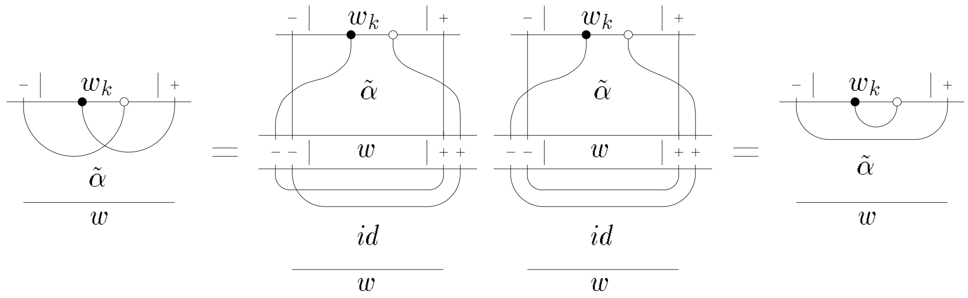
(SA)



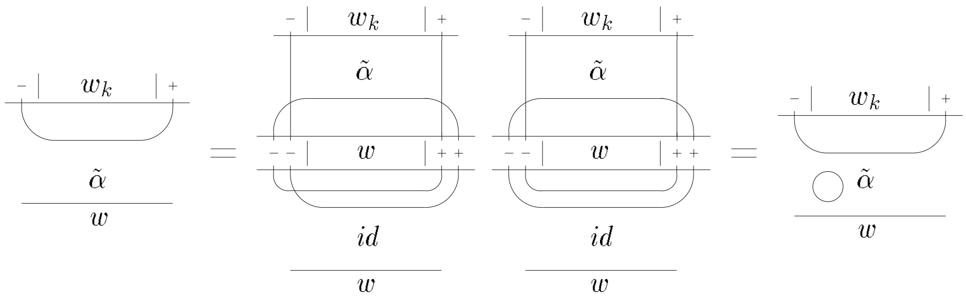
(AS)



(AA)



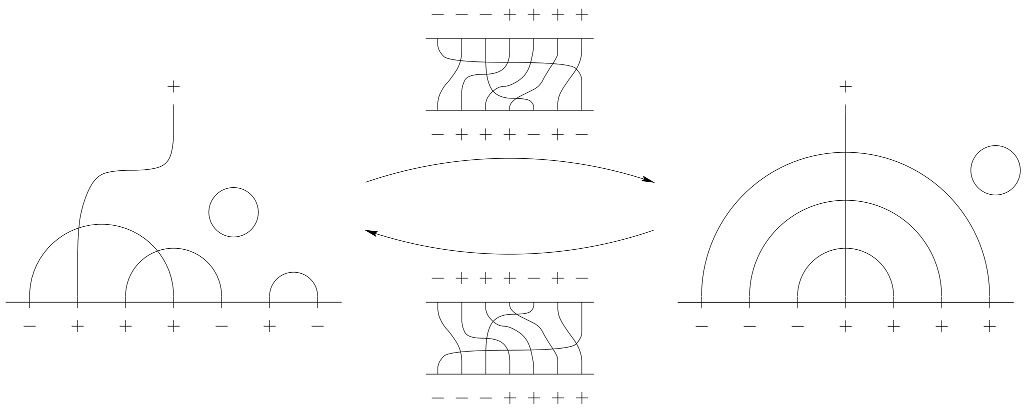
(C)



Note that this also covers the case $k = 0$, i.e. $w_{k+1} = -++$, where the object types (SA) and (AA) do not occur. Again, this only reduces the number of cases which have to be checked. This completes the proof of 3.9 .—

3.10 Remarks:
Note that the analogous statement of 3.9 for the component of the object 0 is false; the inclusion $I : \mathcal{G}^{(0)} \longrightarrow \mathcal{F}^{(0)}_{P,D}$ does not fullfil the condition of Theorem B, because $I/0$ is not connected, while all other I/w are (in $I/0$ the object $(0 \uparrow_{id} 0)$ is not linked with any other object). But any other component except the zeroth could probably also have been chosen.
We have worked with the dual formulation of Theorem B. It is not possible to use the original formulation, since all categories $w\backslash I$ already have completely different sets of path components .—

Theorem B can be applied to the inclusion $I : \mathcal{G}^{(1)} \longrightarrow \mathcal{F}^{(1)}_{P,D}$, all categories I/w are homotopy equivalent and yield categorical models for the homotopy fibre of the inclusion $B\mathcal{G}^{(1)} \hookrightarrow B\mathcal{F}^{(1)}_{P,D}$. For an analysis of this homotopy fibre, it is sufficient to choose the technically most simple one, which is $I/+$, because every object $(w \uparrow_\alpha +)$ consists of only lower arcs, cycles and exactly one vertical string.
A further step of simplification is the passage to an equivalent subcategory of $I/+$ by choosing a skeleton, i.e. a full subcategory such that any object of $I/+$ is isomorphic to exactly one object of this subcategory (cf. [8]).
The isomorphisms of $I/+$ are defined by isomorphisms in $\mathcal{G}$. Hence any two objects $(w \uparrow_\alpha +)$, $(w' \uparrow_{\alpha'} +)$ are isomorphic if and only if the numbers of lower arcs and cycles coincide. We define $\mathcal{H}$ to be the full subcategory of $I/+$ which consists of objects $(w \uparrow_\alpha +)$ in which w is ordered, the vertical string links the upper "+" with the "+" in the middle of w and all lower arcs have no intersections among them. This category $\mathcal{H}$ is a skeleton of $I/+$.
The next picture shows the link between an arbitrary object of $I/+$ and the corresponding object from $\mathcal{H}$ by an isomorphism (which is not unique):



We have the following corollary from 3.9:

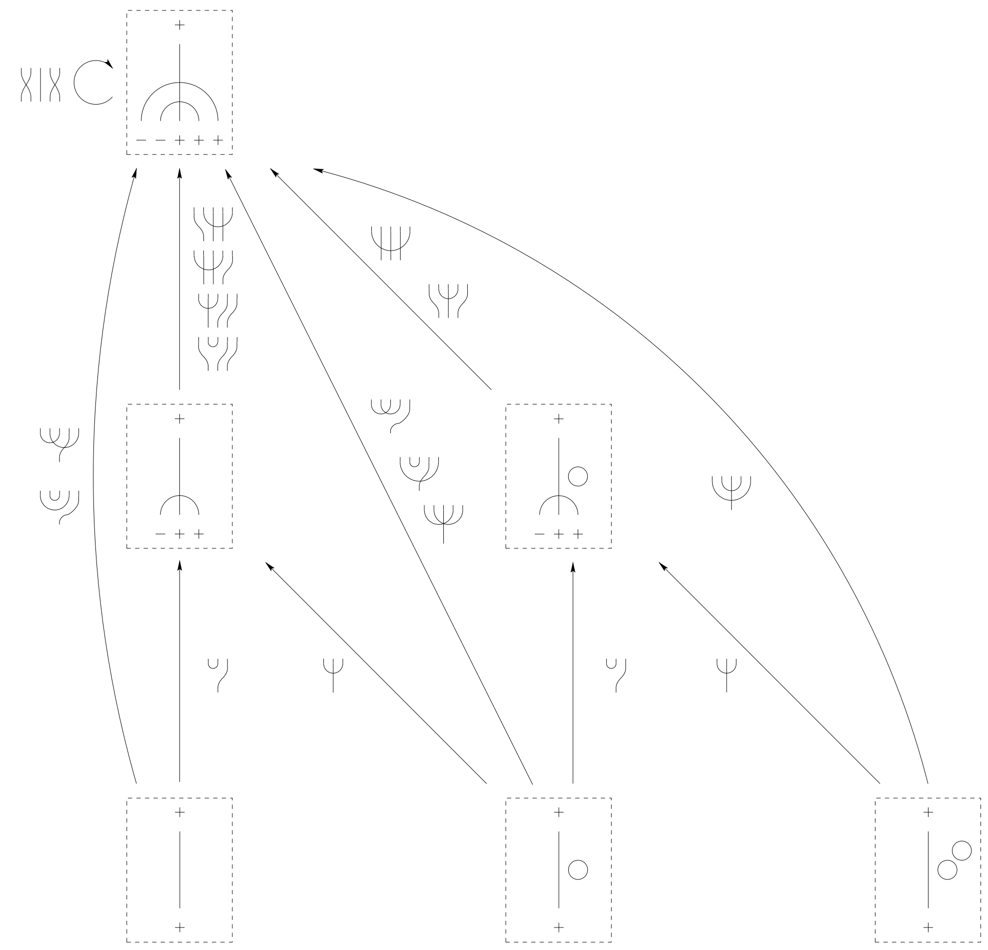
3.11 Corollary: For any choice of basepoint in $\mathcal{H}$ there is an exact sequence

$$\cdots \xrightarrow{I_*} \pi_{i+1}B\mathcal{F}^{(1)}_{P,D} \longrightarrow \pi_i B\mathcal{H} \xrightarrow{p_*} \pi_i B\mathcal{G}^{(1)} \xrightarrow{I_*} \pi_i B\mathcal{F}^{(1)}_{P,D} \longrightarrow \cdots$$

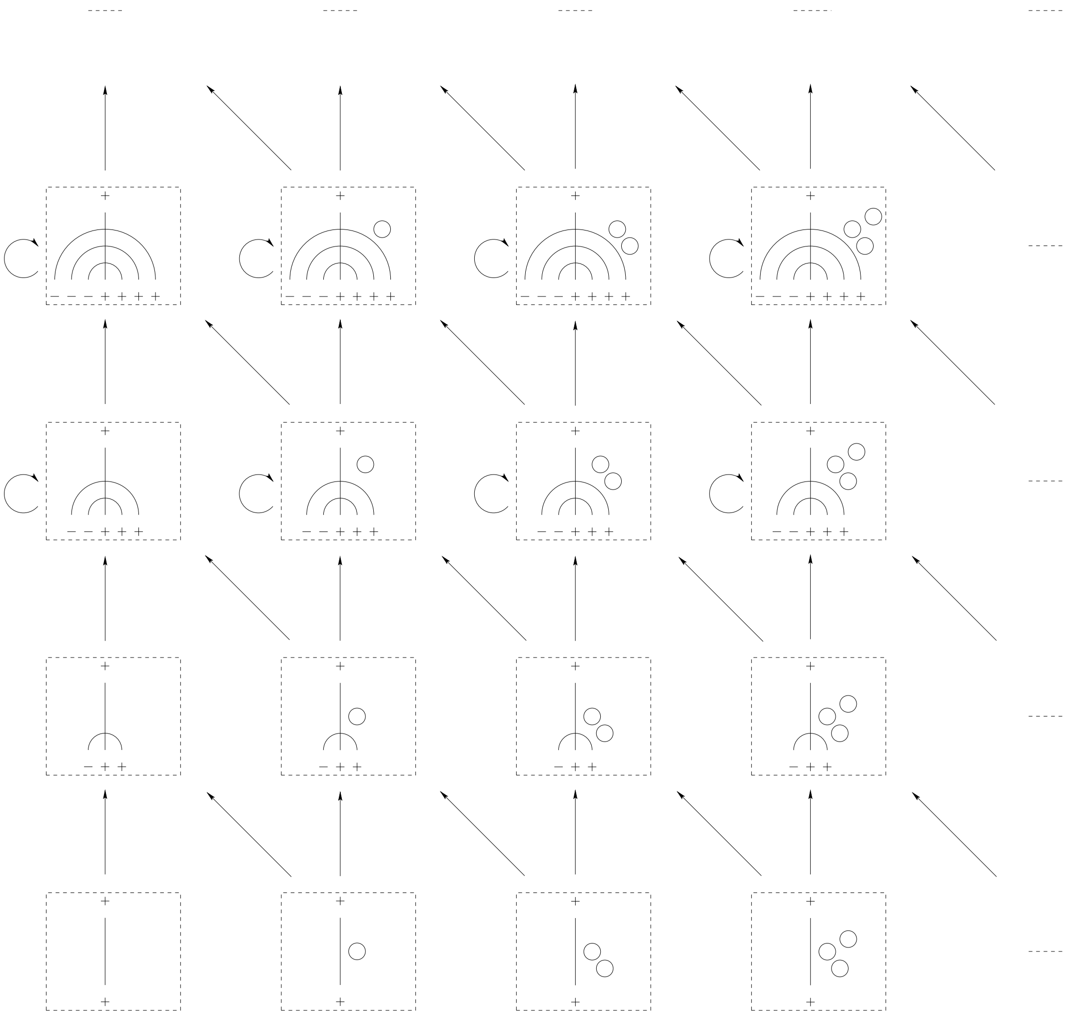
where $p : \mathcal{H} \longrightarrow \mathcal{G}^{(1)}$ is the functor given by

$$p(w \uparrow_\alpha +) = w, \qquad p(\varphi) = \varphi \qquad .\!—$$

We start with the analysis of $\mathcal{H}$. To get used to the structure of $\mathcal{H}$, the following picture displays some objects and all non-identity morphisms between them:



This leads to the following scheme (each arrow stands for a collection of morphisms):



Obviously, the scheme contains all objects. The arrows indicate generating morphisms, hence there is a morphism from one object to another if and only if there is a path along some arrows in the scheme (cf. 3.15 below).
The diagonal arrows stand for morphisms which open cycles to lower arcs. The vertical arrows indicate morphisms which generate a lower arc without decreasing the number of cycles by decomposing the vertical string or another lower arc in a suitable way. Finally, the circular arrows stand for non-trivial automorphisms by permuting the lower arcs. We arrive at a first observation:

3.12 Observation: $\mathcal{H}$ is path-connected, i.e. $\pi_0(B\mathcal{H}) = 0$.—

The following result, which is our next goal, is more complicated:

3.13 Theorem: $\mathcal{H}$ is simply-connected, i.e. $\pi_1(B\mathcal{H}) = 0$.—

Before starting with the proof of the theorem, we record some crucial properties of $\mathcal{H}$ in the following lemma. Let (c, a) denote the object of $\mathcal{H}$ with c cycles and a arcs, and Δ_n the full subcategory of $\mathcal{H}$ which consists of all objects (c, a) with $c + a = n$.

3.14 Lemma:

(a) Δ_n is contractible.

(b) Each morphism $(c, a) \longrightarrow (0, n)$ in $\mathcal{H}$ can be factored (non-uniquely) as

$$(c, a) \longrightarrow (0, c + a) \longrightarrow (0, n).$$

(c) Each morphism $(0, n') \longrightarrow (0, n)$ in $\mathcal{H}$ can be factored (non-uniquely) as

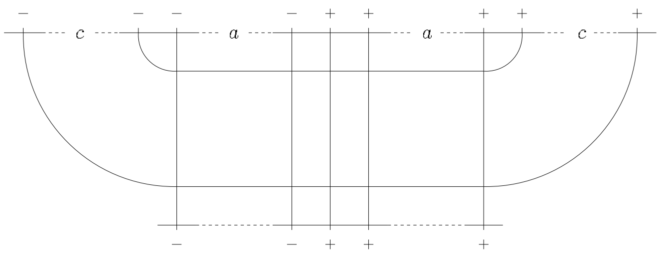
$$(0, n') \longrightarrow (0, k) \longrightarrow (0, n),$$

where $(0, k)$ is any chosen object with $n' \leq k \leq n$.

(d) To each pair $\varphi_1, \varphi_2 : (0, 0) \longrightarrow (0, n)$ in $\mathcal{H}$ there exists an automorphism $\sigma : (0, n) \longrightarrow (0, n)$ with $\sigma \circ \varphi_1 = \varphi_2$.—

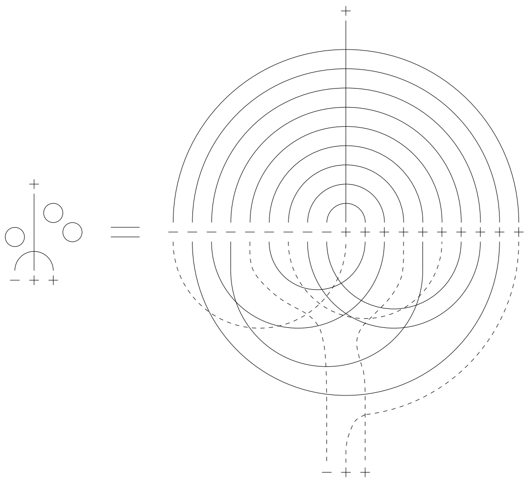
Proof: (a) is trivial since $(n, 0)$ is an initial object in Δ_n.

(b) The factorization mentioned in the lemma is not unique at all, and we give only one construction, which of course contains various choices.
Let $\varphi : (c, a) \longrightarrow (0, n)$ be any given morphism and $\psi : (c, a) \longrightarrow (0, c + a)$ the morphism, which is defined by the following picture.

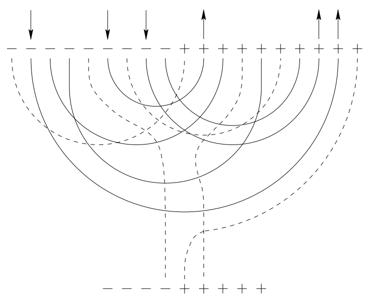
$\psi :=$ 

We will construct a morphism $\tilde{\varphi} : (0, c + a) \longrightarrow (0, n)$ with $\varphi = \tilde{\varphi} \circ \psi$.
Recall that $(0, n)$ and (c, a) can be interpreted as morphisms in $\mathcal{F}_{P,D}$ with $(c, a) = (0, n) \circ \varphi$. The strings of φ can be divided into two classes: those upper arcs which contribute to cycles when composing φ with $(0, n)$ (called

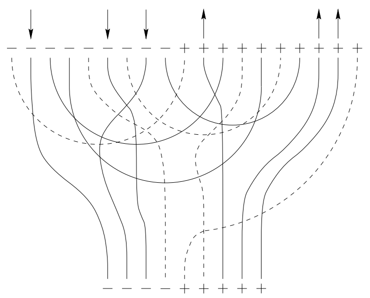
"cyclic arcs"), and all other strings which contribute to the vertical string or lower arcs of (c, a), respectively (called the "remainder"). The definition of $\tilde{\varphi}$ preserves the structure of the remainder and several cyclic arcs, while certain cyclic arcs are "opened" to become closed again by ψ. A more exact description is as follows:



Divide φ into cyclic arcs and the remainder. (The remainder is represented by dashed lines in the picture.)



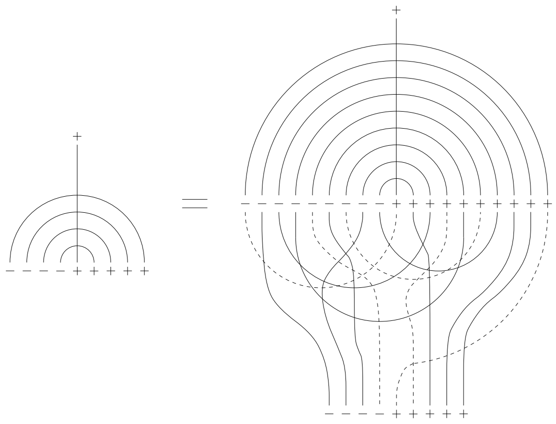
Choose c cyclic arcs by taking one from each set of arcs which belong to the same cycle after composing with $(0, n)$, and add c letters "$-$" and c letters "$+$" to the bottom base.

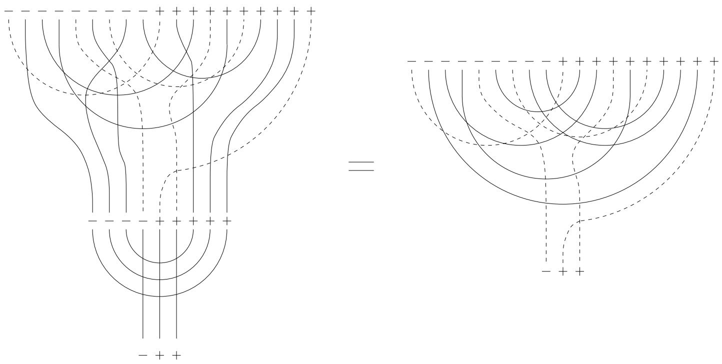


"Open" the marked arcs by connecting the arrowed positions with the additional letters in the bottom base such that the ith letter "$-$" from the left and the ith letter "$+$" from the right belong to the same opened arc. Preserve the rest of the cyclic arcs and the remainder.

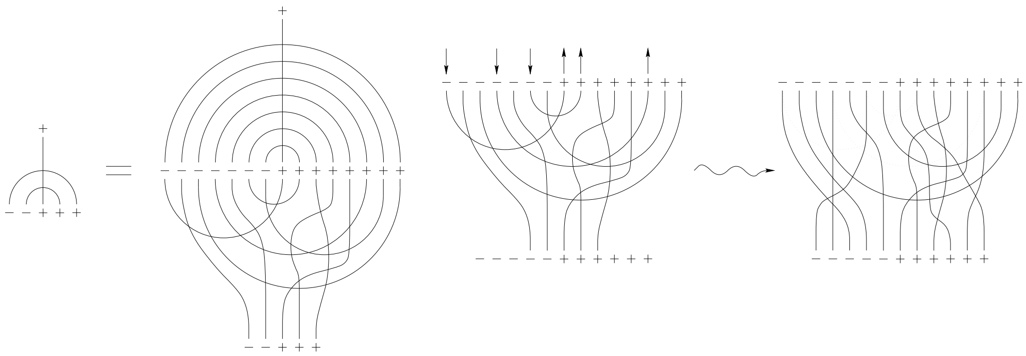
The resulting pseudo-tangle is the desired $\tilde{\varphi}$ because it is indeed a morphism from $(0, c + a)$ to $(0, n)$ and it satisfies the equation $\tilde{\varphi} \circ \psi = \varphi$.

The following pictures clarify these properties in our concrete example.





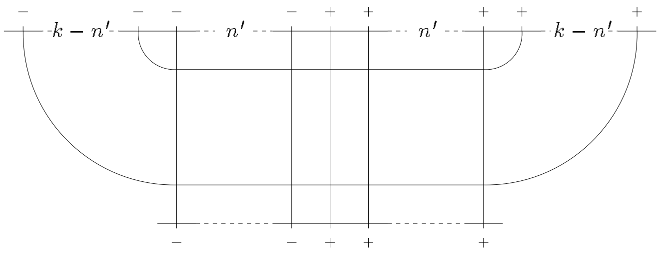
(c) Let $\varphi : (0, n') \longrightarrow (0, n)$ be any given morphism and $(0, k)$ another object with $n' \leq k \leq n$. Create a pseudo-tangle $\bar{\varphi}$ as follows:
Add $k - n'$ letters "$-$" and "$+$", respectively, to the bottom base of φ.
Choose $k - n'$ upper arcs and open them in the sense of the preceding proof.



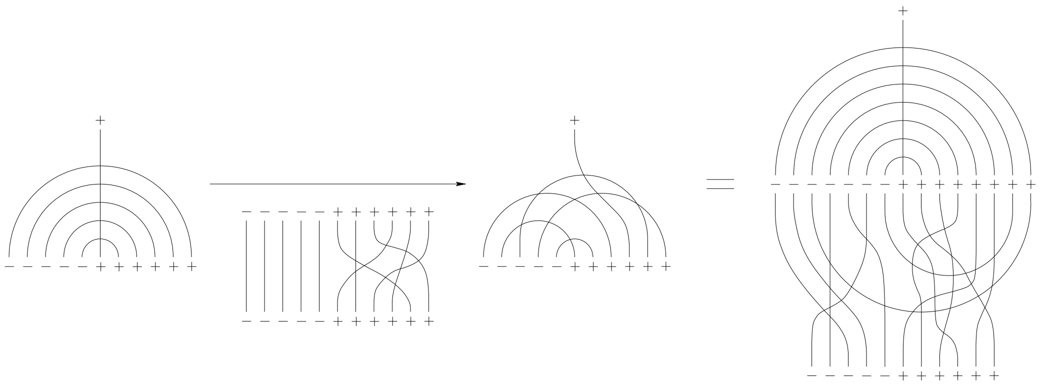
(In this example we have $n = 7,\;\; n' = 2,\;\; k = 5.$)

Obviously, if $\bar{\varphi}$ is composed with $\bar{\psi}$, where $\bar{\psi}$ is defined by

$\bar{\psi}$:=



then we arrive at φ again. One problem remains: If $\bar{\varphi}$ is interpreted as a morphism with target $(0, n)$, then the source object is not necessarily $(0, k)$. But it is an object $((-k \oplus k+1) \uparrow_\alpha +)$ in $I/+$ which is isomorphic to $(0, k)$, since it contains k lower arcs and no cycles.

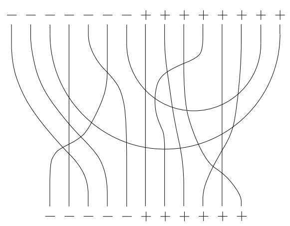


Hence if we choose an isomorphism $(0, k) \xrightarrow{\sigma} (-k \oplus (k+1) \uparrow_\alpha +)$ in $I/+$ (which has already been done in the upper picture) and define

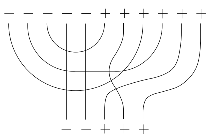
$$\tilde{\varphi} := \bar{\varphi} \circ \sigma, \qquad \psi := \sigma^{-1} \circ \bar{\psi},$$

then we obtain $(0, n') \xrightarrow{\psi} (0, k) \xrightarrow{\tilde{\varphi}} (0, n)$ with $\tilde{\varphi} \circ \psi = \varphi$.

$\tilde{\varphi}$ =

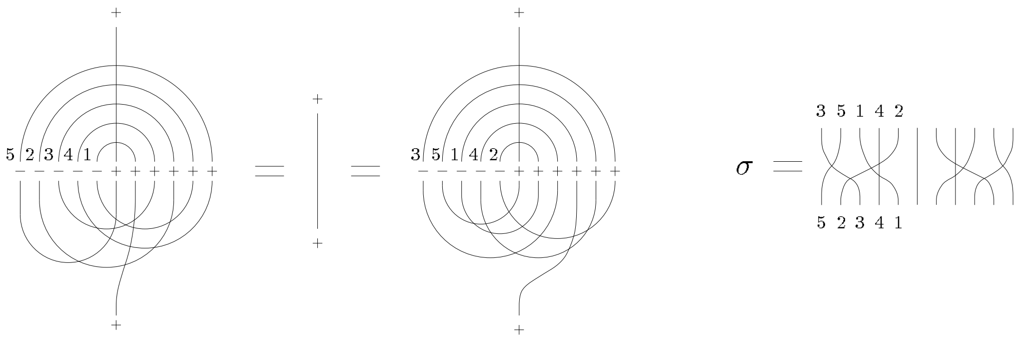


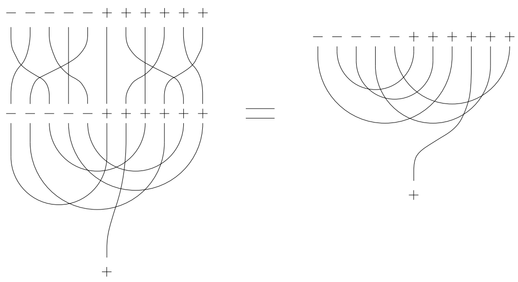
ψ =



The reader is invited to check all properties in the given example.

(d) A morphism $(0,0) \longrightarrow (0,n)$ links all lower arcs of $(0,n)$ to a single vertical string. Hence any morphism determines and is determined by the order of the lower arcs of $(0,n)$, which is obtained by running through this single string. Obviously, this leads to $n!$ different morphisms.
The endomorphisms of $(0,n)$ are permutations of the lower arcs of $(0,n)$. Hence all endomorphisms are invertible and we get $End(0,n) \cong \Sigma_n$ with again $n!$ morphisms.
Two morphisms $\varphi_1, \varphi_2 : (0,0) \longrightarrow (0,n)$ differ only by the order they determine on the lower arcs of $(0,n)$; the permutation $\sigma : (0,n) \longrightarrow (0,n)$, which leads from the first order to the second, satisfies $\sigma \circ \varphi_1 = \varphi_2$.





The lemma is proven .—

Proof of 3.13: It has to be shown that every closed path in $B\mathcal{H}$ is null homotopic. $\mathcal{H}$ has a canonical filtration $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ by its "lower left corners", i.e. $\mathcal{H}_n$ is the full subcategory of objects (c,a) with $c + a \leq n$. This leads to a filtration of the classifying space and it follows that

$$\pi_1 B\mathcal{H} = \varinjlim \pi_1 B\mathcal{H}_n.$$

Hence it is sufficient to prove $\pi_1 B\mathcal{H}_n = 0$. This is trivial for $\mathcal{H}_0$. We assume by induction this statement to be true for $\mathcal{H}_{n-1}$ and will deduce it for $\mathcal{H}_n$. As a basepoint in $B\mathcal{H}_n$ we choose the point corresponding to the object $(0,n)$. Any path in $B\mathcal{H}_n$ may be represented by a finite combination of morphisms in $\mathcal{H}_n$. After composing all morphisms which can be composed, the arrows point in turn in different directions, and the only morphisms which have $(0,n)$ as a source are automorphisms which can be inverted. Therefore any closed path ω can be given in the following way

$$\omega : \quad (0,n) \xleftarrow{\varphi_1} (c_1,a_1) \xrightarrow{\varphi_2} (c_2,a_2) \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_{k-1}} (c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n),$$

where no arrow is an endomorphism, with one small exception which is treated in the following step 1.

<u>*Step 1:*</u> ω of the form $(0,n) \xrightarrow{\varphi} (0,n)$ is null homotopic.

There is nothing to show, since ω is a path in Δ_n, which is contractible.

<u>*Step 2:*</u> ω of the form $(0,n) \xleftarrow{\varphi_1} (0,n') \xrightarrow{\varphi_2} (0,n)$ is null homotopic.

Choose any $\psi : (0,0) \longrightarrow (0,n')$ and an automorphism $\sigma : (0,n) \longrightarrow (0,n)$ with $\sigma \circ (\varphi_1 \circ \psi) = \varphi_2 \circ \psi$ according to 3.14 (d). Then the following diagram gives a homotopy to a path treated in step 1.

$$\begin{tikzcd}
(0,n) \arrow[dd, equal] & (0,n') \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_2"] & (0,n) \arrow[dd, equal] \\
& (0,0) \arrow[u, "\psi"'] \arrow[dl, "\varphi_1 \circ \psi"'] \arrow[dr, "\varphi_2 \circ \psi"] & \\
(0,n) \arrow[rr, "\sigma"'] & & (0,n)
\end{tikzcd}$$

<u>*Step 3:*</u> Every path $\omega' : (c_i,a_i) \xrightarrow{\varphi_{i+1}} (c_{i+1},a_{i+1}) \xleftarrow{\varphi_{i+2}} \cdots \xleftarrow{\varphi_j} (c_j,a_j)$ with $(c_i,a_i),(c_j,a_j) \in \mathcal{H}_{n-1}$ is homotopic rel boundary points to a path in $\mathcal{H}_{n-1}$.

Without loss of generality we can assume $(c_k,a_k) \in \Delta_n$ for all $i < k < j$, since ω' can be decomposed into such paths and paths which already belong completely to $\mathcal{H}_{n-1}$. Furthermore, we can assume $c_i + a_i \geq c_j + a_j$ because of symmetry. Choose morphisms $\psi_1 : (c_{i+1},a_{i+1}) \longrightarrow (0,n)$, $\psi_2 : (c_{j-1},a_{j-1}) \longrightarrow (0,n)$. All further arrows in the following diagram are obtained from composition and application of 3.14 (b) and (c). The two triangles, which do not necessarily commute, can somehow be filled by the contractibility of Δ_n (labelled with (a)) and by the assertion of step 2 (indicated by (2)), respectively.

$$
\begin{tikzcd}
(c_i,a_i) \arrow[r, "\varphi_{i+1}"] \arrow[dd, equal] \arrow[dr, "\psi_1\circ\varphi_{i+1}"'] & (c_{i+1},a_{i+1}) \arrow[d, "\psi_1"] \arrow[r, dotted, no head] \arrow[dr, phantom, "(a)"] & (c_{j-1},a_{j-1}) \arrow[dl, "\psi_2"] \arrow[dr, phantom, "(b)"] & (c_j,a_j) \arrow[l, "\varphi_j"'] \arrow[d, equal] \\
{} \arrow[r, phantom, "(b)"] & (0,n) \arrow[dr, phantom, "(c)"] & (0,c_j+a_j) \arrow[l] \arrow[d] & (c_j,a_j) \arrow[l] \arrow[d, equal] \\
(c_i,a_i) \arrow[r] & (0,c_i+a_i) \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "(2)"] & (0,c_i+a_i) \arrow[ul] & (c_j,a_j) \arrow[l]
\end{tikzcd}
$$

Step 4: ω of the form $(0,n) \xleftarrow{\varphi_1} (c_1,a_1)\cdots(c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n)$ with $(c_1,a_1),(c_{k-1},a_{k-1}) \in \mathcal{H}_{n-1}$ is null homotopic.

Because of symmetry we may assume $c_1+a_1 \geq c_{k-1}+a_{k-1}$. Then look at the following diagram in which again (b) and (c) mark the corresponding statements in 3.14, and $(*)$ marks that part of the diagram which can be filled by the assertion of step 3 and the induction hypothesis:

$$
\begin{tikzcd}
(0,n) \arrow[dd, equal] \arrow[ddr, phantom, "(b)"] & (c_1,a_1) \arrow[l, "\varphi_1"'] \arrow[dd] \arrow[r, dotted, no head] \arrow[ddr, phantom, "(*)"] & (c_{k-1},a_{k-1}) \arrow[r, "\varphi_k"] \arrow[d] \arrow[dr, phantom, "(b)"] & (0,n) \arrow[d, equal] \\
& & (0,c_{k-1}+a_{k-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(c)"] & (0,n) \arrow[d, equal] \\
(0,n) & (0,c_1+a_1) \arrow[l, "\psi_1"'] \arrow[r, equal] & (0,c_1+a_1) \arrow[r, "\psi_2"] & (0,n)
\end{tikzcd}
$$

This is a reduction to step 2.

Step 5: An arbitrary $\omega : (0,n) \xleftarrow{\varphi_1} (c_1,a_1) \xrightarrow{\varphi_2} (c_2,a_2)\cdots(c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n)$ is null homotopic.

Assume $c_1+a_1=n$, i.e. φ_1 is a morphism of Δ_n. Then φ_2 is also a morphism and (c_2,a_2) an object of Δ_n. Choose any morphism $(0,n) \xleftarrow{\psi} (c_2,a_2)$. Since Δ_n is contractible, the path defined by $(0,n) \xleftarrow{\varphi_1} (c_1,a_1) \xrightarrow{\varphi_2} (c_2,a_2)$ is homotopic to ψ, and ω is homotopic to $(0,n) \xleftarrow{\psi\circ\varphi_3} (c_3,a_3)\cdots(c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n)$. In this way ω can be shortened gradually until the first remaining object (c_i,a_i) does not belong to Δ_n, and the same can be done with the other end of ω. This leads to step 4. If all (c_i,a_i) belong to Δ_n, we have immediately finished, because of its contractibility.

This completes the proof of 3.13 .—

3.15 Remark: A morphism $(c,a) \longrightarrow (c',a')$ in $\mathcal{H}$ is given by a morphism $\varphi : -a \oplus a+1 \longrightarrow -a' \oplus a'+1$ in $\mathcal{G}$, but the same φ also defines morphisms $(k+c-c',a) \longrightarrow (k,a')$ for all $k \in \mathbb{N}$.
Hence Lemma 3.14 essentially shows that all morphisms of $\mathcal{H}$ are generated by morphisms $(c,a) \longrightarrow (c,a)$, $(c,a) \longrightarrow (c,a{+}1)$ and $(c,a) \longrightarrow (c{-}1,a{+}1)$, as indicated in the scheme on page 58. The only assertion which has to be added and is easy to prove is that each morphism $(k,n-k) \longrightarrow (0,n)$ can be factored as $(k,n-k) \longrightarrow (l,n-l) \longrightarrow (0,n)$ with any chosen $k \leq l \leq n$. Then if any morphism $(c,a) \longrightarrow (c',a')$ is given, interpret it as morphism $(c-c',a) \longrightarrow (0,a')$, factor it by the lemma and the above assertion, shift the factors back by c' cycles and arrive at a factorization

$$(c,a) \longrightarrow (c{-}1,a{+}1)\cdots \longrightarrow (c',a{+}c{-}c') \longrightarrow (c',a{+}c{-}c'{+}1)\cdots \longrightarrow (c',a').$$

Further, note that the functor $(\beta_- \circ \beta_+)_* : \mathcal{H} \longrightarrow \mathcal{H}$, which adds one cycle to each object, is homotopic to the identity functor, as shown in the proof of 3.9. Another look at the mentioned scheme shows in addition that $\mathcal{H}$ is isomorphic to the image of that functor, which is the full subcategory of objects (c,a) with $c \geq 1$. Iteration is possible .—

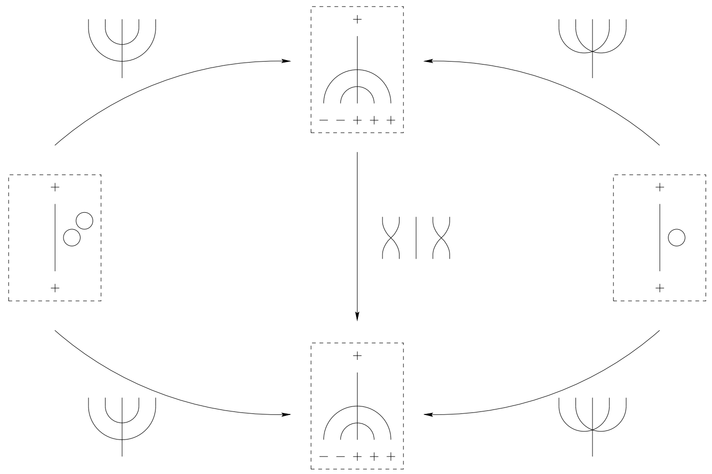
Of course, it would have been desirable to have higher connectivity for $\mathcal{H}$, maybe even contractibility. Unfortunately, this is not the case, since $B\mathcal{H}$ is not even 2-connected, as the following result shows:

3.16 Proposition: The homomorphism

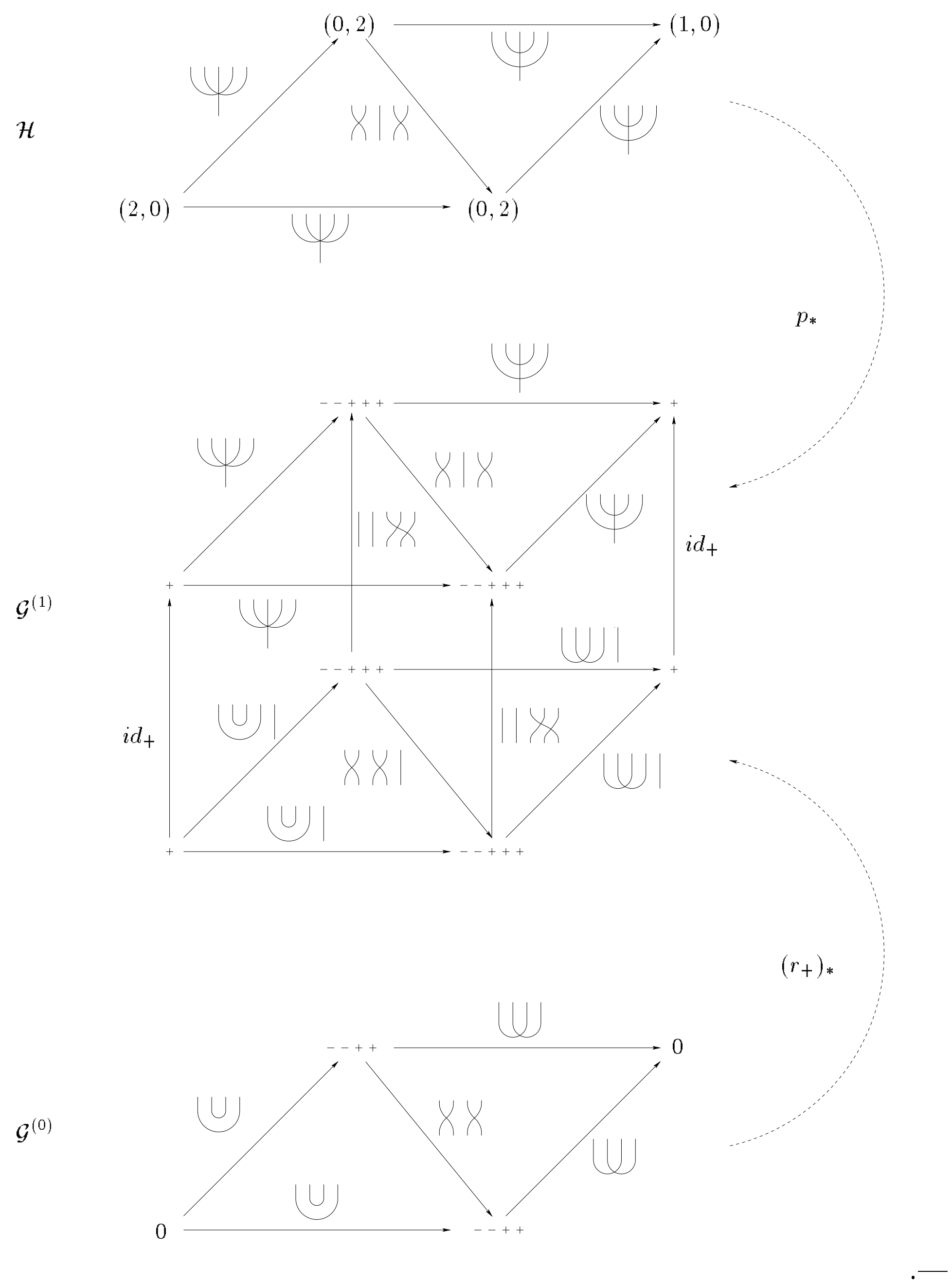
$$p_* : \pi_2(B\mathcal{H}, (+ \uparrow_{|\circ} +)) \longrightarrow \pi_2(B\mathcal{G}, +),$$

which is induced by the functor $p : \mathcal{H} \longrightarrow \mathcal{G}, \;\; p(w \uparrow_\alpha +) = w$ (cf. 3.11), is surjective .—

Proof: From 2.3 we have $\pi_2(B\mathcal{G}, 0) = \pi_2(\Omega^\infty S^\infty, *)$, which is well known to be $\mathbb{Z}/2$. The 2-sphere Θ of 2.12 represents the non-trivial element, and the non-trivial element of $\pi_2(B\mathcal{G}, +)$ is represented by $(r_+)_*(\Theta)$ (cf. 2.11). Hence it suffices to lift $(r_+)_*(\Theta)$ up to homotopy into $B\mathcal{H}$.
This is done by the realization of the following picture which defines a 2-sphere Θ' in $B\mathcal{H}$ (compare the picture on page 57):



The picture on the next page clarifies the connection between Θ' and Θ. After realization the cube in the middle defines a homotopy between $p_*(\Theta')$ and $(r_+)_*(\Theta)$.



In this way we obtain $\pi_2(B\mathcal{H}) \neq 0$ and conclude that $B\mathcal{G}$ and $B\mathcal{F}_{P,D}$ are not of the same homotopy type, i.e. the addition of a free duality structure to $\mathcal{F}_P$ does not give a topological group completion on the level of classifying spaces.

3.3 The lower homotopy groups of $\mathcal{F}_{P,D}$ and the generator for $\pi_1(B\mathcal{F}_{P,D}, 0)$

3.17 Proposition: The functors $l_+,\ l_- : \mathcal{F}_{P,D} \longrightarrow \mathcal{F}_{P,D}$, defined by

$$l_+(w) := + \oplus w,\ l_+(\alpha) := id_+ \oplus \alpha, \quad l_-(w) := - \oplus w,\ l_-(\alpha) := id_- \oplus \alpha,$$

map $\mathcal{F}^{(k)}_{P,D}$ into $\mathcal{F}^{(k+1)}_{P,D}$ and vice versa and induce homotopy inverse maps on classifying spaces. The same is true for $r_+, r_- : \mathcal{F}_{P,D} \longrightarrow \mathcal{F}_{P,D}$ given by

$$r_+(w) := w \oplus +,\ r_+(\alpha) := \alpha \oplus id_+,\ r_-(w) := w \oplus -,\ r_-(\alpha) := \alpha \oplus id_- \ .\text{—}$$

This is proven in exactly the same way as 2.11 .—

Hence the following diagram commutes up to natural transformation so that the inclusion $\mathcal{G}^{(k)} \hookrightarrow \mathcal{F}^{(k)}_{P,D}$ has the same homotopy theoretic properties as $\mathcal{G}^{(1)} \hookrightarrow \mathcal{F}^{(1)}_{P,D}$.

$$\begin{tikzcd}
\mathcal{G}^{(k+1)} \arrow[r, "I"] & \mathcal{F}^{(k+1)}_{P,D} \arrow[d, "l_-"] \\
\mathcal{G}^{(k)} \arrow[u, "l_+"] \arrow[r, "I"] & \mathcal{F}^{(k)}_{P,D}
\end{tikzcd}$$

Invoking 3.12 and 3.13, the exact sequence of 3.11 ends with

$$\cdots \pi_2 B\mathcal{H} \xrightarrow{p_*} \pi_2 B\mathcal{G}^{(1)} \xrightarrow{I_*} \pi_2 B\mathcal{F}^{(1)}_{P,D} \longrightarrow 0 \xrightarrow{p_*} \pi_1 B\mathcal{G}^{(1)} \xrightarrow{I_*} \pi_1 B\mathcal{F}^{(1)}_{P,D} \longrightarrow 0$$

and by 3.16 the homomorphism $I_* : \pi_2 B\mathcal{G}^{(1)} \longrightarrow \pi_2 B\mathcal{F}^{(1)}_{P,D}$ turns out to be the zero map. Hence we can summarize (using also 3.7):

3.18 Theorem:
(a) For any choice of basepoint in $\mathcal{G}$ the inclusion $I : \mathcal{G} \longrightarrow \mathcal{F}_{P,D}$ is 1-connected but not 2-connected, i.e. the induced homomorphism

$$I_* : \pi_i(B\mathcal{G}, w) \longrightarrow \pi_i(B\mathcal{F}_{P,D}, w)$$

is an isomorphism for $i = 0, 1$ and a surjection for $i = 2$.

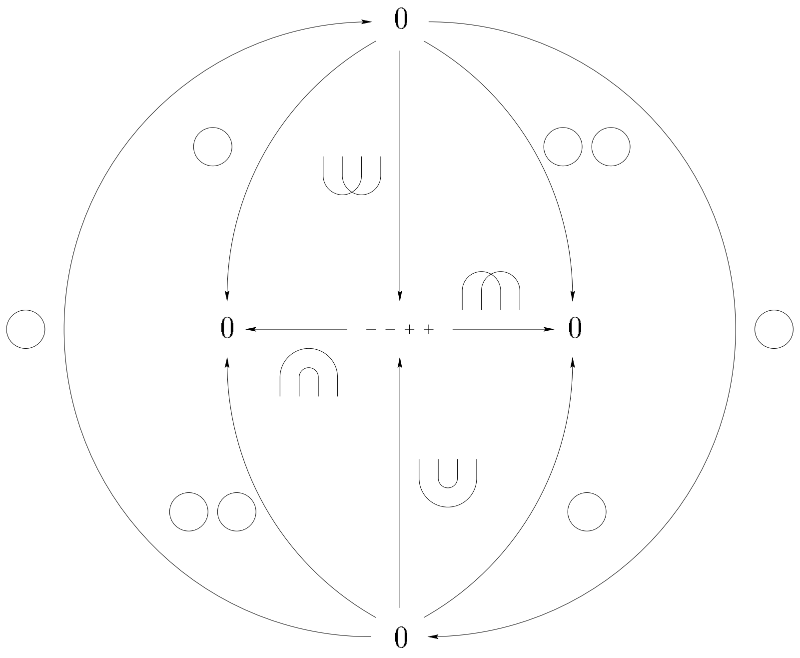
(b) The homotopy groups for $i = 0, 1, 2$ of $\mathcal{F}_{P,D}$ are

$$\begin{aligned}
\pi_0(B\mathcal{F}_{P,D}, w) &= \mathbb{Z}, \\
\pi_1(B\mathcal{F}_{P,D}, w) &= \mathbb{Z}/2, \\
\pi_2(B\mathcal{F}_{P,D}, w) &= 0 \qquad .\text{—}
\end{aligned}$$

By the above theorem, the generator of $\pi_1(B\mathcal{G}, 0)$ generates $\pi_1(B\mathcal{F}_{P,D}, 0)$ as well, but there is another very nice representative:

3.19 Proposition: The endomorphism $0 \longrightarrow 0$ of $\mathcal{F}_{P,D}$, which is depicted by a single circle, represents the generator of $\pi_1(B\mathcal{F}_{P,D}, 0)$.—

Proof: The picture below displays a homotopy between the path κ of 2.10 and the single circle. It also gives a direct proof that the path defined by two circles is null homotopic.



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References

[1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, Iterated Monoidal Categories II, Preprint.

[2] Z. Fiedorowicz, The Quillen-Grothendieck construction and extensions of pairings, in Geometric Applications of Homotopy Theory I, Lecture Notes in Mathematics 657, Springer-Verlag 1978, pp. 163-169.

[3] D. Grayson, Higher algebraic K-Theory: II, Lecture Notes in Mathematics 551, Springer-Verlag 1976, pp. 217-240.

[4] G. Karpilovsky, The Schur Multiplier, London Mathematical Society Monographs New Series 2, Oxford University Press 1987.

[5] C. Kassel, Quantum Groups, Graduate Texts in Mathematics 155, Springer-Verlag 1994.

[6] G.M. Kelly, M.L. Laplaza, Coherence for Compact Closed Categories, Journal of Pure and Applied Algebra 19 (1980), pp. 193-213.

[7] K. Lamotke, Semisimpliziale algebraische Topologie, Die Grundlehren der math. Wiss. in Einzeld. 147, Springer-Verlag 1968.

[8] S. Mac Lane, Categories for the working mathematician, Graduate Texts in Mathematics 5, Springer-Verlag 1971.

[9] J.P. May, The spectra associated to permutative categories, Topology 17 (1987), pp. 225-228.

[10] S. Priddy, On $\Omega^\infty S^\infty$ and the infinite symmetric group, in Proceedings of Symposia in Pure Math. 22, Amer. Math. Soc. (1971), pp. 217-220.

[11] M.C. Shum, Tortile tensor categories, Journal of Pure and Applied Algebra 93 (1994), pp. 57-110.

[12] R.W. Thomason, Beware the phony multiplication on Quillen's $\mathcal{A}^{-1}\mathcal{A}$, Proceedings of the Amer. Math. Soc. 80 (4) (1980), pp. 569-573.

[13] D. Quillen, Higher algebraic K-Theory: I, Lecture Notes in Mathematics 341, Springer-Verlag 1973, pp. 85-147.

[14] G. Segal, Categories and cohomology theories, Topology 13 (1974), pp. 293-312.